COURSE NOTES: HOMOLOGICAL ALGEBRA

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Contents

1. Introduction

Lecture 1, 7 March 2018

Motivating discussion and history: proving the Brouwer Fixed Point Theorem, using the homology functors

$$\mathrm{H}_i : \mathsf{Top} \to \mathsf{Ab},$$

and the structure of the abelian groups $\mathrm{H}_i(\mathbf{B}^n)$ and $\mathrm{H}_i(\mathbf{S}^{n-1})$. This is in the handwritten pages.

What do we keep from this topological story?

- Functors.
- Complexes and their (co)homology.

But the categories that will interest us will be the category of left *A-modules* $\operatorname{Mod} A$ over a ring A, and its relatives. The functors we will work with will be *additive* functors

$$F : \operatorname{Mod} A \to \operatorname{Mod} B.$$

For such a functor F we will study its *left derived functors* $\mathrm{L}_i F$ and its *right derived functors* $\mathrm{R}^i F$.

Here is a sample result on the structure of modules over a commutative ring A that we will prove in the course. This is a theorem that the methods of the course "Commutative Algebra" could not produce for us.

Theorem 1.1. *Let A be a noetherian commutative ring, and let M be a finitely generated A-module. The two conditions below are equivalent.*

(i) *M is flat.*
(ii) *M is projective.*

All the concepts above, with the exception of *projective module*, were studied in the previous course. These concepts shall be explained in our course when the time comes. The proof of the theorem relies on the derived functors

$$\operatorname{Tor}_i^A(-,-) = \mathrm{L}_i(- \otimes_A -)$$

and

$$\operatorname{Ext}_A^i(-,-) = \mathrm{R}^i \operatorname{Hom}_A(-,-).$$

Their role is analogous to the role that the homology groups played in the proof of the Brouwer Theorem.

2. Categories

Definition 2.1. A category C is a mathematical structure consisting of these ingredients:

- A set $\operatorname{Ob}(\mathsf{C})$, whose elements are called the *objects* of C.
- For every pair $C, D \in \operatorname{Ob}(\mathsf{C})$ there is a set $\operatorname{Hom}_{\mathsf{C}}(C, D)$, whose elements are called the *morphisms* from C to D, and are denoted by $f : C \to D$.
- For every triple $C, D, E \in \operatorname{Ob}(\mathsf{C})$ there is a function
$$\operatorname{Hom}_{\mathsf{C}}(D, E) \times \operatorname{Hom}_{\mathsf{C}}(C, D) \to \operatorname{Hom}_{\mathsf{C}}(C, E), \quad (g, f) \mapsto g \circ f$$
called *composition*.
- For every $C \in \operatorname{Ob}(\mathsf{C})$ there is a morphism $\mathrm{id}_C \in \operatorname{Hom}_{\mathsf{C}}(C, C)$ called the *identity morphism*.

There are two axioms:

- (Associativity) For composable morphisms f, g, h there is equality
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- (Identity) For a morphism $f : C \to D$ there is equality
$$f = f \circ \mathrm{id}_C = \mathrm{id}_D \circ f.$$

Remark 2.2. In order to avoid set-theoretic difficulties, we assume that there is a given set U called a universe. The universe U is large enough so as to contain as elements all the mathematical structures that will concern us (rings, groups, topological spaces) and their cartesian products, power sets, and so on. A set S will be called a small set if $S \in \mathsf{U}$.

Our standing assumption is that every category C satisfies $\mathrm{Ob}(\mathsf{C}) \subseteq \mathsf{U}$, and that $\mathrm{Hom}_{\mathsf{C}}(C, D) \in \mathsf{U}$ for every $C, D \in \mathrm{Ob}(\mathsf{C})$.

We will not discuss set theoretical issues beyond the above. More details on this foundational aspect can be found in [Mac2, Section I.6].

Often we write $C \in \mathsf{C}$ as a shortcut for $C \in \mathrm{Ob}(\mathsf{C})$.

We now list a few important examples of categories.

Example 2.3. The category Set has as objects all the small sets. Thus $\mathrm{Ob}(\mathsf{Set}) = \mathsf{U}$. The morphisms $f : S \to T$ in Set are the functions; composition of morphisms is the usual composition of functions; and $\mathrm{id}_S(s) = s$ for every $s \in S$.

Example 2.4. The category Grp has as objects all groups. (Only the small groups, i.e. their underlying sets must be small.) The morphisms $\phi : G \to H$ in Grp are the group homomorphisms. Compositions and identity morphisms are the usual ones.

Example 2.5. The category Ab has as objects all abelian groups. The morphisms $\phi : M \to N$ in Ab are the group homomorphisms. Compositions and identity morphisms are the usual ones.

Example 2.6. The category Ring has as objects all rings. The morphisms $f : A \to B$ in Ring are the ring homomorphisms. (A ring homomorphism f is required to respect unit elements: $f(1_A) = 1_B$.) Compositions and identity morphisms are the usual ones.

Example 2.7. The category $\mathsf{Ring}_{\mathrm{c}}$ has as objects all commutative rings. The morphisms are the ring homomorphisms. Compositions and identity morphisms are the usual ones.

Let A be a ring. Recall that an element $a \in A$ is called *central* if $a \cdot b = b \cdot a$ for all $b \in A$. The center of A is the set of all central elements in it, and we denote it by $\mathrm{Cent}(A)$. It is easy to see that $\mathrm{Cent}(A)$ is a subring of A, and moreover $\mathrm{Cent}(A)$ is a commutative ring. There is a similar story for groups.

The next exercise will help us understand the center better.

Exercise 2.8. Find a ring homomorphism $f : A \to B$ such that $f(\mathrm{Cent}(A))$ is not contained in $\mathrm{Cent}(B)$. Do this also for groups.

Definition 2.9. Let $\mathbb{K}$ be a commutative ring.

(1) A $\mathbb{K}$-*ring* is a pair (A, f_A) consisting of a ring A and a ring homomorphism $f_A : \mathbb{K} \to A$. The ring homomorphism f_A is called the *structural homomorphism* of A, and usually it will be kept implicit, and we shall just say that A is a $\mathbb{K}$-ring.

(2) Suppose (A, f_A) and (B, f_B) are $\mathbb{K}$-rings. A $\mathbb{K}$-*ring homomorphism*

$$g : (A, f_A) \to (B, f_B)$$

is a ring homomorphism $g : A \to B$ such that $g \circ f_A = f_B$. I.e. the diagram

$$\begin{tikzcd} \mathbb{K} \arrow[d, "f_A"'] \arrow[dr, "f_B"] & \\ A \arrow[r, "g"] & B \end{tikzcd}$$

in Ring is commutative.

(3) A *central* $\mathbb{K}$-*ring* is a $\mathbb{K}$-ring (A, f_A) such that $f_A(\mathbb{K}) \subseteq \mathrm{Cent}(A)$.

(4) The *category of central $\mathbb{K}$-rings* is the category $\mathsf{Ring}/_{\mathrm{c}}\,\mathbb{K}$, whose objects are the central $\mathbb{K}$-rings (item (3)), and whose morphism are the $\mathbb{K}$-ring homomorphisms (item (2)).

Exercise 2.10. Show that every ring A admits a unique ring homomorphism $\mathbb{Z} \to A$, and this makes A into a central $\mathbb{Z}$-ring. Conclude that

$$\mathsf{Ring}/_{\mathrm{c}}\,\mathbb{Z} = \mathsf{Ring}\,.$$

So far our examples involved large categories (i.e. they had many objects). This is not always the case:

Example 2.11. Let A be a ring. We define a category A as follows. There is a single object x, so $\mathrm{Ob}(\mathsf{A}) = \{x\}$. The set of morphisms is

$$\mathrm{Hom}_{\mathsf{A}}(x, x) := A.$$

Composition is multiplication in A, and $\mathrm{id}_x := 1_A$.

Definition 2.12. Let A be a ring. A *left A-module* is an abelian group M, together with a function

$$A \times M \to M, \quad (a, m) \mapsto a \cdot m$$

called multiplication, satisfying these conditions for all $a, b \in A$ and $m, n \in M$:

- Associativity: $(a \cdot b) \cdot m = a \cdot (b \cdot m)$.
- Distributivity: $(a + b) \cdot m = (a \cdot m) + (b \cdot m)$ and $a \cdot (m + n) = (a \cdot m) + (a \cdot n)$.
- Unit: $1_A \cdot m = m$.

Exercise 2.13. Let A be a ring.

(1) Define what is a right A-module (relying on Definition 2.12).
(2) Assume A is a commutative ring. Show that there is no difference between left and right A-modules.

Definition 2.14. Let A be a ring. Given left A-modules M and N, an *A-linear homomorphism* $\phi : M \to N$ is a homomorphism of abelian groups such that

$$\phi(a \cdot m) = a \cdot \phi(m)$$

for all $a \in A$ and $m \in M$.

The set of A-linear homomorphism $\phi : M \to N$ is denoted by $\mathrm{Hom}_A(M, N)$.

Convention 2.15. From here on we fix a nonzero commutative base ring $\mathbb{K}$. We assume by default that all rings are central $\mathbb{K}$-rings, and all ring homomorphisms are over $\mathbb{K}$. Given a ring A, by default A-modules are left A-modules.

Definition 2.16. Let A be a ring. The category $\mathsf{Mod}\,A$ has the (left) A-modules as its objects, and the morphisms are the A-linear homomorphisms.

Thus

$$\mathrm{Hom}_{\mathsf{Mod}\,A}(M, N) = \mathrm{Hom}_A(M, N).$$

Exercise 2.17. Let A be a central $\mathbb{K}$-ring and let $M, N \in \mathsf{Mod}\,A$. Show that $\mathrm{Hom}_A(M, N)$ has a canonical $\mathbb{K}$-module structure.

Exercise 2.18. Let $\mathbb{K}$ be a commutative ring, A a central $\mathbb{K}$-ring and M a left A-module. Define $\mathrm{End}_A(M) := \mathrm{Hom}_A(M, M)$.

(1) Show that $\mathrm{End}_A(M)$ is a ring, in which multiplication is composition.
(2) Show that the function $\mathbb{K} \to \mathrm{End}_A(M)$, $\lambda \mapsto \lambda \cdot \mathrm{id}_M$, is a ring homomorphism, and moreover it makes $\mathrm{End}_A(M)$ into a central $\mathbb{K}$-ring.

The next exercise says that the converse is also true:

Exercise 2.19. Let $\mathbb{K}$ be a commutative ring, A a central $\mathbb{K}$-ring and M a $\mathbb{K}$-module. Suppose $g : A \to \mathrm{End}_{\mathbb{K}}(M)$ is a $\mathbb{K}$-ring homomorphism. For $a \in A$ and $m \in M$ define $a \cdot m := g(a)(m)$. Prove that this makes M into a left A-module.

The discussion of $\mathbb{K}$-linear categories, that was started yesterday, will resume next week.

Lecture 2, 14 March 2018

Before continuing with the material, I want to say a few words on the direction the course will take. The some of the students attended the course "Commutative Algebra" last semester, and there they learned a lot about categories and functors. Other students may not have seen this material before, and thus they need to be introduced to it gradually and effectively.

In order not to bore the first group of students, and still be accessible to the second group, I have decided to concentrate on the *noncommutative aspects* of categories and functors. In particular, we will learn:

- Right B-modules and the ring B^{op}; A-B-bimodules and the ring $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$.
- The tensor product $M \otimes_A N$ for a right A-module M and a left A-module N.
- A "baby" case of *Morita equivalence*, between $\operatorname{Mod} A$ and $\operatorname{Mod} B$, where A is a ring and $B := \operatorname{Mat}_r(A)$, the ring of $r \times r$ matrices for $r \geq 1$. This will be done with concrete formulas. (We may do the general Morita equivalence later.)

Now to the material.

Let $M, N, P \in \operatorname{Mod} \mathbb{K}$. Recall that a function

$$\beta : M \times N \to P$$

is called $\mathbb{K}$-bilinear if it's additive in the two arguments:

$$\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n),$$

$$\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$$

and respects multiplication by elements of $\mathbb{K}$:

$$\beta(\lambda \cdot m, n) = \beta(m, \lambda \cdot n) = \lambda \cdot \beta(m, n)$$

for all $m, m_1, m_2 \in M$, all $n, n_1, n_2 \in N$ and all $\lambda \in \mathbb{K}$.

Definition 2.20. Let $\mathbb{K}$ be a nonzero commutative ring. A $\mathbb{K}$-*linear category* is a category M, together with a $\mathbb{K}$-module structure on each of the morphism sets $\operatorname{Hom}_{\mathsf{M}}(M, N)$. The condition is that for every triple of objects $L, M, N \in \mathsf{M}$ the composition function

$$\operatorname{Hom}_{\mathsf{M}}(M, N) \times \operatorname{Hom}_{\mathsf{M}}(L, M) \to \operatorname{Hom}_{\mathsf{M}}(L, N)$$

is $\mathbb{K}$-bilinear.

Proposition 2.21. *If A is a central $\mathbb{K}$-ring, then $\operatorname{Mod} A$ is a $\mathbb{K}$-linear category.*

Proof. Let's write $\mathsf{M} := \operatorname{Mod} A$. We start by specifying the $\mathbb{K}$-module structure on each of the morphism sets $\operatorname{Hom}_{\mathsf{M}}(M, N)$. (This was Exercise 2.17.) The zero element of $\operatorname{Hom}_{\mathsf{M}}(M, N)$ is the zero homomorphism $0 : M \to N$, i.e. the constant function $m \mapsto 0$. Given A-linear homomorphisms $\phi, \psi : M \to N$, their sum $\phi + \psi$ is the A-linear homomorphism

$$(\phi + \psi)(m) := \phi(m) + \psi(m).$$

Given an A-linear homomorphism $\phi : M \to N$ and an element $\lambda \in \mathbb{K}$, let $\lambda \cdot \phi : M \to N$ be the homomorphism

$$(\lambda \cdot \phi)(m) := \lambda \cdot \phi(m).$$

We must check that $\lambda \cdot \phi$ is A-linear, and this is where we use the fact that A is a central $\mathbb{K}$-ring. Take an element $a \in A$. Then

$$(\lambda \cdot \phi)(a \cdot m) = \lambda \cdot \phi(a \cdot m) = \lambda \cdot a \cdot \phi(m) = a \cdot \lambda \cdot \phi(m) = a \cdot (\lambda \cdot \phi)(m)$$

for every $m \in M$.

