20 Dec 2017

Prime Ideals

[commutative theory]

Recall that a ring A is an integral domain if $A \neq 0$, and for every nonzero $a, b \in A$ the product $a \cdot b$ is nonzero.

<u>Def</u> An ideal $\mathfrak{p} \subseteq A$ is called <u>prime</u> if the quotient ring $A/\mathfrak{p}$ is an integral domain.

In other word, $\mathfrak{p}$ is prime if $\mathfrak{p} \neq A$, and $a, b \notin \mathfrak{p} \Rightarrow a \cdot b \notin \mathfrak{p}$.

<u>Exa</u> ~~If~~ A is an integral domain, ~~then~~ iff $\mathfrak{p} := (0)$ is a prime ideal.

<u>Exa</u> If $\mathfrak{m}$ is a maximal ideal then it is prime.

<u>Exa</u> The ring $A = \mathbb{Z}$ has two kinds of prime ideals:

- The prime ideal $\mathfrak{p} = (0)$.
- The maximal ideals $\mathfrak{m} = (p)$, p a prime number.

If $\mathfrak{p} \subseteq A$ is a prime ideal, then its complement

$$S := A \setminus \mathfrak{p}$$

is a m.c.s.

<u>Notation:</u> The localized ring is denoted by

$$A_{\mathfrak{p}} := A_S.$$

For an A-mod. M we write

$$M_{\mathfrak{p}} := M_S \cong A_{\mathfrak{p}} \otimes_A M.$$

<u>Prop.</u> Let $f: A \to B$ be a ring hom. Given a prime ideal $\mathfrak{q} \subseteq B$, the ideal

$$\mathfrak{p} := f^{-1}(\mathfrak{q}) \subseteq A$$

is prime.

Pf. f induces an injective ring hom

$$\bar{f}: \frac{A}{\mathfrak{p}} \hookrightarrow \frac{B}{\mathfrak{q}}.$$

So $A/\mathfrak{p}$ is isom. to a subring of the integral domain $B/\mathfrak{q}$, and hence $A/\mathfrak{p}$ is an int. dom. $\square$

Def Let A be a ring. The set of prime ideals of A is called the <u>spectrum</u> of A, and is denoted by $\operatorname{Spec}(A)$.

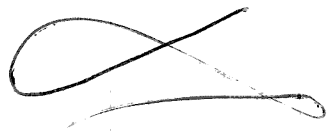
The prev. prop. shows that a ring hom $f: A \to B$ induces a function

$$(*) \qquad \operatorname{Spec}(f) : \begin{array}{ccc} \operatorname{Spec}(B) & \to & \operatorname{Spec}(A) \\ \cup\!\!| & & \cup\!\!| \\ \mathfrak{q} & \longmapsto & f^{-1}(\mathfrak{q}) \end{array}$$

Actually the set $\operatorname{Spec}(A)$ is a top. space. Its top. is called the <u>Zariski topology</u>.

The function $\operatorname{Spec}(f)$ above is continuous for this topology.

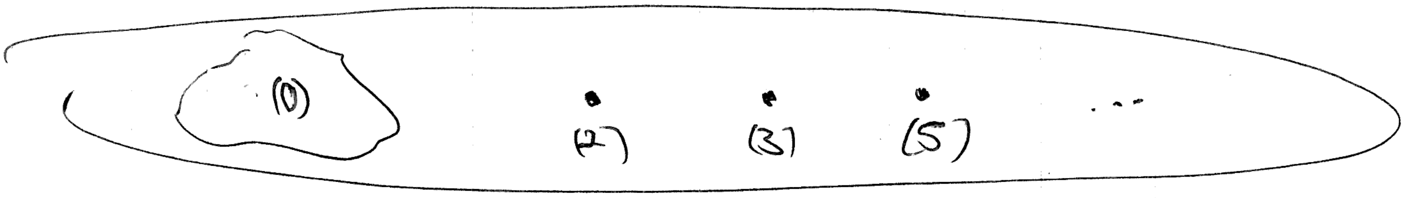
The topological space $\mathrm{Spec}(A)$ has more structure. It is an affine scheme.
These are the building blocks for schemes, in the algebraic geometry ~~introduced~~ by Grothendieck around 1960.