We now have to prove that composition in M is $\mathbb{K}$-bilinear. Let $\phi : L \to M$ and $\psi : M \to N$ be morphisms in M. For every $\lambda \in \mathbb{K}$ and $l \in L$ we have

$$((\lambda \cdot \phi) \circ \psi)(l) = (\lambda \cdot \phi)(\psi(l)) = \lambda \cdot \phi(\psi(l)) = \lambda \cdot (\phi \circ \psi)(l) = (\lambda \cdot (\phi \circ \psi))(l).$$

Therefore

$$(\lambda \cdot \phi) \circ \psi = \lambda \cdot (\phi \circ \psi).$$

The remainder of the proof (checking the other three equations) is left as an exercise. □

Exercise 2.22. Finish the proof of the last proposition, by verifying that

$$\phi \circ (\lambda \cdot \psi) = \lambda \cdot (\phi \circ \psi)$$

$$(\phi_1 + \phi_2) \circ \psi = (\phi_1 \circ \psi) + (\phi_2 \circ \psi)$$

and

$$\phi \circ (\psi_1 + \psi_2) = (\phi \circ \psi_1) + (\phi \circ \psi_2).$$

Most categories that we talked about do not admit a group structure on their morphisms sets; at least not in any "natural" way. The next two exercises explore this point.

Exercise 2.23. Consider the category of groups Grp. For a pair of groups G, H the set $\mathrm{Hom}_{\mathsf{Grp}}(G, H)$ has a special element, namely the constant homomorphism $\phi_1 : G \to H$, $\phi_1(g) := 1_H$. You might think that there's a group structure on $\mathrm{Hom}_{\mathsf{Grp}}(G, H)$, in which ϕ_1 is the unit element.

Show that this is false, by calculating $\phi_1 \cdot \phi_1$ for nonabelian G and H (e.g. take both to be S_3).

Exercise 2.24. Consider the category of rings Ring. Find a pair of rings A, B such that the set $\mathrm{Hom}_{\mathsf{Ring}}(A, B)$ is empty. So it can't be a group.

Exercise 2.25. This exercise reverses Proposition 2.21. Let M be a $\mathbb{K}$-linear category, and let $M \in \mathsf{M}$. Show that

$$\mathrm{End}_{\mathsf{M}}(M) := \mathrm{Hom}_{\mathsf{M}}(M, M)$$

is a central $\mathbb{K}$-ring.

We now leave linear categories for a while.

Definition 2.26. Let C be a category. A *subcategory* C' of C consists of a subset $\mathrm{Ob}(\mathsf{C}') \subseteq \mathrm{Ob}(\mathsf{C})$, and for each pair of objects $C, D \in \mathrm{Ob}(\mathsf{C}')$ a subset

$$\mathrm{Hom}_{\mathsf{C}'}(C, D) \subseteq \mathrm{Hom}_{\mathsf{C}}(C, D). \tag{2.27}$$

There is no condition on the subset $\mathrm{Ob}(\mathsf{C}')$.

There are two conditions on the morphism subsets $\mathrm{Hom}_{\mathsf{C}'}(C, D)$:

- (Identities) $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{C}'}(C, C)$ for every $C \in \mathrm{Ob}(\mathsf{C}')$.
- (Closure under composition) If $C, D, E \in \mathrm{Ob}(\mathsf{C}')$, $f \in \mathrm{Hom}_{\mathsf{C}'}(C, D)$ and $g \in \mathrm{Hom}_{\mathsf{C}'}(D, E)$, then $g \circ f \in \mathrm{Hom}_{\mathsf{C}'}(C, E)$.

We write $\mathsf{C}' \subseteq \mathsf{C}$.

Of course C' on its own is a category, with the operations inherited from C.

Definition 2.28. A subcategory $\mathsf{C}' \subseteq \mathsf{C}$ is called a *full subcategory* if for every pair of objects $C, D \in \operatorname{Ob}(\mathsf{C}')$ there is equality

$$\operatorname{Hom}_{\mathsf{C}'}(C, D) = \operatorname{Hom}_{\mathsf{C}}(C, D).$$

In other words, a full subcategory $\mathsf{C}' \subseteq \mathsf{C}$ is determined by selecting a subset $\operatorname{Ob}(\mathsf{C}')$ of $\operatorname{Ob}(\mathsf{C})$. This subset could be empty (not interesting), finite, etc.

Example 2.29. The category $\mathsf{Ab}_{\mathrm{fin}}$ of finite abelian groups is a full subcategory of Ab.

Example 2.30. Let us choose one group from each isomorphisms class of finite abelian groups. This gives us a countable subset $S \subseteq \operatorname{Ob}(\mathsf{Ab}_{\mathrm{fin}})$. Let $\mathsf{C} \subseteq \mathsf{Ab}_{\mathrm{fin}}$ be the full subcategory such that $\operatorname{Ob}(\mathsf{C}) = S$. In some sense "everything happens" already inside C. This observation will be made precise later.

Example 2.31. Consider the category of rings Ring. Let C be the following subcategory of Ring : it has all the objects, but the the morphisms $f : A \to B$ in C are the those ring homomorphisms that satisfy

$$f(\operatorname{Cent}(A)) \subseteq \operatorname{Cent}(B).$$

It is easy to see that C is indeed a subcategory. Exercise 2.8 shows that there are less morphisms in C. So this is not a full subcategory.

Example 2.32. Here's a variation on Example 2.11. Let $A' \subsetneq A$ be a subring. Define the subcategory $\mathsf{A}' \subseteq \mathsf{A}$ with $\operatorname{Ob}(\mathsf{A}') = \{x\}$ and

$$\operatorname{Hom}_{\mathsf{A}'}(x, x) := A'.$$

This is not a full subcategory.

Definition 2.33. Let C be a category. A morphism $f : C \to D$ in C is called an *isomorphism* if there is a morphism $g : D \to C$ such that $f \circ g = \mathrm{id}_C$ and $g \circ f = \mathrm{id}_D$.

If an exercise is labeled "optional" (like the next one), then you should solve it only if you think it is not trivial. In any case, do not submit it in writing.

Exercise 2.34. (Optional) If $f : C \to D$ is an isomorphism, then the morphism g in the definition above is unique.

The morphism g in Definition 2.33 is called the inverse of f, and is denoted by f^{-1}.

Definition 2.35. A category G is called a *groupoid* if all the morphisms in it are isomorphisms.

Exercise 2.36. Let C be a category. Show that there is a subcategory $\mathsf{C}^{\times}$ of C, that has all the objects, and whose morphisms are the isomorphisms in C . Show that $\mathsf{C}^{\times}$ is a groupoid.

Here is an important special case.

Exercise 2.37. Let M be a $\mathbb{K}$-linear category, and consider the subcategory $\mathsf{M}^{\times} \subseteq \mathsf{M}$ as in the previous exercise. Take an object $M \in \mathsf{M}$. We know from Exercise 2.25 that $A := \operatorname{End}_{\mathsf{M}}(M)$ is a central $\mathbb{K}$-ring. Show that $\operatorname{End}_{\mathsf{M}^{\times}}(M)$ is the group $A^{\times}$ of invertible elements of the ring A.

Definition 2.38. Let C be a category.

(1) An object $I \in \mathsf{C}$ is called an *initial object* if for every object $C \in \mathsf{C}$ there is a unique morphism $I \to C$.
(2) An object $T \in \mathsf{C}$ is called a *terminal object* if for every object $C \in \mathsf{C}$ there is a unique morphism $C \to T$.

Exercise 2.39. Let C be a category.

(1) If I and I' are both initial objects of C, then there is a unique isomorphism $I \to I'$.
(2) If T and T' are both terminal objects of C, then there is a unique isomorphism $T \to T'$.

Exercise 2.40.

(1) Find initial and terminal objects in the category Set.
(2) Find initial and terminal objects in the category Grp.
(3) Find initial and terminal objects in the category Ring.
(4) Find initial and terminal objects in the category $\mathsf{Mod}\, A$ for a ring A.

We end this section with a discussion of products and coproducts.

Let C be a category. By a *collection* of objects of C, indexed by a set I, we mean a function

$$\gamma : I \to \mathrm{Ob}(\mathsf{C}).$$

We usually denote this collection by $\{C_i\}_{i \in I}$, where $C_i := \gamma(i) \in \mathrm{Ob}(\mathsf{C})$.

Definition 2.41. Let C be a category, and let $\{C_i\}_{i \in I}$ be a collection of objects of C. A *product* of this collection is a pair

$$\big(C, \{p_i\}_{i \in I}\big)$$

where C is an object of C, and $\{p_i\}_{i \in I}$ is a collection of morphisms $p_i : C \to C_i$ in C, called projections. The pair $\big(C, \{p_i\}_{i \in I}\big)$ should have the following universal property:

(P) Given an object $D \in \mathsf{C}$, and a collection $\{f_i\}_{i \in I}$ of morphisms $f_i : D \to C_i$, there exists a unique morphism $f : D \to C$ such that

$$f_i = p_i \circ f.$$

The notation for the object C is $\prod_{i \in I} C_i$. And usually we leave the morphisms p_i implicit.

Here is a depiction of Definition 2.41 where $I = \{1, 2, 3\}$.

$$\begin{tikzcd}
 & C \arrow[dl, "p_1"'] \arrow[d, "p_2"'] \arrow[dr, "p_3"] & \\
C_1 & C_2 & C_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D \arrow[r, dashed, "f"] \arrow[d, dashed, "f_1"'] & C \arrow[dl] \arrow[d] \arrow[dr] \arrow[ddl, dashed, "f_2"] \arrow[ddr, dashed, "f_3"] & \\
C_1 & C_2 & C_3
\end{tikzcd}$$

A product is unique:

Proposition 2.42. *Let C be a category and let $\{C_i\}_{i \in I}$ be a collection of objects of C. Suppose that $\big(C, \{p_i\}_{i \in I}\big)$ and $\big(C', \{p'_i\}_{i \in I}\big)$ are both products of this collection of objects. Then there is a unique isomorphism $g : C \xrightarrow{\simeq} C'$ such that $p'_i \circ g = p_i$ for all i.*

Exercise 2.43. Prove Proposition 2.42.

From now on we say *the product.*

Example 2.44. In the category Set all products exist; they are the usual cartesian products, with the usual projections on the coordinates.

Exercise 2.45. Let A be a ring and let $\{M_i\}_{i \in I}$ be a collection of objects of $\operatorname{Mod} A$. Prove that the product $\big(M, \{p_i\}_{i \in I}\big)$ in $\operatorname{Mod} A$ exists. (Hint: take the product $M := \prod_{i \in I} M_i$ in Set, and show that the set M has an A-module structure, such the projections p_i are homomorphisms.)

Exercise 2.46. Show that the category $\mathsf{Ab}_{\mathrm{fin}}$ of finite abelian groups has finite products (i.e. products indexed by finite sets I), but not infinite products. (Hints: for the first part use Exercise 2.45. For the second part find a concrete counterexample.)

Solution 2.47 (of Exercise 2.46)**.** Suppose $\{M_i\}_{i \in I}$ is a collection of nontrivial finite groups, indexed by an infinite set I. We can't argue that the product in Ab is infinite, and thus it does not belong to $\mathsf{Ab}_{\mathrm{fin}}$; see Remark 2.58.

Here is a correct proof. Let I be the set of positive integers, and for every $i \in I$ choose a finite abelian group M_i of size i (e.g. the cyclic group $\mathbb{Z}/(i)$). Assume that the product $\big(M, \{p_i\}_{i \in I}\big)$ in $\mathsf{Ab}_{\mathrm{fin}}$ exists. For every i consider the collection of homomorphisms $\{\phi_{i,j}\}_{j \in I}$, $\phi_{i,j} : M_j \to M_i$, defined as follows:

$$\phi_{i,j} := \begin{cases} \mathrm{id}_{M_i} & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

By the universal property we get a homomorphism $\phi : M \to M_i$, and

$$p_i \circ \phi = \phi_{i,i} = \mathrm{id}_{M_i} .$$

Hence $p_i : M \to M_i$ is surjective. This implies $|M| \geq |M_i| = i$. So M is infinite. Contradiction.

Lecture 3, 21 March 2018

Here is the notion dual to product.

Definition 2.48. Let C be a category, and let $\{C_i\}_{i \in I}$ be a collection of objects of C. A *coproduct* of this collection is a pair

$$\big(C, \{e_i\}_{i \in I}\big)$$

where C is an object of C, and $\{e_i\}_{i \in I}$ is a collection of morphisms $e_i : C_i \to C$. This pair $\big(C, \{e_i\}_{i \in I}\big)$ should have the following universal property:

(C) Given an object $D \in \mathsf{C}$, and a collection $\{g_i\}_{i \in I}$ of morphisms $g_i : C_i \to D$, there exists a unique morphism $g : C \to D$ such that

$$g_i = g \circ e_i.$$

The notation for the object C is $\coprod_{i \in I} C_i$.

Here is a depiction of Definition 2.48 where $I = \{1, 2, 3\}$.

$$\begin{tikzcd}
 & C & \\
C_1 \arrow[ur, "e_1"] & C_2 \arrow[u, "e_2"] & C_3 \arrow[ul, "e_3"']
\end{tikzcd}
\qquad
\begin{tikzcd}
D & & C \arrow[ll, dashed, "g"'] \\
 & & \\
C_1 \arrow[uu, dashed, "g_1"] \arrow[uur] \arrow[uuurr, dashed, "g_2"] & C_2 \arrow[uu] \arrow[uuur, dashed, "g_3"] & C_3 \arrow[uuul]
\end{tikzcd}$$

Exercise 2.49. Let C be a category and let $\{C_i\}_{i \in I}$ be a collection of objects of C. Suppose that $\big(C, \{e_i\}_{i \in I}\big)$ and $\big(C', \{e'_i\}_{i \in I}\big)$ are both coproducts of the collection of objects $\{C_i\}_{i \in I}$. Prove that there is a unique isomorphism $h : C \xrightarrow{\simeq} C'$ such that $e'_i = h \circ e_i$.

Example 2.50. In the category Set all coproducts exist; they are the disjoint union. Given a collection of sets $\{S_i\}_{i \in I}$, the functions

$$e_j : S_j \to \coprod\nolimits_{i \in I} S_i$$

are the inclusions.