Example

$A = \mathbb{Z}$

$X = \mathrm{Spec}(\mathbb{Z}) =$



closed sets: X, $Z = \{$finite set of max. ideals$\}$

open sets: $\emptyset$, $U = \{$complement of finite set of max ideals$\}$

U open nonempty $\Rightarrow (0) \in U$.

So closure of $\{(0)\} = X$.

$\uparrow$ generic pt of $\mathrm{Spec}\,\mathbb{Z}$.

Def. A chain of prime ideals in A is a sequence $(\mathbb{P}_0, \ldots, \mathbb{P}_n)$ of prime ideals, s.t. $\mathbb{P}_i \subsetneq \mathbb{P}_{i+1}$. The length of this chain is n.

Def The Krull dimension of A is the supremum of the lengths of chains of prime ideals in A. It's denoted by dim (A).

→ Exercise. Let A be either $\mathbb{Z}$, or the polynomial ring $\mathbb{k}[t]$ in one variable over a field $\mathbb{k}$. Show that $\dim(A) = 1$.

→ Exercise. $A := \mathbb{Z}/(24)$. Find Spec (A) and dim(A).

→ Exercise. Let $\mathbb{k}$ be an algebraically closed field, Let V be a fin. gen. $\mathbb{k}$-mod, let $T \in \mathrm{End}_{\mathbb{k}}(V)$, and let

$A := \mathbb{k}[T]$, the $\mathbb{k}$-subring of $\mathrm{End}_{\mathbb{k}}(V)$ generated by T.

Find Spec (A). Relate it to the eigenvalues of T.

→ Exercise. Let $\mathbb{k}$ be a field and $A := \mathbb{k}[t_1, \ldots, t_n]$, the poly. ring in n variables. Show that
$\dim(A) \geq n$.
(The thm. on next page says that $\dim(A) = n$.)

→ Exercise. Here $A := \mathbb{Z}[t_1, \ldots, t_n]$.
Show that $\dim(A) \geq n+1$.
(There's equality here, but harder to prove.)

§3.3

Hopefully we will have time to prove the next thm.

Thm Let $\mathbb{k}$ be a field, and let A be a finitely generated $\mathbb{k}$-ring that's an integral domain. Then

$$\dim(A) = \operatorname{tr.deg}_{\mathbb{k}}(L),$$

where $L := \operatorname{Frac}(A)$.

Above, tr.deg is transcendence degree of a field extension.

<u>Lemma.</u> Let $S \subseteq A$ be a m.c.s., and let $\mathfrak{b} \in A_S$ be an ideal. Consider the can. ring hom. $g: A \to A_S$. Let

$$\mathfrak{a} := g^{-1}(\mathfrak{b}) \subseteq A.$$

Then $\boxed{\mathfrak{b} = \mathfrak{a} \cdot A_S.}$

<u>Pf.</u>

Since $g(\mathfrak{a}) \subseteq \mathfrak{b}$ we have

$$\mathfrak{a} \cdot A_S \subseteq \mathfrak{b}.$$

For the opposite inclusion, take some

$\frac{a}{s} \in \mathfrak{b}$. So $\quad g(a) = \frac{a}{1} = \frac{s}{1} \cdot \frac{a}{s} \in \mathfrak{b}$

$\Rightarrow \quad a \in \mathfrak{a} \quad \Rightarrow \quad \frac{a}{s} = \frac{1}{s} \cdot g(a) \in A_S \cdot \mathfrak{a}.$

$\square$

Thm Let $S \subseteq A$ be a m.c.s., with loc. hom. $g: A \to A_S$. Then the function

$$\mathrm{Spec}(g): \mathrm{Spec}(A_S) \to \mathrm{Spec}(A)$$

gives a bijection

$$\mathrm{Spec}(A_S) \xrightarrow{\simeq} \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\}.$$

The inverse function is $\mathfrak{p} \mapsto A_S \cdot \mathfrak{p}$.

Pf.

Step 1. $\mathrm{Spec}(g)$ is injective: if

$$\underbrace{g^{-1}(\mathfrak{q}_1)}_{\mathfrak{p}_1} = \underbrace{g^{-1}(\mathfrak{q}_2)}_{\mathfrak{p}_2}$$

then $\mathfrak{q}_1 = A_S \cdot \mathfrak{p}_1 = A_S \cdot \mathfrak{p}_2 = \mathfrak{q}_2$.

by the lemma

Step 2. Let $\mathfrak{q} \in \mathrm{Spec}(A_S)$ and $\mathfrak{p} := g^{-1}(\mathfrak{q})$.

If there exists $s \in \mathfrak{p} \cap S$, then

$1 = s \cdot \frac{1}{s} \in \mathfrak{p} \cdot A_S = \mathfrak{q}$, which is false.

So $\mathfrak{p} \cap S = \emptyset$.

Step 3 Let

$$\mathfrak{p} \in \operatorname{Spec}(A), \quad \mathfrak{p} \cap S = \emptyset.$$

Let $\mathfrak{q} := \mathfrak{p} \cdot A_S \subseteq A_S$. We will prove that $\mathfrak{q}$ is prime.

Consider the ex. seq

$$0 \to \mathfrak{p} \to A \to \underset{\bar{A}}{\underset{\|}{A/\mathfrak{p}}} \to 0$$

Since A_S is flat over A, get ex. seq.

$$0 \to A_S \otimes_A \mathfrak{p} \to A_S \to A_S \otimes_A \bar{A} \to 0.$$

the image of $A_S \otimes_A \mathfrak{p}$ in A_S is $A_S \cdot \mathfrak{p} = \mathfrak{q}$. On the other hand,

$$A_S \otimes_A \bar{A} \cong \bar{A}_{\bar{S}},$$

the localiz. of $\bar{A}$ at the m.c.s. $\bar{S}$ $:= (\{\text{image of } S \text{ in } \bar{A}\})$. So have ex. seq.

$$(\dagger) \quad 0 \to \mathfrak{q} \to A_S \to \bar{A}_{\bar{S}} \to 0$$

Now. $S \cap \mathfrak{p} = \emptyset \Rightarrow \bar{S} \subseteq \bar{A} - \{0\}$, so

$\bar{A}_{\bar{S}} \subseteq \operatorname{Frac}(\bar{A})$, the field of fractions

$\Rightarrow$ $\bar{A}_{\bar{S}}$ integ. domain.

$\Rightarrow$ (from $\dagger$) $\mathfrak{q}$ is a prime ideal.

Step 4. With $\mathfrak{q}$, $\mathfrak{P}$ from step 3, we prove that

$$\mathfrak{P} = g^{-1}(\mathfrak{q}).$$

Have comm. diag. with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{P} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "g"] \arrow[dr, dashed, "h"] & \bar{A} \arrow[r] \arrow[d, "\text{injective}"] & 0 \\
0 \arrow[r] & \mathfrak{q} \arrow[r] & A_S \arrow[r] & \bar{A}_{\bar{S}} \arrow[r] & 0
\end{tikzcd}$$

, because $\bar{A}$ int. dom, $\bar{S} \subseteq \bar{A} - \{0\}$

So $g^{-1}(\mathfrak{q}) = \{a \in A \mid h(a) = 0\} = \mathfrak{P}$.

by diagram chase

$\square$

Local Rings

Def A ring A is called *local* if it has exactly one maximal ideal.

Def Let A be local, with max. ideal $\mathfrak{m}$. The field $A/\mathfrak{m}$ is called the *residue field* of A.