Example 2.51. In the category Grp all coproducts exist, but they are very nasty. Given a finite collection of sets $\{G_i\}_{i \in I}$, indexed by $I = \{1, \ldots, n\}$, the coproduct is the group

$$\coprod\nolimits_{i \in I} G_i := G_1 * \cdots * G_n.$$

Even if the G_i are abelian, the coproduct isn't (unless all but one of the G_i are trivial).

For instance, if $G_i = \mathbb{Z}$, then $G_1 * \cdots * G_n$ is the free group on n generators. It very nonabelian (for $n > 1$): its center is trivial.

For an infinite indexing set I, the coproduct is the union:

$$\coprod\nolimits_{i \in I} G_i = \bigcup\nolimits_{J \subseteq I} \coprod\nolimits_{j \in J} G_j,$$

where J runs over the finite subsets of I.

Example 2.52. In the category $\mathsf{Ring}_{\mathrm{c}}/\mathbb{K}$ of commutative $\mathbb{K}$-rings (see Examples 2.7 and 2.9) the coproduct is the tensor product. This was studied in the previous course (see [Ye2, Lecture 5, page 52]).

Given a finite collection $\{A_i\}_{i \in I}$ in $\mathsf{Ring}_{\mathrm{c}}/\mathbb{K}$, indexed by $I = \{1, \ldots, n\}$, the coproduct is the ring

$$\coprod\nolimits_{i \in I} A_i := A_1 \otimes_{\mathbb{K}} \ldots \otimes_{\mathbb{K}} A_n .$$

For instance, if $A_i = \mathbb{K}[t_i]$, the polynomial ring in a variable t_i, then

$$A_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} A_n = \mathbb{K}[t_1, \ldots, t_n],$$

the commutative polynomial ring in n variables.

For an infinite indexing set I, the coproduct is the union:

$$\coprod\nolimits_{i \in I} A_i = \bigcup\nolimits_{J \subseteq I} \coprod\nolimits_{j \in J} A_j ,$$

where J runs over the finite subsets of I.

Example 2.53. In the category $\mathsf{Ring}/_{\mathrm{c}}\, \mathbb{K}$ of central $\mathbb{K}$-rings (see Example 2.9) the coproduct exists, and it is as nasty as in Example 2.51.

For instance, if $I = \{1, \ldots, n\}$, and $A_i = \mathbb{K}[t_i]$, the commutative polynomial ring in a variable t_i, then

$$\coprod\nolimits_{i \in I} A_i = \mathbb{K}\langle t_1, \ldots, t_n \rangle ,$$

the noncommutative polynomial ring in n variables. If $n > 1$ then the center is $\mathbb{K}$.

In a $\mathbb{K}$-linear category M we usually say *direct sum* instead of coproduct, and write

$$\bigoplus_{i \in I} M_i := \coprod_{i \in I} M_i .$$

Exercise 2.54. Let A be a ring and let $\{M_i\}_{i \in I}$ be a collection of objects of $\mathsf{Mod}\, A$. Describe the coproduct, or direct sum, $\bigoplus_{i \in I} M_i$. (Hint: it is a submodule of the product.)

Exercise 2.55. Prove that $\mathsf{Ab}_{\mathrm{fin}}$ does not have infinite direct sums (i.e. coproducts). (Warning: recall the comment on Exercise 2.46, and read Remark 2.58 below.)

$\mathbb{K}$-linear categories were introduced in Definition 2.20.

Proposition 2.56. *Let M be a $\mathbb{K}$-linear category, and let $\{M_i\}_{i \in I}$ be a collection of objects of M, indexed by a finite set I. Assume that the direct sum $\big(M, \{e_i\}_{i \in I}\big)$ of the collection $\{M_i\}_{i \in I}$ exists in M. Then:*

(1) *The object M is also the product of the collection $\{M_i\}_{i \in I}$. Namely there are morphisms $p_i : M \to M_i$, such that the pair $\big(M, \{p_i\}_{i \in I}\big)$ is the product of the collection $\{M_i\}_{i \in I}$.*
(2) *The collections of morphisms $\{e_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ satisfy*

$$p_i \circ e_i = \mathrm{id}_{M_i}$$

and

$$\sum_{i \in I} e_i \circ p_i = \mathrm{id}_M .$$

This proposition applies to $\mathsf{M} = \mathsf{Mod}\, A$ for a central $\mathbb{K}$-ring A.

Exercise 2.57. Prove Proposition 2.56. (Hint: use the universal property (C) of the coproduct to construct the morphisms p_i.) If you find it easier, prove the proposition for $\mathsf{M} = \mathsf{Mod}\, A$, $A \in \mathsf{Ring}/_{\mathsf{c}}\, \mathbb{K}$.

Remark 2.58. Regarding Exercise 2.46, I said that it could happen that for a full subcategory $\mathsf{C}' \subseteq \mathsf{C}$, products and coproducts might be different in each category.

This can be easily seen for the coproduct of abelian groups: given nontrivial abelian groups G_1 and G_2, their coproduct in Ab is not their coproduct in Grp, even though $\mathsf{Ab} \subseteq \mathsf{Grp}$ is a full subcategory. Indeed, the coproduct $G_1 * G_2$ in Grp is nonabelian.

3. Free Modules

We now leave the abstraction of categories for a while, to talk about an important concrete construction. In this section A is some nonzero ring (possibly noncommutative). Recall that modules are left modules, by default.

Definition 3.1. Let M be an A-module and let X be a set.

(1) The *support* of a function $f : X \to M$ is the set
$$\operatorname{Supp}(f) := \{x \in X \mid f(x) \neq 0\} \subseteq X.$$
(2) We denote by $\mathrm{F}_{\mathrm{fin}}(X, M)$ the set of *finitely supported functions* $f : X \to M$.
(3) Given $f \in \mathrm{F}_{\mathrm{fin}}(X, M)$, its sum
$$\sum_{x \in X} f(x) \in M$$
is defined to be
$$(\dagger) \qquad \sum_{x \in X} f(x) := \sum_{x \in \operatorname{Supp}(f)} f(x).$$

Note that the second sum in (†) is finite, so we are not doing anything illegal.

Exercise 3.2. Let M be a module over the ring A. We know that the set $\mathrm{F}(X, M)$ of all functions $f : X \to M$ is an A-module, by pointwise operations. Show that
$$\mathrm{F}_{\mathrm{fin}}(X, M) \subseteq \mathrm{F}(X, M)$$
is an A-submodule.

As before, a function $f : X \to M$ can be considered as a collection $\{m_x\}_{x \in X}$ of element of M, where $m_x := f(x)$. Under the guise of a collection, we sometimes use the notation
$$\boldsymbol{m} := \{m_x\}_{x \in X}.$$
Thus $\boldsymbol{m} = f$, two ways of referring to the same thing.

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Recall $\mathbb{K}$ is a nonzero commutative base ring, A is some nonzero central $\mathbb{K}$-ring (possibly noncommutative), and all modules are by default left modules.

Definition 3.3. Let M be an A-module. Given a collection $\boldsymbol{m} := \{m_x\}_{x \in X}$ of elements of M, and a finitely supported collection $\boldsymbol{a} := \{a_x\}_{x \in X}$ of elements of A, both indexed by the same indexing set X, the collection $\{a_x \cdot m_x\}_{x \in X}$ is a finitely supported collection of elements of M. Thus the sum

$$\boldsymbol{a} \cdot \boldsymbol{m} := \sum_{x \in X} a_x \cdot m_x \in M$$

exists.

Definition 3.4. Let M be an A-module and let $\boldsymbol{m} := \{m_x\}_{x \in X}$ be a collection of elements of M; i.e. $\boldsymbol{m} \in \mathrm{F}(X, M)$.

(1) We say that the collection $\boldsymbol{m}$ *generates* the module M if for every element $m \in M$ there exists some $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that $m = \boldsymbol{a} \cdot \boldsymbol{m}$.
(2) We say that the collection $\boldsymbol{m}$ is *linearly independent* if the only element $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ such that $\boldsymbol{a} \cdot \boldsymbol{m} = 0$ is $\boldsymbol{a} = 0$.
(3) We say that the collection $\boldsymbol{m}$ is a *basis* of the module M if it generates M and it is linearly independent.

Definition 3.5. An A-module M is called *free* if it has a basis.

For a set X and an element $x \in X$ we denote by $\delta_x : X \to A$ the function

$$\delta_x(y) := \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases} \tag{3.6}$$

The support of δ_x is $\{x\}$.

Proposition 3.7. *Let X be a set. Consider the collection $\boldsymbol{\delta} := \{\delta_x\}_{x \in X}$ of elements of $\mathrm{F}_{\mathrm{fin}}(X, A)$.*

(1) *For every $\boldsymbol{a} \in \mathrm{F}_{\mathrm{fin}}(X, A)$ there is equality*

$$\boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{\delta}$$

in $\mathrm{F}_{\mathrm{fin}}(X, A)$.
(2) *The A-module $\mathrm{F}_{\mathrm{fin}}(X, A)$ is free, with basis $\boldsymbol{\delta}$.*

Exercise 3.8. Prove this proposition.

Example 3.9. Suppose the indexing set is $X = \{1, \ldots, r\}$ for some natural number r. Then

$$\mathrm{F}_{\mathrm{fin}}(X, A) = \mathrm{F}(X, A) = A^r.$$

The standard basis of A^r is, in our notation,

$$\boldsymbol{\delta} = (\delta_1, \ldots, \delta_r).$$

Proposition 3.10. *Let M be an A-module, and let $\boldsymbol{m} := \{m_x\}_{x \in X}$ be a collection of elements of M. The function*

$$\phi_{\boldsymbol{m}} : \mathrm{F}_{\mathrm{fin}}(X, A) \to M, \quad \phi_{\boldsymbol{m}}(\boldsymbol{a}) := \boldsymbol{a} \cdot \boldsymbol{m}$$

is an A-module homomorphism. Moreover, it is the unique A-module homomorphism

$$\phi : \mathrm{F}_{\mathrm{fin}}(X, A) \to M$$

such that $\phi(\delta_x) = m_x$ for all $x \in X$.

Exercise 3.11. Prove this proposition.

Proposition 3.12. *Let M be an A-module, and let $\boldsymbol{m} := \{m_x\}_{x \in X}$ be a collection of elements of M. Consider the homomorphism*

$$\phi_{\boldsymbol{m}} : \mathrm{F}_{\mathrm{fin}}(X, A) \to M.$$

(1) *$\boldsymbol{m}$ generates M iff $\phi_{\boldsymbol{m}}$ is surjective.*
(2) *$\boldsymbol{m}$ is linearly independent iff $\phi_{\boldsymbol{m}}$ is injective.*
(3) *$\boldsymbol{m}$ is a basis of M iff $\phi_{\boldsymbol{m}}$ is bijective.*

Exercise 3.13. Prove this proposition.

We see that:

Corollary 3.14. *An A module M is free iff $M \cong \mathrm{F}_{\mathrm{fin}}(X, A)$ for some set X.*

Proposition 3.15. *Let $f : X \to Y$ be a function between sets. There is a unique A-module homomorphism*

$$\mathrm{tr}_f : \mathrm{F}_{\mathrm{fin}}(X, A) \to \mathrm{F}_{\mathrm{fin}}(Y, A)$$

such that

$$\mathrm{tr}_f(\delta_x) = \delta_{f(x)}$$

for every $x \in X$.

Proof. We give two proofs. First: use Proposition 3.10 with $M := \mathrm{F}_{\mathrm{fin}}(Y, A)$, and with the collection of elements $\boldsymbol{m} = \{m_x\}_{x \in X}$ defined by $m_x := \delta_{f(x)}$. Then $\mathrm{tr}_f := \phi_{\boldsymbol{m}}$ works.

Second proof: for $\psi \in \mathrm{F}_{\mathrm{fin}}(X, A)$ define

$$\mathrm{tr}_f(\psi)(y) := \sum_{x \in f^{-1}(y)} \psi(x) \in A.$$

□

Remark 3.16. We can put the discrete topologies on the sets X and Y. The corresponding Lebesgue, or Borel, measures are the atomic measures that give each point the measure 1. The integrable functions $X \to A$ and $Y \to A$ are then the finitely supported functions. These are also the compactly supported functions.

The homomorphism

$$\mathrm{tr}_f : \mathrm{F}_{\mathrm{fin}}(X, A) \to \mathrm{F}_{\mathrm{fin}}(Y, A)$$

is then the standard pushforward (or integration on fibers)

$$\mathrm{tr}_f = \int_f$$

of measures, or of compactly supported functions (or of distributions). See any (probably nonexistent) modern textbook on measure theory; or any modern textbook on analysis (e.g. [KaSc]).

Here are two theorems about commutative rings, that were proved in the previous course.

Theorem 3.17. *Let K be a field. Every K-module M has a basis.*

See [Ye2, Lecture 3, page 28]. The proof relies om Zorn's Lemma (i.e. the Axiom of Choice).

Theorem 3.18. *Let A be a nonzero commutative ring, and let M be a free A-module. If $\boldsymbol{m} = \{m_x\}_{x \in X}$ and $\boldsymbol{m}' = \{m'_{x'}\}_{x' \in X'}$ are two bases of M, then the sets X and X' have the same cardinality.*

See [Ye2, Lecture 3, page 32]. The proof also relies on the Axiom of Choice (via the fact that a nonzero commutative ring A has a maximal ideal $\mathfrak{m}$). In the situation of Theorem 3.18, the cardinality of a basis of M is called the rank of M, and it is denoted by $\operatorname{rank}_A(M)$.

Example 3.19. Theorem 3.18 is false in general for noncommutative rings. Here is a counterexample. Let $\mathbb{K}$ be a field, and let $V := \mathrm{F}_{\mathrm{fin}}(\mathbb{N}, \mathbb{K})$, which is a free $\mathbb{K}$-module of countable rank. Define the central $\mathbb{K}$-ring

$$A := \operatorname{End}_{\mathbb{K}}(V).$$

We can view an element $v \in V$ as a column with finitely many nonzero entries. Then an element $a \in A$ becomes an $\mathbb{N} \times \mathbb{N}$ matrix with entries in $\mathbb{K}$, that has finitely supported columns (i.e. the columns of a are elements of V). The action of A on V is by matrix multiplication.

$$a \cdot v = \begin{bmatrix} \lambda_{0,0} & \lambda_{0,1} & \cdots \\ \lambda_{1,0} & \lambda_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \end{bmatrix}$$

Consider the free A-module $M := A$. Let $M_0 \subseteq M$ be the submodule consisting of the matrices a in which only the even numbered columns can be nonzero; and let $M_1 \subseteq M$ be the submodule consisting of the matrices a in which only the odd numbered columns can be nonzero. Then $M = M_0 \oplus M_1$ as left A-modules. On the other hand $M_0 \cong M_2 \cong M$ as left A-modules. We see that $A \cong A^2$ as left A-modules. This shows that free A-modules do not have well-defined ranks.