Thm. Let $\mathfrak{p}$ be a prime ideal in A. Then $A_{\mathfrak{p}}$ is a local ring, with max. ideal $\mathfrak{p}_{\mathfrak{p}}$, and residue field

$$A_{\mathfrak{p}} / \mathfrak{p}_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p}).$$

Pf. Consider the short exact seq.

$$(\dagger) \qquad 0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0 .$$

Let $S := A - \mathfrak{p}$. The image $\bar{S}$ of S in $\bar{A} := A/\mathfrak{p}$ is

$$\bar{S} = \bar{A} - \{0\}.$$

So $\bar{A}_{\bar{S}} = \mathrm{Frac}(\bar{A})$. But $\bar{A}_{\bar{S}} \cong A_{\mathbb{P}} \otimes_A \bar{A}$.
Now we apply $A_{\mathbb{P}} \otimes_A -$ to (†).
Get sh. ex. seq.

$$(\dagger) \qquad 0 \to \mathbb{P}_{\mathbb{P}} \to A_{\mathbb{P}} \longrightarrow \mathrm{Frac}(\bar{A}) \to 0.$$

We see that $\mathbb{P}_{\mathbb{P}}$ is a max ideal of $A_{\mathbb{P}}$.

Using the bijection from Thm. [p. 85], the max. ideal $\mathbb{P}_{\mathbb{P}}$ corresponds to

$$\mathbb{P} \in \{ \text{prime ideals } \mathfrak{r} \subseteq A \text{ s.t. } \underset{\substack{\Updownarrow \\ \mathfrak{r} \subseteq \mathbb{P}}}{\mathfrak{r} \cap S = \emptyset} \}$$

$S = A - \mathbb{P}$

and $\mathbb{P}$ is the max elt. in this set. Thus every prime ideal of $A_{\mathbb{P}}$ is contained in $\mathbb{P}_{\mathbb{P}}$. We see that $\mathbb{P}_{\mathbb{P}}$ is the only max. ideal of $A_{\mathbb{P}}$.

$\square$

The Nakayama Lemma

Lem. Let A be local, with max. ideal $\mathfrak{m}$. If $a \in \mathfrak{m}$ then $1-a$ is an invertible element.

Pf. If $b := 1-a$ is not invertible, then the ideal $\mathfrak{b} := (b)$ is $\subsetneq A$. So $A/\mathfrak{b} \neq 0$, so $\exists$ max ideal $\bar{\mathfrak{m}}$ in $A/\mathfrak{b} =: \bar{A}$. Let $\mathfrak{n} \subseteq A$ be the preimage of $\bar{\mathfrak{m}}$. So

$$\frac{A}{\mathfrak{n}} \cong \frac{\bar{A}}{\bar{\mathfrak{m}}} \text{ is a field,}$$

so $\mathfrak{n} \subseteq A$ max, so $\mathfrak{n} = \mathfrak{m}$.

Thus $\mathfrak{b} \subseteq \mathfrak{n} = \mathfrak{m} \Rightarrow \underset{1-a}{\overset{\shortparallel}{b}} \in \mathfrak{m}$

$\Rightarrow 1 = a+b \in \mathfrak{m}$. Contradiction $\square$

Thm (Nakayama Lemma).
Let A be a local ring with max. ideal $\mathfrak{m}$. Let M be a fin. gen. A-mod. If $\mathfrak{m} \cdot M = M$ then $M = 0$.

There's a fancy proof using the Cayley-Hamilton Thm. We'll give an elementary proof.

pf Let $r \geq 0$ be the minimal number of generators of M. We will prove that $r=0$. Otherwise, if $r \geq 1$, then let $m_1, \ldots, m_r$ be generators of M. Since $M = m \cdot M$, we can find $a_1, \ldots, a_r \in m$ s.t.

$$m_r = a_1 m_1 + \cdots + a_r \cdot m_r.$$

So

$$m_r = (1-a_r)^{-1} \cdot (a_1 m_1 + \cdots + a_{r-1} m_{r-1})$$

By the lemma, $1-a_r$ is invertible.

We see that $m_1, \ldots, m_{r-1}$ also generate M, contrary to the minimality of r. $\square$

to here lecture 20.12.17

to students: continue reading all the way to page 98, and solve the exercises.

→ __Exercise__ find a local ring $(A, \mathfrak{m})$, and a nonzero A-mod. M, s.t. $M = \mathfrak{m} \cdot M$ (of course M is not fin. gen.)
(Hint: $A = \mathbb{Z}_{(p)}$, p a prime number, and $M = \mathbb{Q}$)

(in next corollaries) Here A is a local ring, $\mathfrak{m}$ the max. ideal, and $K := A/\mathfrak{m}$ the res. fld.

__Cor 1.__ If M is a nonzero f.g. A-mod then $K \otimes_A M \neq 0$.

__pf__

$K \otimes_A M \cong M/\mathfrak{m} \cdot M$. It's nonzero by Nakayama. $\square$

__Cor 2.__ Let M be a fin. gen. A-mod, and let $\underline{m} = (m_1, \dots, m_r)$ be a seq. in M, with image $\bar{\underline{m}} = (\bar{m}_1, \dots, \bar{m}_r)$ in $\bar{M} := K \otimes_A M$. TFAE:

(i) $\underline{m}$ generates M.

(ii) $\bar{\underline{m}}$ generates $\bar{M}$ (as K-module).

Pf. Let $\varphi: A^r \to M$ be the hom. corresponding to $\underline{\underline{m}}$. Let $N := \mathrm{Coker}(\varphi)$, which is fin. gen. too.

Have ex. seq of A-mods:

$$A^r \xrightarrow{\varphi} M \to N \to 0$$

Applying $k \otimes_A -$ get ex. seq. of k-mods:

$$k^r \xrightarrow{\bar{\varphi}} \bar{M} \to \bar{N} \to 0$$

and $\bar{\varphi}$ corresponds to $\bar{\underline{\underline{m}}}$.

So: $\underline{\underline{m}}$ generates $\Leftrightarrow N = 0 \Leftrightarrow \bar{N} = 0$

$\Leftrightarrow \bar{\underline{\underline{m}}}$ generates

by Cor. 1

$\square$

Cor. 3. In the situation of Cor. 2, TFAE:

(i) $\underline{\underline{m}}$ is a minimal generating seq. of M. (i.e. of minimal length)

(ii) $\bar{\underline{\underline{m}}}$ is a basis of $\bar{M}$.

Pf. Note:

$\bar{\underline{\underline{m}}}$ is lin. indep. iff $\mathrm{rank}_k(\bar{M}) = r$,

iff $\bar{M}$ can't be gen. by $< r$ elts,

iff M " " " " " "

(by Cor. 2)

$\square$

Under both conditions $\underline{\underline{m}}$ gen's M & $\bar{\underline{\underline{m}}}$ gen's $\bar{M}$, by Cor. 2.

Cor. 4. Let M & N be nonzero f.g. A-mods. Then $M \otimes_A N \neq 0$.

Pf. $\bar{M} := k \otimes_A M$ & $\bar{N} := k \otimes_A N$ are nonzero (free of ranks >0, so) k-mods, so $\bar{M} \otimes_k \bar{N} \neq 0$.

But $\bar{M} \otimes_k \bar{N} \cong k \otimes_A (M \otimes_A N)$. $\square$

Using Nakayama & some homological algebra, we will prove (in next course):

Thm. Let A be a noeth. local ring, and let P be a fin. gen. flat A-mod. Then P is free.

→ Exercise. (hard!) A is a local ring, M & N are fin. gen. A-mods, and
$$M \otimes_A N \cong A.$$
Then $M \cong N \cong A$.
(Hint: use the corollaries to deduce that M & N are cyclic A-mods)
$\parallel$
(single generator)

Support of Modules

Def. Let M be an A-module. The support of M is the set
$$\mathrm{Supp}(M) := \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0 \}.$$

Theorem. Let M be an A-module.
TFAE:

(i) $M \neq 0$.
(ii) $\mathrm{Supp}(M) \neq \emptyset$.

pf. (ii) $\Rightarrow$ (i): $\mathrm{Supp}(M) \neq \emptyset \Rightarrow \exists\, \mathfrak{p}$ s.t.
$M_{\mathfrak{p}} \neq 0 \Rightarrow M \neq 0$.