The next few definitions and results (up to Exercise 3.27, about left noetherian rings) were not in the lecture.

Definition 3.20. Let A be a ring. An A-module M is called a *noetherian module* if every submodule $M' \subseteq M$ is finitely generated.

Proposition 3.21. *Let A be a ring and M an A-module. The following two conditions are equivalent*:

(i) *M is a noetherian module.*
(ii) *M satisfies the* ascending chain condition: *if*

$$M_0 \subseteq M_1 \subseteq M_1 \subseteq \cdots$$

is a chain of submodules of M, then $M_i = M_{i+1}$ for $i \gg 0$.

Exercise 3.22. Prove this proposition.

Definition 3.23. A ring A is called a *left noetherian ring* if A is a noetherian left module module over itself; i.e. if every left ideal $\mathfrak{a} \subseteq A$ is finitely generated as a left A-module.

Theorem 3.24. *Let A be a ring. The following two conditions are equivalent*:

(i) *A is a left noetherian ring.*
(ii) *Every finitely generated A-module M is a noetherian module.*

Exercise 3.25. Prove Theorem 3.24. (Hint: See the commutative case in [Ye2, Lecture 10, page 99].)

Exercise 3.26. Let A be a nonzero left noetherian ring. Show that Theorem 3.18 holds for free A-modules. (Hint: study Example 3.19.)

Exercise 3.27. Let A be a ring for which there exists a nonzero ring homomorphism $f : A \to B$ to a commutative ring B. Then Theorem 3.18 holds for free A-modules. (Hint: modify the proof of Theorem 3.18, see [Ye2, Lecture 3, page 32].)

Exercise 3.28. Let X be a set and M an A-module. Consider the collection of modules $\{M_x\}_{x \in X}$ with $M_x := M$. Show that

$$\mathrm{F}_{\mathrm{fin}}(X, M) = \bigoplus_{x \in X} M_x$$

and

$$\mathrm{F}(X, M) = \prod_{X \in X} M_x .$$

4. Functors

Definition 4.1. Let C and D be categories. A *functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of these ingredients:

- A function
$$F_{\mathrm{Ob}} : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D}).$$
- For every pair $C_1, C_2 \in \mathrm{Ob}(\mathsf{C})$, a function
$$F_{C_1, C_2} : \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big).$$

There are two conditions:

- (Composition) For all composable morphisms $C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3$ in C there is equality
$$F_{C_2, C_3}(f_2) \circ F_{C_1, C_2}(f_1) = F_{C_1, C_3}(f_2 \circ f_1)$$
of morphisms $F_{\mathrm{Ob}}(C_1) \to F_{\mathrm{Ob}}(C_3)$ in D.
- (Identity) For every object C of C there is equality
$$F_{C,C}(\mathrm{id}_C) = \mathrm{id}_{F_{\mathrm{Ob}}(C)}$$
of morphisms from $F_{\mathrm{Ob}}(C) \to F_{\mathrm{Ob}}(C)$ in D.

Definition 4.2. Abbreviations: we will usually write F instead of F_{Ob} and F_{C_1, C_2}, since this will be clear from the context.

Example 4.3. Let A be a central $\mathbb{K}$-ring. Given an A-module M, let $F(M)$ be its underlying $\mathbb{K}$-module. The resulting functor

$$F : \mathsf{Mod}\, A \to \mathsf{Mod}\, \mathbb{K}$$

is called a *forgetful functor*, because it "forgets" some structure.

We can also consider the underlying set $G(M)$ of the module M. The resulting functor

$$G : \mathsf{Mod}\, A \to \mathsf{Set}$$

is also a forgetful functor.

The functor G is a composition of functors (see definition below):

$$G = H \circ F\,,$$

where

$$H : \mathsf{Mod}\, \mathbb{K} \to \mathsf{Set}$$

is this forgetful functor.

Definition 4.4. Let C, D and E be categories, and let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ be functors. The *composed functor*

$$G \circ F : \mathsf{C} \to \mathsf{E}$$

has ingredients

$$(G \circ F)_{\mathrm{Ob}} : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{E})\,,$$
$$(G \circ F)_{\mathrm{Ob}} := G_{\mathrm{Ob}} \circ F_{\mathrm{Ob}}$$

and

$$(G \circ F)_{C_1, C_2} : \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathsf{E}}\big((G \circ F)_{\mathrm{Ob}}(C_1), (G \circ F)_{\mathrm{Ob}}(C_2)\big)\,,$$
$$(G \circ F)_{C_1, C_2} := G_{F_{\mathrm{Ob}}(C_1), F_{\mathrm{Ob}}(C_2)} \circ F_{C_1, C_2}\,.$$

The relevant diagram is

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \mathsf{D} \arrow[r, "G"] & \mathsf{E} \end{tikzcd} \tag{4.5}$$

The material from here until Exercise 4.8 (more example of functors) was not in the lecture.

Exercise 4.6. Verify that $G \circ F$ is indeed a functor.

Exercise 4.7. We know (by Exercise 2.8) that the center is not a functor from Grp to Ab. However, given a group G, let $[G, G]$ be the subgroup of G generated by the commutators

$$[g, h] := g \cdot h \cdot g^{-1} \cdot h^{-1}.$$

(1) Show that $[G, G]$ is a normal subgroup.
(2) Show that
$$\mathrm{Ab}(G) := G/[G, G]$$
is an abelian group.
(3) Show that
$$\mathrm{Ab} : \mathsf{Grp} \to \mathsf{Ab}$$
is a functor. It is called the *abelianization functor*.

(4) Try to find the relation between the functor Ab and the embedding functor

$$\mathsf{Emb} : \mathsf{Ab} \to \mathsf{Grp}\,.$$

(Recall that Ab is a full subcategory of Grp).

Exercise 4.8. Let A be a nonzero ring. For a set X let

$$\mathsf{Free}(X) := \mathrm{F}_{\mathrm{fin}}(X, A) \in \mathsf{Mod}\, A.$$

To a function $f : X \to Y$ between sets we assign the A-module homomorphism

$$\mathsf{Free}(f) := \mathrm{tr}_f : \mathrm{F}_{\mathrm{fin}}(X, A) \to \mathrm{F}_{\mathrm{fin}}(Y, A)$$

from Proposition 3.15.

(1) Prove that

$$\mathsf{Free} : \mathsf{Set} \to \mathsf{Mod}\, A$$

is a functor.

(2) Try to find the relation between the functor Free and the forgetful functor

$$\mathsf{Forg} : \mathsf{Mod}\, A \to \mathsf{Set}\,.$$

The relation between the functors in Exercises 4.7 and 4.8 is called *adjunction*. We will discuss it later.

Lecture 5, 11 April 2018. Version 2 of notes.

Definition 4.9. Let M and N be $\mathbb{K}$-linear categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is called a $\mathbb{K}$-*linear functor* if for every pair of objects $M_1, M_2 \in \mathsf{M}$ the function

$$F : \operatorname{Hom}_{\mathsf{M}}(M_1, M_2) \to \operatorname{Hom}_{\mathsf{N}}\big(F(M_1), F(M_2)\big)$$

is $\mathbb{K}$-linear.

If $\mathbb{K} = \mathbb{Z}$ then F is just called a *linear functor*, or an *additive functor*.

Exercise 4.10. Let $f : A \to B$ be a $\mathbb{K}$-ring homomorphism. A B-module N can be made into an A-module by $a \cdot n := f(a) \cdot n$ for $a \in A$ and $n \in N$. Show that the formula above gives rise to a $\mathbb{K}$-linear functor

$$\operatorname{Rest}_f : \operatorname{Mod} B \to \operatorname{Mod} A.$$

It is called the *restriction functor* corresponding to f. We sometimes write

$$\operatorname{Rest}_{B/A} := \operatorname{Rest}_f .$$

Note that the forgetful functor F from Example 4.3 is also a restriction functor.

Exercise 4.11. Let A be a central $\mathbb{K}$-ring. We know that $\operatorname{Mod} A$ is a $\mathbb{K}$-linear category. Fix an A-module M. Thus for each $N \in \operatorname{Mod} A$ we have a $\mathbb{K}$-module

$$F_M(N) := \operatorname{Hom}_A(M, N),$$

and each homomorphism $\phi : N \to N'$ in $\operatorname{Mod} A$ induces a homomorphism

$$F_M(\phi) := \operatorname{Hom}(\mathrm{id}_M, \phi) : F(N) \to F(N').$$

(1) Show that

$$F_M : \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}$$

is a $\mathbb{K}$-linear functor.

(2) Assume that A is a commutative ring. Shows that the functor F from item (1) can be upgraded to an A-linear functor

$$F'_M : \operatorname{Mod} A \to \operatorname{Mod} A,$$

such that

$$F_M = \operatorname{Rest}_{A/\mathbb{K}} \circ F'_M.$$

Remark 4.12. Suppose that in the situation of Exercise 4.11 we were to define

$$G_M(N) := \operatorname{Hom}_A(N, M)$$

for $N \in \operatorname{Mod} A$. Then $G_M(N)$ belongs to $\operatorname{Mod} \mathbb{K}$. But for a homomorphism $\phi : N \to N'$, the homomorphism

$$G_M(\phi) := \operatorname{Hom}(\phi, \mathrm{id}_M)$$

goes in the wrong direction. We will see soon that G_M is a *contravariant* $\mathbb{K}$-*linear functor*. This is a confusing, yet unavoidable, feature of category theory.

We will study contravariant functors, as well as opposite categories, later – just before talking about adjoint functors.

Exercise 4.13. Let $F : \mathsf{M} \to \mathsf{N}$ and $G : \mathsf{N} \to \mathsf{P}$ be $\mathbb{K}$-linear functors between $\mathbb{K}$-linear categories. Show that $G \circ F$ is a $\mathbb{K}$-linear functor.

Exercise 4.14. Let M and N be $\mathbb{K}$-linear categories, and let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. We know already that for every object $M \in \mathsf{M}$ the set of endomorphisms $\operatorname{End}_{\mathsf{M}}(M)$ is a central $\mathbb{K}$-ring. Show that for every $M \in \mathsf{M}$ the function

$$F : \operatorname{End}_{\mathsf{M}}(M) \to \operatorname{End}_{\mathsf{N}}(F(M))$$

is a $\mathbb{K}$-ring homomorphism.

Left A-modules were defined in Definition 2.12, and you were supposed to define right A-modules in Exercise 2.13. The next definition combines the previous two in a rather complicated way, that requires getting used to. Recall that rings are by default $\mathbb{K}$-central.

Definition 4.15. Let A and B be rings. A *$\mathbb{K}$-central A-B-bimodule* is a $\mathbb{K}$-module M, equipped with a left A-module structure and a right B-module structure, that commute with each other and respect the given $\mathbb{K}$-module structure of M. Namely

$$(a \cdot m) \cdot b = a \cdot (m \cdot b)$$

and

$$(\lambda \cdot 1_A) \cdot m = \lambda \cdot m = m \cdot (\lambda \cdot 1_B)$$

for all $m \in M$, $a \in A$, $b \in B$ and $\lambda \in \mathbb{K}$.

Example 4.16. Let A be a nonzero ring. Fix positive integers r and s. Define the rings

$$B := \operatorname{Mat}_r(A) = \begin{bmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{bmatrix}$$

and $C := \operatorname{Mat}_s(A)$. Define the $\mathbb{K}$-module

$$M := \operatorname{Mat}_{s \times r}(A).$$

Matrix multiplication gives a left action of C on M:

$$(c, m) \mapsto c \cdot m\,,$$

and also a right action of B on M:

$$(m, b) \mapsto m \cdot b.$$

The usual calculation in the linear algebra course shows that

$$(c \cdot m) \cdot b = c \cdot (m \cdot b)\,,$$

regardless of the fact that the ring A is noncommutative. We see that M is a C-B-bimodule.

another

Exercise 4.17. Let M be an A-B-bimodule.

(1) Show that for every $N \in \operatorname{Mod} A$, the $\mathbb{K}$-module $\operatorname{Hom}_A(M, N)$ has a B-module structure, with formula

$$(b \cdot \phi)(m) := \phi(m \cdot b)$$

for $\phi \in \operatorname{Hom}_A(M, N)$, $b \in B$ and $m \in M$.

(2) Show that in this way

$$\operatorname{Hom}_A(M, -) : \operatorname{Mod} A \to \operatorname{Mod} B$$

is a $\mathbb{K}$-linear functor.

Remark 4.18. Suppose that $s = 1$ in the previous example, so that $C = A$ and M is an A-B-bimodule. Later in the course (I hope) we will see that the functor

$$\operatorname{Hom}_A(M, -) : \operatorname{Mod} A \to \operatorname{Mod} B$$

is an equivalence of $\mathbb{K}$-linear categories. This is what I call "baby Morita Equivalence."

Exercise 4.19. Let M and N be $\mathbb{K}$-linear categories, and let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. Pick a pair of objects $M_1, M_2 \in \mathsf{M}$, and define $A_i := \operatorname{End}_{\mathsf{M}}(M_i)$ and $B_i := \operatorname{End}_{\mathsf{N}}(F(N_i))$. We know that $F : A_i \to B_i$ are ring homomorphisms (by Exercise 4.14), and that there are restriction functors

$$\operatorname{Rest}_{B_i/A_i} : \operatorname{Mod} B_i \to \operatorname{Mod} A_i .$$

(Exercise 4.10). Show that:

(1) The $\mathbb{K}$-module $\operatorname{Hom}_{\mathsf{M}}(M_1, M_2)$ is a $\mathbb{K}$-central A_2-A_1-bimodule.
(2) Show that

$$F : \operatorname{Hom}_{\mathsf{M}}(M_1, M_2) \to \operatorname{Hom}_{\mathsf{N}}\big(F(M_1), F(M_2)\big)$$

is a homomorphism of A_2-A_1-bimodules.

Convention 4.20. From here on we assume by default that all linear categories are $\mathbb{K}$-linear, and all linear functors are $\mathbb{K}$-linear. Also we assume that all bimodules are $\mathbb{K}$-central.

Of course if $\mathbb{K} = \mathbb{Z}$ then the convention above is automatic.

comment: To here in class lecture 5, 11 April. Continue reading, and solving the exercises, all the way to page 26.

5. Natural Transformations

Definition 5.1. Let C and D be categories, and let

$$F, G : \mathsf{C} \to \mathsf{D}$$

be functors. A *natural transformation*, or a *morphism of functors*

$$\eta : F \to G$$

is a collection

$$\eta = \{\eta_C\}_{C \in \operatorname{Ob}(\mathsf{C})}$$

of morphisms

$$\eta_C : F(C) \to G(C)$$

in the category D. The condition is this:

(N) For each morphism $\phi : C_1 \to C_2$ in C there is equality

$$\eta_{C_2} \circ F(\phi) = G(\phi) \circ \eta_{C_1}$$

of morphisms in D. In other words, the diagram

$$\begin{tikzcd} F(C_1) \arrow[r, "F(\phi)"] \arrow[d, "\eta_{C_1}"'] & F(C_2) \arrow[d, "\eta_{C_2}"] \\ G(C_1) \arrow[r, "G(\phi)"] & G(C_2) \end{tikzcd}$$

in D is commutative.

Here is the diagram of functors:

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathsf{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd} \tag{5.2}$$

It is customary to draw morphisms of functors in such diagrams as doubled arrows.

Definition 5.3. In the situation of Definition 5.1, the morphism of functors η is called an *isomorphism of functors* if for every object $C \in \mathsf{C}$ the morphism

$$\eta_C : F(C) \to G(C)$$

in the category D is an isomorphism.

Here are a few examples of morphisms of functors.

Example 5.4. Continuing with Example 4.10, let $A \xrightarrow{f} B \xrightarrow{g} C$ be $\mathbb{K}$-ring homomorphisms. Then there is an isomorphism of functors

$$\eta : \mathrm{Rest}_f \circ \mathrm{Rest}_g \xrightarrow{\simeq} \mathrm{Rest}_{g \circ f}$$

of $\mathbb{K}$-linear functors

$$\mathsf{Mod}\, C \to \mathsf{Mod}\, A.$$

The isomorphism η_N, for $N \in \mathsf{Mod}\, C$, is the identity on the underlying $\mathbb{K}$-module.

Exercise 5.5. Continuing with Exercise 4.11, let A be a central $\mathbb{K}$-ring, and let $\psi : M_1 \to M_2$ be some fixed homomorphism in $\mathsf{Mod}\, A$. We have $\mathbb{K}$-linear functors

$$F_1, F_2 : \mathsf{Mod}\, A \to \mathsf{Mod}\, \mathbb{K}$$

defined by

$$F_i := \mathrm{Hom}_A(M_i, -).$$

For each $N \in \mathsf{Mod}\, A$ there is a morphism

$$\eta_N : F_2(N) \to F_1(N), \quad \eta_N := \mathrm{Hom}_A(\psi, \mathrm{id}_N)$$

in $\mathsf{Mod}\, \mathbb{K}$. Prove that the collection of morphisms

$$\eta = \{\eta_N\}_{N \in \mathrm{Ob}(\mathsf{Mod} A)}$$

is a morphism of functors

$$\eta : F_2 \to F_1.$$

Example 5.6. We continue with Exercise 4.8. So A is a nonzero ring, and we have the functors

$$\mathrm{Free} : \mathsf{Set} \to \mathsf{Mod}\, A, \quad \mathrm{Free}(X) := \mathrm{F}_{\mathrm{fin}}(X, A)$$

and the forgetful functor

$$\mathrm{Forg} : \mathsf{Mod}\, A \to \mathsf{Set}\,.$$

The composed functor

$$\mathrm{Forg} \circ \mathrm{Free} : \mathsf{Set} \to \mathsf{Set}$$

sends a set X to the underlying set of the A-module $\mathrm{F}_{\mathrm{fin}}(X, A)$.

For a set X let

$$\eta_X : X \to \mathrm{F}_{\mathrm{fin}}(X, A)$$

be the function

$$\eta_X(x) := \delta_x \in \mathrm{F}_{\mathrm{fin}}(X, A).$$

We want to prove that

$$\eta = \{\eta_X\}_{X \in \mathrm{Ob}(\mathrm{Set})}$$

is a morphism of functors

$$\eta : \mathrm{Id}_{\mathrm{Set}} \to \mathrm{Forg} \circ \mathrm{Free}\,.$$

So we have to check that for every morphism $g : X \to Y$ in Set, i.e. a function, the diagram

$$\begin{tikzcd} \mathrm{Id}(X) \arrow[r, "\mathrm{Id}(g)"] \arrow[d, "\eta_X"'] & \mathrm{Id}(Y) \arrow[d, "\eta_Y"] \\ (\mathrm{Forg} \circ \mathrm{Free})(X) \arrow[r, "(\mathrm{Forg} \circ \mathrm{Free})(g)"] & (\mathrm{Forg} \circ \mathrm{Free})(Y) \end{tikzcd} \tag{5.7}$$

in Set is commutative.

Let us translate this to a diagram with the actual objects and morphisms:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \mathrm{F}_{\mathrm{fin}}(X, A) \arrow[r, "\mathrm{tr}_g"] & \mathrm{F}_{\mathrm{fin}}(X, A) \end{tikzcd}$$

Now for an element $x \in X$ we have

$$(\eta_Y \circ g)(x) = \eta_Y(g(x)) = \delta_{g(x)}$$

and

$$(\mathrm{tr}_g \circ \eta_X)(x) = \mathrm{tr}_g(\delta_x) = \delta_{g(x)}.$$

These are equal, so diagram (5.7) is indeed commutative.

Exercise 5.8. Suppose we are given three categories $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$; pairs of functors

$$F_i, G_i : \mathrm{C}_i \to \mathrm{C}_{i+1};$$

and morphisms of functors

$$\eta_i : F_i \to G_i.$$

Give the formula for the composed morphism

$$\eta_2 \circ \eta_1 : (F_2 \circ F_1) \to (G_2 \circ G_1)$$

of functors $\mathrm{C}_1 \to \mathrm{C}_3$, and prove that it is indeed a morphism of functors.

This operation is called *horizontal composition of morphisms of functors.*

Here are the relevant diagrams of functors:

$$\begin{tikzcd} \mathrm{C}_1 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\eta_1 \Downarrow"] & \mathrm{C}_2 \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\eta_2 \Downarrow"] & \mathrm{C}_3 \end{tikzcd} \tag{5.9}$$

and

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, bend left=60, "F_2 \circ F_1"] \arrow[r, bend right=60, "G_2 \circ G_1"'] \arrow[r, phantom, "\eta_2 \circ \eta_1 \Downarrow"] & \mathsf{C}_3 \end{tikzcd} \tag{5.10}$$

Exercise 5.11. Suppose we are given categories C_1 and C_2; three functors

$$F, G, H : \mathsf{C}_1 \to \mathsf{C}_2;$$

and morphisms of functors

$$\eta : F \to G$$

and

$$\theta : G \to H.$$

For every object $C \in \mathsf{C}_1$ define the morphism

$$(\theta * \eta)_C := \theta_C \circ \eta_C : F(C) \to H(C)$$

in C_2. Show that the collection

$$\theta * \eta = \big\{ (\theta * \eta)_C \big\}_{C \in \mathrm{Ob}(\mathsf{C}_1)}$$

is a morphism of functors

$$\theta * \eta : F \to H.$$

This operation is called *vertical composition of morphisms of functors.*

Here are the relevant diagrams of functors:

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, "G"'] \arrow[r, phantom, shift left=3, "\eta \Downarrow"] \arrow[r, phantom, shift right=3, "\theta \Downarrow"] & \mathsf{C}_2 \end{tikzcd} \tag{5.12}$$

and

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\theta * \eta \Downarrow"] & \mathsf{C}_1 \end{tikzcd} \tag{5.13}$$

The operations from the the previous two exercises can be combined. This is very confusing, but in the end quite elementary. The data is this: we are given three categories $\mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3$; triples of functors

$$F_i, G_i, H_i : \mathsf{C}_i \to \mathsf{C}_{i+1};$$

and pairs of morphisms of functors

$$\eta_i : F_i \to G_i, \quad \theta_i : G_i \to H_i.$$

It can be shown (by a very tedious calculation) that the *exchange property* holds:

$$(\theta_2 * \eta_2) \circ (\theta_1 * \eta_1) = (\theta_2 \circ \theta_1) * (\eta_2 \circ \eta_1) \tag{5.14}$$

as morphisms

$$F_2 \circ F_1 \to H_2 \circ H_1$$

of functors $\mathsf{C}_1 \to \mathsf{C}_3$.

This situation is shown in the next diagram.

(5.15)
$$\begin{tikzcd}
\mathsf{C}_1 \arrow[r, bend left=60, "F_1"{name=U1}] \arrow[r, bend right=60, "H_1"'{name=D1}] \arrow[r, "G_1"] \arrow[from=U1, to=D1, Rightarrow, "\eta_1"] \arrow[from=U1, to=D1, Rightarrow, "\theta_1"'] & \mathsf{C}_2 \arrow[r, bend left=60, "F_2"{name=U2}] \arrow[r, bend right=60, "H_2"'{name=D2}] \arrow[r, "G_2"] \arrow[from=U2, to=D2, Rightarrow, "\eta_2"] \arrow[from=U2, to=D2, Rightarrow, "\theta_2"'] & \mathsf{C}_3
\end{tikzcd}$$

Remark 5.16. All these operations are explained in [Mac2, Section XII.3], as part of the discussion of *the* 2*-category* **Cat** *of all* U*-categories.* We do not need to know about 2-categories; all we need is to have a good understanding of the properties of morphisms of functors.

The horizontal and vertical composition of morphisms of functors is made concrete in the next exercise.

Exercise 5.17. Let A, B and C be rings. Let us introduce temporary notation for the category of A-B-bimodules: $\mathsf{BiMod}(A, B)$. (Later, after we talk about opposite rings and tensor products of rings, this category will be denoted by $\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}}$.)

We are given $M_i \in \mathsf{BiMod}(A, B)$ and $N_i \in \mathsf{BiMod}(B, C)$, for $i = 1, 2, 3$. In Exercise 4.17 we saw that this data gives rise to $\mathbb{K}$-linear functors

$$F_i := \mathrm{Hom}_A(M_i, -) : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$$

and

$$G_i := \mathrm{Hom}_B(N_i, -) : \mathsf{Mod}\, B \to \mathsf{Mod}\, C.$$

We are also given homomorphisms $\phi_i : M_i \to M_{i+1}$ in $\mathsf{BiMod}(A, B)$ and $\psi_i : N_i \to N_{i+1}$ in $\mathsf{BiMod}(B, C)$.

(1) Show that ϕ_i induces a morphism of functors

$$\eta_i : F_{i+1} \to F_i\,,$$

and ψ_i induces a morphism of functors

$$\theta_i : G_{i+1} \to G_i.$$

(2) Show that the homomorphism $\phi_2 \circ \phi_1$ induces the morphism of functors $\eta_1 * \eta_2$, and that the homomorphism $\psi_2 \circ \psi_1$ induces the morphism of functors $\theta_1 * \theta_2$.

(3) Show that for every i the homomorphisms ϕ_i and ψ_i induce the morphism of functors $\theta_i \circ \eta_i$.

Remark 5.18. After we learn about tensor products, we will see that there is an isomorphism of functors

$$G_i \circ F_i \cong \mathrm{Hom}_A(P_i, -),$$

where

$$P_i := M_i \otimes_B N_i \in \mathsf{BiMod}(A, C).$$

And then there is a morphism

$$\phi_i \otimes \psi_i : P_i \to P_{i+1},$$

which induces the morphism of functors $\theta_i \circ \eta_i$.

comment: Read to here for lecture 5.

comment: Next topic: Equivalence of Categories

Lecture 6, 25 April 2018.

6. Equivalence of Categories

Definition 6.1. A functor

$$F : \mathsf{C} \to \mathsf{D}$$

is called an *isomorphism of categories* if the functions

$$F_{\mathrm{Ob}} : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$$

and

$$F_{C_1, C_2} : \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big)$$

are all bijective.

If F is an isomorphism of categories, then it has an inverse functor

$$F^{-1} : \mathsf{D} \to \mathsf{C},$$

with the obvious formulas. The inverse F^{-1} is unique.

It turns out that the notion of isomorphism of categories is too restrictive, and that we need a more relaxed notion: *equivalence of categories.* As we will see, the notion of equivalence of categories is analogous to the notion of *homotopy equivalence of topological spaces.*

Example 6.2. Let X be the origin in the real plane $\mathbb{R}^2$, and let Y be the closed unit disk in $\mathbb{R}^2$. We give $\mathbb{R}^2$ the standard metric topology, and X, Y have the subspace topologies.

The inclusion $f : X \to Y$ is not an isomorphism in Top; it is not even bijective. Yet f is a homotopy equivalence: the constant function $g : Y \to X$ is a homotopy-inverse of f. There is equality $g \circ f = \mathrm{id}_X$, and there is a homotopy $f \circ g \to \mathrm{id}_Y$.

In the categorical setting the role of homotopies is played by the morphisms of functors.

Definition 6.3. A functor

$$F : \mathsf{C} \to \mathsf{D}$$

is called an *equivalence of categories* if there is a functor

$$G : \mathsf{D} \to \mathsf{C},$$

and isomorphisms of functors

$$\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}}$$

and

$$\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}} .$$

The functor G is called a *quasi-inverse* of F.

The situation is symmetric: the quasi-inverse

$$G : \mathsf{D} \to \mathsf{C}$$

is also an equivalence of categories.

Let me give an important example.

Example 6.4. Let $\mathsf{D} := \mathsf{Set}_{\mathrm{fin}}$ be the category of finite sets. Define C to the full subcategory of D on the set of objects $\{S_i\}_{i \in \mathbb{N}}$, where

$$S_i := \{1, \ldots, i\}.$$

The functor

$$F : \mathsf{C} \to \mathsf{D}$$

is the inclusion. We will prove that F is an equivalence.

Since there is exactly one object from each isomorphism class in C, there is no choice in the definition of the quasi-inverse functor

$$G : \mathsf{D} \to \mathsf{C}$$

on objects. For a finite set S of cardinality i we must take $G(S) := S_i$. What we need to choose an isomorphism

$$\eta_S : G(S) \xrightarrow{\simeq} S_i$$

in C. We make matters simple by choosing

$$\eta_{S_i} := \mathrm{id}_{S_i} : S_i \xrightarrow{\simeq} S_i \tag{6.5}$$

for $S_i \in \mathsf{C}$.