(i) $\Rightarrow$ (ii).
Step 1. First assume M is cyclic, say
$M = A\cdot m$. Let
$\mathfrak{a} := \mathrm{Ann}(M) = \{ a \in A \mid a\cdot m = 0 \}$.
So $M \cong A/\mathfrak{a}$ as A-mods.
The ring $\bar{A} := A/\mathfrak{a}$ is nonzero, so it has a max. ideal $\bar{\mathfrak{m}} \subseteq \bar{A}$.

Let $k := \bar{A}/\bar{\mathfrak{m}}$. So
$$k \otimes_A M \cong k \otimes_{\bar{A}} \bar{A} \cong k \quad \text{as } A\text{-mods}$$

Let $\mathfrak{m} \subseteq A$ be the pre-image of $\bar{\mathfrak{m}}$.

So $A/_{\mathfrak{m}} \cong \bar{A}/_{\bar{\mathfrak{m}}} = k$.

$\mathfrak{m}$ is maximal, so prime.

$$M_{\mathfrak{m}} = A_{\mathfrak{m}} \otimes_A M \longrightarrow K \otimes_A M \cong K$$

And $A_{\mathfrak{m}} / \mathfrak{m}_{\mathfrak{m}} \cong A/_{\mathfrak{m}} \cong k$.
Get hom

So $M_{\mathfrak{m}} \neq 0$.

Step 1. Now $M \neq 0$ arbitrary. Take $m \in M$, $m \neq 0$. Then $M' := A \cdot m \subseteq M$ is nonzero & cyclic so

$$0 \neq M'_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}.$$

(flatness of localiz.)

$\square$

→ <u>Exercise</u> Let M be a finite ab. grp, with decomposition into cyclic grps:

$$M \cong \bigoplus_{i=1}^{r} \mathbb{Z}/(p_i^{e_i})$$

for prime numbers $p_1, \ldots, p_r$ (poss. w. repetition) and $e_i \geq 1$.

Then

$$\mathrm{Supp}(M) = \{ (p_1), \ldots, (p_r) \}.$$
$$\subseteq \mathrm{Spec}(\mathbb{Z})$$

<u>Example</u>. Let M and N be nonzero fin. gen. ab. grp. Let's analyze when $M \otimes_{\mathbb{Z}} N \neq 0$.

<u>Case 1</u>. M (or N) is infinite.

So $M \cong \mathbb{Z} \oplus M'$. For any $\mathfrak{p} \in \operatorname{Spec}(\mathbb{Z})$
Have $M_{\mathfrak{p}} \neq 0$.
I.e. $\operatorname{Supp}(M) = \operatorname{Spec}(\mathbb{Z})$.

$N \neq 0 \Rightarrow \exists\ \mathfrak{p} \in \operatorname{Supp}(N)$.
So $N_{\mathfrak{p}} \neq 0$, $M_{\mathfrak{p}} \neq 0$.
By Cor. 4 [p. 94], $M_{\mathfrak{p}} \otimes_{\mathbb{Z}_{\mathfrak{p}}} N_{\mathfrak{p}} \neq 0$.
But ... $\cong (M \otimes_{\mathbb{Z}} N)_{\mathfrak{p}}$.

So $M \otimes_{\mathbb{Z}} N \neq 0$.

<u>Case 2</u> M & N are finite, and

$$\operatorname{Supp}(M) \cap \operatorname{Supp}(N) \neq \emptyset.$$

(e.g. $M = \mathbb{Z}/(6)$, $N = \mathbb{Z}/(8)$
$\operatorname{Supp} = \{(2), (3)\}$ $\operatorname{Supp} = \{(2)\}$.)

Again $M \otimes_{\mathbb{Z}} N \neq 0$.

<u>Case 3</u>. $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) = \emptyset$.
(e.g. $M = \mathbb{Z}/(6)$, $N = \mathbb{Z}/(25)$). Then $M \otimes_{\mathbb{Z}} N = 0$.