We need to say what G does on morphisms. Given a morphism $\phi : S \to T$ in D we define

$$G(\phi) : G(S) \to G(T)$$

by

$$G(\phi) := \eta_T^{-1} \circ \phi \circ \eta_S.$$

This make the diagram diagram

$$\begin{tikzcd} G(S) \arrow[r, "G(\phi)"] \arrow[d, "\eta_S"'] & G(T) \arrow[d, "\eta_T"] \\ S \arrow[r, "\phi"] & T \end{tikzcd} \tag{6.6}$$

commutative.

It is easy to see that G is a functor. Note that the action of G on morphisms is determined by our choice of isomorphisms η_S.

By our choice in (6.5) we see that $G \circ F = \mathrm{Id}_{\mathsf{C}}$, so we take the isomorphism of functors

$$\zeta := \mathrm{id} : G \circ F \to \mathrm{Id}_{\mathsf{C}} .$$

And the collection of morphisms

$$\eta := \{\eta_S\}_{S \in \mathsf{D}}$$

is an isomorphism of functors

$$\eta : F \circ G \to \mathrm{Id}_{\mathsf{D}} .$$

Proposition 6.7. *If $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories, and if $G, G' : \mathsf{D} \to \mathsf{C}$ are both quasi-inverses of F, then there is an isomorphism of functors $G \cong G'$.*

Exercise 6.8. Prove this proposition.

Proposition 6.9. *If*

$$F : \mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, then for every pair of objects $C_1, C_2 \in \mathsf{C}$ the function

$$F : \operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \to \operatorname{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big)$$

is bijective.

Exercise 6.10. Try to prove this proposition. It is a bit tricky. If you can't, then there is a proof here: Solution 7.18.

Another important example of an equivalence is in the next exercise.

Exercise 6.11. Let $\mathbb{K}$ be a field, let $\mathsf{D} := \mathsf{Mod}_{\mathrm{fin}}\, \mathbb{K}$ be the category of finitely generated $\mathbb{K}$-modules (aka finite dimensional vector spaces), and let C be the full subcategory of D on the set of objects $\{\mathbb{K}^i\}_{i \in \mathbb{N}}$. The functor

$$F : \mathsf{C} \to \mathsf{D}$$

is the inclusion. Prove that F is an equivalence.

Remark 6.12. In the last example, let us view $\mathbb{K}^i$ as a column module (for $i > 0$). Then the endomorphism ring of $\mathbb{K}^i$ is the ring of matrices $\operatorname{Mat}_i(\mathbb{K})$. If $M \in \mathsf{Mod}_{\mathrm{fin}}\, \mathbb{K}$ is some rank i module, then its endomorphism ring is isomorphic to $\operatorname{Mat}_i(\mathbb{K})$, and the isomorphism depends on our choice of quasi-inverse $G : \mathsf{D} \to \mathsf{C}$. Indeed, by Propositions 6.9 and 6.16, and Exercise 4.14, we have a $\mathbb{K}$-ring isomorphism

$$G : \operatorname{End}_{\mathbb{K}}(M) \xrightarrow{\simeq} \operatorname{End}_{\mathbb{K}}(\mathbb{K}^i) = \operatorname{Mat}_i(\mathbb{K}).$$

Definition 6.13. Let

$$F : \mathsf{C} \to \mathsf{D}$$

be a functor.

(1) The functor F is called *full* (resp. *faithful*) if for every pair of objects $C_1, C_2 \in \mathsf{C}$ the function

$$F : \operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \to \operatorname{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big)$$

is surjective (resp. injective).

(2) The functor F is called *essentially surjective on objects* if for every object $D \in \mathsf{D}$ there is an object $C \in \mathsf{C}$ with an isomorphism $F(C) \xrightarrow{\simeq} D$ in D.

Theorem 6.14. *Let*

$$F : \mathsf{C} \to \mathsf{D}$$

be a functor. The following two conditions are equivalent.

(i) *F is an equivalence of categories.*
(ii) *F is full, faithful and essentially surjective on objects.*

Exercise 6.15. Prove this theorem. (Hint: in Proposition 6.9 we saw that an equivalence F is full and faithful; so this is almost the implication (i) $\Rightarrow$ (ii). For the opposite implication try to imitate Example 6.4.)

The next proposition was not in the lecture.

Proposition 6.16. *Let M and N be $\mathbb{K}$-linear categories, and let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. Assume that F is an equivalence, and $G : \mathsf{N} \to \mathsf{M}$ is a quasi-inverse of F. Then G is a $\mathbb{K}$-linear functor.*

Exercise 6.17. Prove this proposition. (Hint: see my proof of Proposition 6.9, i.e. Solution 7.18.)

7. Opposite Rings and Tensor Products

Definition 7.1. Let A be a $\mathbb{K}$-ring. The *opposite ring* of A is the $\mathbb{K}$-ring A^{op}, that has the same underlying $\mathbb{K}$-module structure, but with multiplication

$$a_1 \cdot^{\mathrm{op}} a_2 := a_2 \cdot a_2.$$

The unit element remains the same.

The identity function

$$\mathrm{op} : A^{\mathrm{op}} \to A \tag{7.2}$$

is a *ring anti-isomorphism.*

We can view right A-modules as left A^{op}-modules. Indeed, given a right A-module M, define a left multiplication by elements of A^{op} as follows:

$$a \cdot^{\mathrm{op}} m := m \cdot a.$$

It is clear that the unit element acts as the identity automorphism of M. As for associativity:

$$\begin{aligned}(a_1 \cdot^{\mathrm{op}} a_2) \cdot^{\mathrm{op}} m &= m \cdot (a_1 \cdot^{\mathrm{op}} a_2) = m \cdot (a_2 \cdot a_1) = (m \cdot a_2) \cdot a_1 \\ &= a_1 \cdot^{\mathrm{op}} (m \cdot a_2) = a_1 \cdot^{\mathrm{op}} (a_2 \cdot^{\mathrm{op}} m).\end{aligned}$$

We can make it very formal using our fancy language:

Proposition 7.3. *There is a $\mathbb{K}$-linear isomorphism of categories*

$$F : (\text{right } A\text{-modules}) \to \mathsf{Mod}\, A^{\mathrm{op}}$$

such that the diagram

$$\begin{tikzcd} (\text{right } A\text{-modules}) \arrow[r, "F"] \arrow[dr, "\mathrm{Forg}"'] & \mathsf{Mod}\, A^{\mathrm{op}} \arrow[d, "\mathrm{Forg}"] \\ & \mathsf{Mod}\, \mathbb{K} \end{tikzcd}$$

is commutative.

Note that $A = A^{\mathrm{op}}$ iff A is commutative.

We now talk about tensor products.

Definition 7.4. Let A be a $\mathbb{K}$-ring, $M \in \mathsf{Mod}\, A^{\mathrm{op}}$, $N \in \mathsf{Mod}\, A$ and $P \in \mathsf{Mod}\, \mathbb{K}$. An A-bilinear function

$$\beta : M \times N \to P$$

is a function with the following four properties:

- $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$
- $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$
- $\beta(m \cdot a, n) = \beta(m, a \cdot n)$

- $\beta(\lambda \cdot m, n) = \beta(m, \lambda \cdot n) = \lambda \cdot \beta(m, n)$

These must hold for every $m, m_i \in M$; $n, n_i \in N$; $a \in A$ and $\lambda \in \mathbb{K}$.

Example 7.5. If $M = A$ and $P = N$, then $\beta(a, n) := a \cdot n$ is an A-bilinear function.

Definition 7.6. Let A be a $\mathbb{K}$-ring, $M \in \operatorname{Mod} A^{\mathrm{op}}$ and $N \in \operatorname{Mod} A$. A *tensor product* of M and N over A is a pair (P, β), where $P \in \operatorname{Mod} \mathbb{K}$, and

$$\beta : M \times N \to P$$

is an A-bilinear function. The pair (P, β) must have this universal property:

(T) For every pair (P', β') of this sort, there is a unique $\mathbb{K}$-linear homomorphism $\phi : P \to P'$ such that

$$\beta' = \phi \circ \beta.$$

Theorem 7.7. *Let A be a $\mathbb{K}$-ring, $M \in \operatorname{Mod} A^{\mathrm{op}}$ and $N \in \operatorname{Mod} A$. A tensor product (P, β) of M and N over A exists, and it is unique up to a unique isomorphism.*

Proof. Uniqueness: suppose (P, β) and (P', β') are both tensor products of M and N over A. By property (T) there are unique homomorphisms $\phi : P \to P'$ and $\phi' : P' \to P$ that interact with β and β' as specified. The standard argument shows that ϕ and ϕ' are inverse to each other.

Existence: Let $\tilde{P}$ be the free $\mathbb{K}$-module on the set $M \times N$. Consider the $\mathbb{K}$-submodule $R \subseteq \tilde{P}$ generated by these four types of elements:

- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$
- $(m \cdot a, n) - (m, a \cdot n)$
- $(\lambda \cdot m, n) - \lambda \cdot (m, n)$

Define the $\mathbb{K}$-module $P := \tilde{P}/R$ and the function

$$\beta : M \times N \to P, \quad \beta(m, n) := (m, n) + R.$$

The end of the proof is left as an exercise. □

Exercise 7.8. Finish the proof. (Hint: see proof of the commutative theorem, [Ye2, lecture 4, page 39].)

Definition 7.9. The tensor product gets this notation:

$$M \otimes_A N := P$$

and

$$m \otimes n := \beta(m, n).$$

The elements $m \otimes n$ are called *pure tensors*.

Proposition 7.10. *The $\mathbb{K}$-module $M \otimes_A N$ is generated by the pure tensors.*

Exercise 7.11. Prove this proposition. (Hint: study the proof of Theorem 7.7.)

Proposition 7.12. *Let $\phi : M_1 \to M_2$ be a homomorphism in $\operatorname{Mod} A^{\mathrm{op}}$, and let $\psi : N_1 \to N_2$ be a homomorphism in $\operatorname{Mod} A$. Then there is a unique homomorphism*

$$\phi \otimes \psi : M_1 \otimes_A N_1 \to M_2 \otimes_A N_2$$

in $\operatorname{Mod} \mathbb{K}$, such that

$$(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$$

for every $m \in M$ and $n \in N$.

Exercise 7.13. Prove this proposition. Hint: find a bilinear function

$$\beta : M_1 \times N_1 \to M_2 \otimes_A N_2.$$

Exercise 7.14. In the previous course we considered commutative rings. For a commutative ring A, and for A-modules M and N, we defined the tensor product $M \otimes_A N$. See [Ye2, Lecture 4].

Show that if A is a commutative ring, and we take $\mathbb{K} := A$, then the tensor product from Definition 7.6 above coincides with the commutative tensor product.

Theorem 7.15. *Let A and B be central $\mathbb{K}$-rings. Then the $\mathbb{K}$-module $A \otimes_{\mathbb{K}} B$ is a central $\mathbb{K}$-ring, with multiplication*

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$$

and unit element

$$1_{A \otimes_{\mathbb{K}} B} := 1_A \otimes 1_B.$$

Exercise 7.16. Prove this theorem. (Hint: see the proof of the commutative theorem, [Ye2, lecture 5, page 50].)

Example 7.17. Let A be a $\mathbb{K}$-ring and $n \geq 1$. Then

$$\operatorname{Mat}_n(A) \cong A \otimes_{\mathbb{K}} \operatorname{Mat}_n(\mathbb{K})$$

as $\mathbb{K}$-rings.

Here is why: Let $\boldsymbol{e}_{i,j} \in \operatorname{Mat}_n(\mathbb{K})$ be the matrix with 1 in the (i, j) position, and 0 elsewhere. Given a matrix

$$\boldsymbol{a} = [a_{i,j}] \in \operatorname{Mat}_n(A),$$

with entries $a_{i,j} \in A$, we send it to the tensor

$$f(\boldsymbol{a}) := \sum_{i,j} a_{i,j} \otimes \boldsymbol{e}_{i,j} \in A \otimes_{\mathbb{K}} \operatorname{Mat}_n(\mathbb{K}).$$

We get a $\mathbb{K}$-module isomorphism

$$f : \operatorname{Mat}_n(A) \xrightarrow{\simeq} A \otimes_{\mathbb{K}} \operatorname{Mat}_n(\mathbb{K}).$$

Let us show that f respects multiplication. Because of bilinearity, it is enough to look at matrices

$$\boldsymbol{a} = a \cdot \boldsymbol{e}_{i,j} \in \operatorname{Mat}_n(A)$$

and

$$\boldsymbol{b} = b \cdot \boldsymbol{e}_{k,l} \in \operatorname{Mat}_n(A)$$

for some $a, b \in A$ and indices i, j, k, l. Here we also view $\boldsymbol{e}_{i,j}, \boldsymbol{e}_{k,l} \in \operatorname{Mat}_n(A)$. Then the product in $\operatorname{Mat}_n(A)$ is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a \cdot b \cdot \boldsymbol{e}_{i,l}$$

if $j = k$, and 0 otherwise.

Now

$$f(\boldsymbol{a}) = a \otimes \boldsymbol{e}_{i,j} \in A \otimes_{\mathbb{K}} \operatorname{Mat}_n(\mathbb{K})$$

and

$$f(\boldsymbol{b}) = b \otimes \boldsymbol{e}_{k,l} \in A \otimes_{\mathbb{K}} \operatorname{Mat}_n(\mathbb{K}).$$

The product in $A \otimes_{\mathbb{K}} \mathrm{Mat}_n(\mathbb{K})$ is

$$(a \otimes \boldsymbol{e}_{i,j}) \cdot (b \otimes \boldsymbol{e}_{k,l}) = (a \cdot b) \otimes (\boldsymbol{e}_{i,j} \cdot \boldsymbol{e}_{k,l}) = \begin{cases} (a \cdot b) \otimes \boldsymbol{e}_{i,l} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Solution 7.18 (Solution of Exercise 6.10)**.** Take a pair of objects $C_1, C_2 \in \mathsf{C}$, and consider the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \arrow[r, "F"] \arrow[dr, "G \circ F"'] \arrow[dd, "\mathrm{id}"', "\cong"] & \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big) \arrow[d, "G"] \\ & \mathrm{Hom}_{\mathsf{C}}\big((G \circ F)(C_1), (G \circ F)F(C_2)\big) \arrow[dl, "\cong", "\mathrm{Hom}_{\mathsf{C}}(\eta_{C_2}^{-1}, \eta_{C_2})"'] \\ \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) & \end{tikzcd} \tag{7.19}$$

in Set. Here

$$\eta_{C_i} : (G \circ F)(C_i) \xrightarrow{\simeq} C_i$$

are the given isomorphisms in C (the components of η). We see that the function $G \circ F$ in the diagram is bijective, the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big)$$

injective, and the function G is surjective.

We can play the game from the other side too (replacing C with D, etc.), and this will tell us that the function

$$G : \mathrm{Hom}_{\mathsf{D}}(D_1, D_2) \to \mathrm{Hom}_{\mathsf{C}}\big(G(D_1), G(D_2)\big)$$

is injective for every pair $D_1, D_2 \in \mathsf{D}$. In particular, for the choice $D_i := F(C_i)$, where C_i as the objects from above, we see that the function G in diagram (7.19) is injective. So the function G in diagram (7.19) is bijective, and hence the function F in diagram (7.19) is bijective.

I wonder if there is a cleaner proof of this proposition?

Lecture 7, 2 May 2018.

Bimodules were introduced in Definition 4.15.

Proposition 7.20. *Suppose A and B are central $\mathbb{K}$-rings, and M is a $\mathbb{K}$-central A-B-bimodule. There is a unique structure of left $(A \otimes_{\mathbb{K}} B^{\mathrm{op}})$-module on M such that*

$$(a \otimes b) \cdot m = a \cdot m \cdot b$$

for all $a \in A$, $b \in B$ and $m \in M$.

The next three exercises are quite similar. Perhaps solving one of them is enough.

Exercise 7.21. Prove Proposition 7.20.

From now on we will identify these two notions. Thus we shall use the notation $\operatorname{Mod} A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ for the category of $\mathbb{K}$-central A-B-bimodules.

Proposition 7.22. *Suppose A, B, C are central $\mathbb{K}$-rings, $M \in \operatorname{Mod} A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ and $N \in \operatorname{Mod} B \otimes_{\mathbb{K}} C^{\mathrm{op}}$. Then there is a unique structure of left $(A \otimes_{\mathbb{K}} C^{\mathrm{op}})$-module on the $\mathbb{K}$-module $M \otimes_B N$, such that*

$$(a \otimes c) \cdot (m \otimes n) = (a \cdot m) \otimes (n \cdot c)$$

for all $a \in A$, $c \in C$, $m \in M$ and $n \in N$.

Exercise 7.23. Prove this proposition.

Proposition 7.24. *Suppose A, B, C are central $\mathbb{K}$-rings, $M \in \operatorname{Mod} B \otimes_{\mathbb{K}} A^{\mathrm{op}}$ and $N \in \operatorname{Mod} B \otimes_{\mathbb{K}} C^{\mathrm{op}}$. Then there is a unique structure of left $(A \otimes_{\mathbb{K}} C^{\mathrm{op}})$-module on the $\mathbb{K}$-module $\operatorname{Hom}_B(M, N)$, such that*

$$((a \otimes c) \cdot \phi)(m) = \phi(m \cdot a) \cdot c$$

for all $a \in A$, $c \in C$, $m \in M$ and $\phi \in \operatorname{Hom}_B(M, N)$.

Exercise 7.25. Prove this proposition.

The next exercise is harder.

Exercise 7.26. By Propositions 7.22 and 7.12, a bimodule $M \in \operatorname{Mod} A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ gives rise to a $\mathbb{K}$-linear functor

$$G_M : \operatorname{Mod} B \to \operatorname{Mod} A, \quad G_M(N) := M \otimes_B N.$$

(1) Can we recover the bimodule M from the functor G_M ?
(2) (Very hard) Is every $\mathbb{K}$-linear functor

$$G : \operatorname{Mod} B \to \operatorname{Mod} A$$

isomorphic to the functor G_M for some bimodule M ?

8. Contravariant Functors and Opposite Categories

Recall that in Remark 4.12 we fixed a module $M \in \operatorname{Mod} A$, and defined

$$G_M(N) := \operatorname{Hom}_A(N, M) \in \operatorname{Mod} \mathbb{K}$$

for $N \in \operatorname{Mod} A$. This looks like a functor. But for a homomorphism $\phi : N_1 \to N_2$ the $\mathbb{K}$-linear homomorphism

$$G_M(\phi) := \operatorname{Hom}(\phi, \mathrm{id}_M) : G(N_2) \to G(N_1)$$

goes in the wrong direction. This is the prototypical example of a:

Definition 8.1. Let C and D be categories. A *contravariant functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of these ingredients:

- A function
$$F : \operatorname{Ob}(\mathsf{C}) \to \operatorname{Ob}(\mathsf{D}).$$
- For every pair of objects $C_1, C_2 \in \mathsf{C}$, a function
$$F : \operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \to \operatorname{Hom}_{\mathsf{D}}\big(F(C_2), F(C_1)\big).$$

There are two conditions:

- (Composition) For all composable morphisms $C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3$ in C there is equality
$$F(f_1) \circ F(f_2) = F(f_2 \circ f_1)$$
of morphisms $F(C_3) \to F(C_1)$ in D.
- (Identity) For every object C of C there is equality
$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$$
of morphisms from $F(C) \to F(C)$ in D.

A functor in the ordinary sense is sometimes called a *covariant functor*.

Here is another example.

Example 8.2. Let $f : A \to B$ be and anti-homomorphism of rings (such as $f(\boldsymbol{a}) := \boldsymbol{a}^{\mathrm{t}}$ for $A = B = \operatorname{Mat}_n(\mathbb{K})$). Let A and B be the corresponding single-object linear categories. We get a contravariant linear functor

$$F : \mathsf{A} \to \mathsf{B}\,.$$

Contravariant functors can be handled effectively by the following construct.

Definition 8.3. Let C be a category. The *opposite category* of C is the category C^{op} defined like this:

- The set of objects of C^{op} is $\operatorname{Ob}(\mathsf{C})$.
- For every pair of objects $C_1, C_2 \in \mathsf{C}^{\mathrm{op}}$, the set of morphisms is
$$\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_1, C_2) := \operatorname{Hom}_{\mathsf{C}}(C_2, C_1).$$
- For every triple of objects $C_1, C_2, C_3 \in \mathsf{C}^{\mathrm{op}}$, the composition
$$\circ^{\mathrm{op}} : \operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_2, C_3) \times \operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_1, C_2) \to \operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_1, C_3)$$
is
$$f_2 \circ^{\mathrm{op}} f_1 := f_1 \circ f_2.$$

- For every object $C \in \mathsf{C}^{\mathrm{op}}$, the identity automorphism id_C is that of C.

For a category C there is a canonical contravariant functor

$$\mathrm{Op} : \mathsf{C}^{\mathrm{op}} \to \mathsf{C} \tag{8.4}$$

that is the identity on objects and morphisms. This is the categorical version of the homomorphism (7.2). Note that the contravariant functor Op is an anti-isomorphism of categories, and that

$$\mathrm{Op} \circ \mathrm{Op} = \mathrm{Id}\,.$$

Proposition 8.5. *Let* C *and* D *be categories. The assignment* $F \mapsto \mathrm{Op} \circ F$ *is a bijection from the set of contravariant functors* $F : \mathsf{C} \to \mathsf{D}$ *to the set of covariant functors* $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

We leave out the easy proof.

Note that if M is a linear category, then so is M^{op}, and the contravariant functor

$$\mathrm{Op} : \mathsf{M}^{\mathrm{op}} \to \mathsf{M}$$

is linear.

Morphisms between contravariant functors take place in the target category, so they are insensitive to variance.

Remark 8.6. Let A be a ring, and let $\mathsf{M} := \mathsf{Mod}\, A$. The opposite category M^{op} is also linear. However, there is never (except when A is the zero ring) a linear equivalence of categories $\mathsf{M}^{\mathrm{op}} \to \mathsf{Mod}\, B$, for a ring B. This is pretty hard to prove.

The linear categories M and M^{op} are *abelian categories*. This is a complicated notion, and we won't talk about it. All we need to know is the concept of *exact sequences* in M and M^{op}. In M it is easy to define, and exactness in M^{op} will be defined in an ad hoc manner.

Example 8.7. Let $\mathbb{K}$ be a field. There is a $\mathbb{K}$-linear equivalence

$$(\mathsf{Mod}\, \mathbb{K})^{\mathrm{op}} \to \mathsf{TopMod}_{\mathrm{pf}}\, \mathbb{K}$$

where the latter is the category of profinite topological $\mathbb{K}$-modules, with continuous homomorphisms. See [Ye3, Section 3]. (This article is mostly a survey, and this particular result is not new. Usually people call these objects "linearly compact vector spaces".)

Exercise 8.8. Let $\mathbb{K}$ be a field, and let $\mathsf{M} := \mathsf{Mod}_{\mathrm{fin}}\, \mathbb{K}$, the category of finitely generated $\mathbb{K}$-modules (aka finite dimensional vector spaces). For $M \in \mathsf{M}$ define

$$D(M) := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}).$$

Prove that

$$D : \mathsf{M}^{\mathrm{op}} \to \mathsf{M}$$

is a $\mathbb{K}$-linear equivalence of categories.

Exercise 8.9. Let A and B be rings, and let $M \in \mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}}$. By Proposition 7.24 there is a $\mathbb{K}$-linear functor

$$F_M := \mathrm{Hom}_A(-, M) : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, B^{\mathrm{op}}.$$

(1) Can we recover the bimodule M from the functor F_M ?
(2) (Hard) Is every $\mathbb{K}$-linear functor

$$F : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, B^{\mathrm{op}}$$

isomorphic to the functor F_M for some bimodule M ?

9. Bifunctors

Definition 9.1. Let C and D be categories. Their *product category* is the category $\mathsf{C} \times \mathsf{D}$ defined like this:

- The set of objects is
$$\operatorname{Ob}(\mathsf{C} \times \mathsf{D}) := \operatorname{Ob}(\mathsf{C}) \times \operatorname{Ob}(D).$$
- For every pair of objects
$$(C_1, D_1), (C_2, D_2) \in \operatorname{Ob}(\mathsf{C} \times \mathsf{D})$$
the set of morphisms is
$$\operatorname{Hom}_{\mathsf{C} \times \mathsf{D}}\big((C_1, D_1), (C_2, D_2)\big) := \operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \times \operatorname{Hom}_{\mathsf{D}}(D_1, D_2).$$
- The composition and identity morphisms are component wise, i.e.
$$(f_2, g_2) \circ (f_1, g_1) := (f_2 \circ f_1, g_2 \circ g_1)$$
and
$$\operatorname{id}_{(C,D)} := (\operatorname{id}_C, \operatorname{id}_D).$$

Definition 9.2. Let C_1, C_2 and D be categories. A *bifunctor* from C_1 and C_2 to D is a functor
$$F : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}\,.$$

Here is the most important example of a bifunctor.

Example 9.3. Let C be a category. The bifunctor
$$\operatorname{Hom}_{\mathsf{C}}(-,-) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$$
assigns to each pair of objects the set $\operatorname{Hom}_{\mathsf{C}}(C_1, C_2)$, and to each pair of morphisms $f_1 : C_1 \to C_1'$ and $f_2 : C_2 \to C_2'$ in C, the morphism
$$\operatorname{Hom}_{\mathsf{C}}(f_1, f_2) : \operatorname{Hom}_{\mathsf{C}}(C_1', C_2) \to \operatorname{Hom}_{\mathsf{C}}(C_1, C_2')$$
in Set.

Exercise 9.4. Suppose M, N and P are $\mathbb{K}$-linear categories.

(1) The bifunctor
$$\operatorname{Hom}_{\mathsf{M}}(-,-) : \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \operatorname{Mod} \mathbb{K}$$
is a $\mathbb{K}$-*bilinear functor*. Try to give a definition of this notion.

(2) Try to define the $\mathbb{K}$-linear category $\mathsf{M} \otimes_{\mathbb{K}} \mathsf{N}$, in such a way that $\mathbb{K}$-bilinear bifunctors
$$\mathsf{M} \times \mathsf{N} \to \mathsf{P}$$
become the $\mathbb{K}$-linear functors
$$\mathsf{M} \otimes_{\mathbb{K}} \mathsf{N} \to \mathsf{P}\,.$$
In particular, there should be a universal $\mathbb{K}$-bilinear bifunctor
$$\mathsf{M} \times \mathsf{N} \to \mathsf{M} \otimes_{\mathbb{K}} \mathsf{N}\,.$$

The next propositions are upgradings of Propositions 7.22 and 7.24.

Proposition 9.5. *Suppose A, B, C are central $\mathbb{K}$-rings. Then*

$$(- \otimes_B -) : (\operatorname{Mod} A \otimes_{\mathbb{K}} B^{\mathrm{op}}) \times (\operatorname{Mod} B \otimes_{\mathbb{K}} C^{\mathrm{op}}) \to \operatorname{Mod} A \otimes_{\mathbb{K}} C^{\mathrm{op}}$$

is a $\mathbb{K}$-bilinear bifunctor.

Proposition 9.6. *Suppose A, B, C are central $\mathbb{K}$-rings. Then*

$$\operatorname{Hom}_B(-,-) : (\operatorname{Mod} B \otimes_{\mathbb{K}} A^{\mathrm{op}})^{\mathrm{op}} \times (\operatorname{Mod} B \otimes_{\mathbb{K}} C^{\mathrm{op}}) \to \operatorname{Mod} A \otimes_{\mathbb{K}} C^{\mathrm{op}}$$

is a $\mathbb{K}$-bilinear bifunctor.

Exercise 9.7. Prove the last two propositions.

10. Morita Theory, Adjoint Functors

This material is **optional**. It is not needed for the remainder of the course, but is good to know it if you have the time...

Recall that all rings are central over $\mathbb{K}$. From now $\otimes := \otimes_{\mathbb{K}}$. The *enveloping ring of A* is $A^{\mathrm{en}} := A \otimes A^{\mathrm{op}}$.

Definition 10.1. Let A and B be rings. A bimodule $P \in \operatorname{Mod} B \otimes A^{\mathrm{op}}$ is called an *invertible B-A-bimodule* if there is some $Q \in \operatorname{Mod} A \otimes B^{\mathrm{op}}$ with isomorphisms $P \otimes_A Q \cong B$ in $\operatorname{Mod} B^{\mathrm{en}}$ and $Q \otimes_B P \cong A$ in $\operatorname{Mod} A^{\mathrm{en}}$. Such Q is called a *quasi-inverse* of P.

It can be shown that the quasi-inverse Q has these isomorphisms:

$$Q \cong \operatorname{Hom}_B(P, B) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(P, A)$$

in $\operatorname{Mod} A \otimes B^{\mathrm{op}}$.

Temporary definition: a module $P \in \operatorname{Mod} A$ is called *projective* if it is a direct summand of a free A-module; i.e. if there's some P' and an isomorphism $P \oplus P' \cong F$ for some free A-module F.

Here is the classical Morita result.

Theorem 10.2. *Let A and B be rings and let $P \in \operatorname{Mod} B \otimes A^{\mathrm{op}}$. The following conditions are equivalent.*

(1) *The functor*
$$P \otimes_A (-) : \operatorname{Mod} A \to \operatorname{Mod} B$$
is an equivalence.

(2) *The B-A-bimodule P is invertible.*

(3) *P is finitely generated projective as a B-module, and the ring homomorphism*
$$A^{\mathrm{op}} \to \operatorname{End}_B(P)$$
is bijective.

The second Morita Theorem is this:

Theorem 10.3. *Let A and B be rings and let*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a $\mathbb{K}$-linear equivalence. Define $P := F(A) \in \operatorname{Mod} B$, and give P the right A-module structure coming from F. Then

$$F \cong P \otimes_A (-)$$

as functors.

For a proof see [Row, Chapter 4].

We will take a closer look at what I call "baby Morita equivalence", where A is any nonzero ring, and $B := \operatorname{Mat}_n(A)$ for some $n \geq 1$. Let $P := A^n$ written as a column module, and let $Q := A^n$ written as a row module. For $n = 2$ this is what we have:

$$Q = \begin{bmatrix} A & A \end{bmatrix}, \quad B = \begin{bmatrix} A & A \\ A & A \end{bmatrix}, \quad P = \begin{bmatrix} A \\ A \end{bmatrix}.$$

The action of B on P and Q is by left and right matrix multiplication, respectively; and the action of A is by scalar multiplication on the other sides.

Exercise 10.4. With the definitions above, and without using the Morita Theorems:

(1) Prove that P and Q are quasi-inverses, in the sense of Definition 10.1.
(2) Prove that P is a projective B-module and a projective A^{op}-module.
(3) Prove that Q is a projective B^{op}-module and projective A-module.
(4) Prove that ring homomorphisms

$$B \to \operatorname{End}_{A^{\mathrm{op}}}(P)$$

and

$$A^{\mathrm{op}} \to \operatorname{End}_B(P)$$

is bijective. Do the same for Q.

The *NC Picard group* of A is the group $\operatorname{Pic}_{\mathbb{K}}(A)$ whose elements are the isomorphism classes (in $\operatorname{Mod} A^{\mathrm{en}}$) of the invertible A-A-bimodules. The operation is induced by $(-\otimes_A -)$. This is a nonabelian group.

If A is a commutative ring, then we have the much more famous *commutative Picard group* $\operatorname{Pic}_A(A)$. Its elements are the isomorphism classes of the *invertible A-modules.* (In our NC terms they can be viewed as the A-central A-A-bimodules.) The operation is induced by $(- \otimes_A -)$. This is an abelian group. It can be shown (without much difficulty) that

$$\operatorname{Pic}_{\mathbb{K}}(A) \cong \operatorname{Aut}_{\mathbb{K}}(A) \ltimes \operatorname{Pic}_A(A),$$

where $\operatorname{Aut}_{\mathbb{K}}(A)$ is the "Galois group", i.e. the group of $\mathbb{K}$-ring automorphisms of A.

Now to adjunctions. Here is a very difficult to absorb definition. It is due to Daniel Kan, circa 1958.

Definition 10.5. Let

$$F : \mathsf{C} \to \mathsf{D}$$

and

$$G : \mathsf{D} \to \mathsf{C}$$

be functors between categories. An *adjunction* between F and G is an isomorphism

$$\xi : \operatorname{Hom}_{\mathsf{D}}\big(F(-), -\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}}\big(-, G(-)\big)$$

of bifunctors

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}\,.$$

The functor F is called a *left adjoint* of G, and the functor G is called a *right adjoint* of F.

Let us try to understand it better. What the definition says is that given objects $C \in \mathsf{C}$ and $D \in \mathsf{D}$, there is a bijection between the sets of morphisms:

$$\xi_{C,D} : \operatorname{Hom}_{\mathsf{D}}\big(F(C), D\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}}\big(C, G(D)\big).$$

But moreover, these isomorphisms are functorial in C and D.

Here are a few examples.

Example 10.6. Suppose

$$F : \mathsf{C} \to \mathsf{D}$$

is an equivalence of categories, with quasi-inverse

$$G : \mathsf{D} \to \mathsf{C},$$

and isomorphisms of functors

$$\eta : G \circ F \xrightarrow{\simeq} \operatorname{Id}_{\mathsf{C}}$$

and

$$\zeta : F \circ G \xrightarrow{\simeq} \operatorname{Id}_{\mathsf{D}} .$$

See Definition 6.3. In this case the functors F and G are adjoints on both sides. One adjunction is gotten this way: we define

$$\xi_{C,D} : \operatorname{Hom}_{\mathsf{D}}\big(F(C), D\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}}\big(C, G(D)\big)$$

to be the composition

$$\operatorname{Hom}_{\mathsf{D}}\big(F(C), D\big) \xrightarrow{G} \operatorname{Hom}_{\mathsf{C}}\big((G \circ F)(C), G(D)\big) \xrightarrow{\operatorname{Hom}_{\mathsf{C}}(\eta_C^{-1}, \operatorname{id}_{G(D)})} \operatorname{Hom}_{\mathsf{C}}\big(C, G(D)\big).$$

Exercise 10.7. Let A be a nonzero ring. The forgetful functor

$$\operatorname{Forg} : \operatorname{Mod} A \to \mathsf{Set}$$

and the free module functor

$$\mathrm{F}_{\mathrm{fin}}(-, A) : \mathsf{Set} \to \operatorname{Mod} A$$

are adjoints to each other. Find out from which sides, and prove it.

Exercise 10.8. Let $f : A \to B$ be a ring homomorphism. There are three functors related to this: restriction

$$\operatorname{Rest}_f : \operatorname{Mod} B \to \operatorname{Mod} A,$$

induction

$$\operatorname{Ind}_f := B \otimes_A (-) : \operatorname{Mod} A \to \operatorname{Mod} B,$$

and coinduction

$$\operatorname{CoInd}_f := \operatorname{Hom}_A(B, -) : \operatorname{Mod} A \to \operatorname{Mod} B.$$

Find the adjunction relations between them.

Lecture 8, 9 May 2018.

11. Exact Functors

Let A be a ring. A diagram

$$S = \big(\cdots M_{-1} \xrightarrow{\phi_{-1}} M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \cdots\big) \tag{11.1}$$

in the category $\operatorname{Mod} A$ is called a *sequence in* $\operatorname{Mod} A$. Here M_i are modules and ϕ_i are homomorphisms. The sequence S can extend infinitely on either side. A module M_i is said to be internal in S if it is not initial (i.e. M_{i-1} exists) and it is not terminal (i.e. M_{i+1} exists).

Definition 11.2. Let A be a ring and let S be a sequence in $\operatorname{Mod} A$, with the notation of (11.1).

(1) Let M_i be an internal object in S. The sequence S is *exact at* M_i if

$$\operatorname{Im}(\phi_{i-1}) = \operatorname{Ker}(\phi_i)$$

as submodule of M_i.

(2) The sequence S is called an *exact sequence* if it is exact at every internal object.

Example 11.3. An exact sequence of this shape

$$S = \big(0 \to M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \to 0\big) \tag{11.4}$$

is called a *short exact sequence*.

- Exactness at M_0 means that ϕ_0 is injective. This is also called a *monomorphism*.
- Exactness at M_2 means that ϕ_1 is surjective. This is also called an *epimorphism*.
- If we identify M_0 with its image in M_1 under ϕ_0, then ϕ_1 induces an isomorphism

$$M_1/M_0 \xrightarrow{\simeq} M_2.$$

Remark 11.5. The source of the name "exact" is in differential geometry. Suppose X is a differentiable manifold (of type C^∞ over $\mathbb{R}$) of dimension n. You can take $X = \mathbb{R}^n$. We denote by Ω^p_X the $\mathbb{R}$-module of global differentiable p-forms on X. Exterior derivation is an $\mathbb{R}$-linear homomorphism

$$\mathrm{d} : \Omega^p_X \to \Omega^{p+1}_X.$$

We know that $\mathrm{d} \circ \mathrm{d} = 0$.

A p-form α is called *closed* if $\mathrm{d}(\alpha) = 0$. The form α is called *exact* if $\alpha = \mathrm{d}(\beta)$ for some $\beta \in \Omega^{p-1}_X$.

The *Poincaré Lemma* says that if X is diffeomorphic to an open ball, then the sequence

$$0 \to \mathbb{R} \to \Omega^0_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots \Omega^n_X \to 0$$

is exact.

Definition 11.6. Let A and B be rings and let

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a $\mathbb{K}$-linear functor.

(1) The functor F is called *left exact* if for every exact sequence $\boldsymbol{S}$ in $\operatorname{Mod} A$, with notation (11.4), the sequence

$$0 \to F(M_0) \xrightarrow{F(\phi_0)} F(M_1) \xrightarrow{F(\phi_1)} F(M_2)$$

in $\operatorname{Mod} B$ is exact.

(2) The functor F is called *right exact* if for every exact sequence $\boldsymbol{S}$ in $\operatorname{Mod} A$, with notation (11.4), the sequence

$$F(M_0) \xrightarrow{F(\phi_0)} F(M_1) \xrightarrow{F(\phi_1)} F(M_2) \to 0$$

in $\operatorname{Mod} B$ is exact.

(3) The functor F is called *exact* if it is both left and right exact.

Example 11.7. Suppose P is a free A-module. Consider the functor

$$F : \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}, \quad F(M) = \operatorname{Hom}_A(P, M).$$

Then F is exact. To see this, let us choose an isomorphism

$$P \cong \mathrm{F}_{\mathrm{fin}}(X, A)$$

for a set X. Then there is an isomorphism

$$\eta_M : F(M) \xrightarrow{\simeq} \mathrm{F}(X, M)$$

in $\operatorname{Mod} \mathbb{K}$, and this is a functorial isomorphism. An exact sequence $\boldsymbol{S}$ gets transformed by η to the sequence

$$0 \to \mathrm{F}(X, M_0) \xrightarrow{\mathrm{F}(X, \phi_0)} \mathrm{F}(X, M_1) \xrightarrow{\mathrm{F}(X, \phi_1)} \mathrm{F}(X, M_2) \to 0.$$

Since this sequence is exact in each coordinate $x \in X$, it is exact.

Definition 11.8. An A-module P is called *flat* if the functor

$$(-) \otimes_A P : \operatorname{Mod} A^{\mathrm{op}} \to \operatorname{Mod} \mathbb{K}$$

is exact.

Example 11.9. Suppose P is a free A-module. We claim that P is flat. To see this, let us choose an isomorphism

$$P \cong \mathrm{F}_{\mathrm{fin}}(X, A)$$

for a set X. For every $M \in \operatorname{Mod} A^{\mathrm{op}}$ there is an isomorphism

$$\eta_M : M \otimes_A P \xrightarrow{\simeq} \mathrm{F}_{\mathrm{fin}}(X, M)$$

in $\operatorname{Mod} \mathbb{K}$, and η is an isomorphism of functors.

An exact sequence $\boldsymbol{S}$ gets transformed by η to

$$0 \to \mathrm{F}_{\mathrm{fin}}(X, M_0) \xrightarrow{\mathrm{F}_{\mathrm{fin}}(X, \phi_0)} \mathrm{F}_{\mathrm{fin}}(X, M_1) \xrightarrow{\mathrm{F}_{\mathrm{fin}}(X, \phi_1)} \mathrm{F}_{\mathrm{fin}}(X, M_2) \to 0.$$

Since this sequence is exact in each coordinate $x \in X$, it is exact.

Example 11.10. Here is an example of a functor that is not exact. Take the ring $A = \mathbb{K} = \mathbb{Z}$ and the A-module (abelian group) $N := \mathbb{Z}/(2)$. Consider the functor

$$G := N \otimes_{\mathbb{Z}} (-) : \operatorname{Mod} \mathbb{Z} \to \operatorname{Mod} \mathbb{Z}.$$

We will test it on the short exact sequence

$$\boldsymbol{S} = \big(0 \to \mathbb{Z} \xrightarrow{2 \cdot (-)} \mathbb{Z} \xrightarrow{\pi} N \to 0\big)$$

where π is the canonical epimorphism.

Upon applying G we get the sequence

$$G(\boldsymbol{S}) = \big(0 \to N \xrightarrow{0} N \xrightarrow{\mathrm{id}} N \to 0\big)$$

This is not exact at the first N. Therefore N is not a flat $\mathbb{Z}$-module.

There is exactness at the second and third N. This is not a coincideence: the tensor functor $G = N \otimes_A (-)$ is always *right exact*. We will prove it later (next week?)

Exercise 11.11. Prove that a $\mathbb{Z}$-module N is flat iff it is torsion-free. (First try to prove this for finitely generated modules.)

Proposition 11.12. *Let*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be an exact functor. Then for every sequence $\boldsymbol{S}$ *in* $\operatorname{Mod} A$ *as in* (11.1)*, the sequence*

$$F(\boldsymbol{S}) = \big(\cdots F(M_{-1}) \xrightarrow{F(\phi_{-1})} F(M_0) \xrightarrow{F(\phi_0)} F(M_1) \xrightarrow{F(\phi_1)} F(M_2) \cdots\big)$$

is exact.

Proof. If $\boldsymbol{S}$ has only two objects then there is no exactness to test. So let's assume it has at least three adjacent objects. But then we can assume the exact sequence $\boldsymbol{S}$ goes infinitely on both sides, by doing this: if M_i was initial is $\boldsymbol{S}$, then we define

$$M_{i-1} := \operatorname{Ker}(\phi_i : M_i \to M_{i+1}),$$

and $\phi_{i-1} : M_{i-1} \to M_i$ is the inclusion. Then we let $M_j := 0$ for $j < i - 1$. Similarly on the right, but using the cokernel of the last homomorphism. Later we can delete the new objects and recover the original sequence.

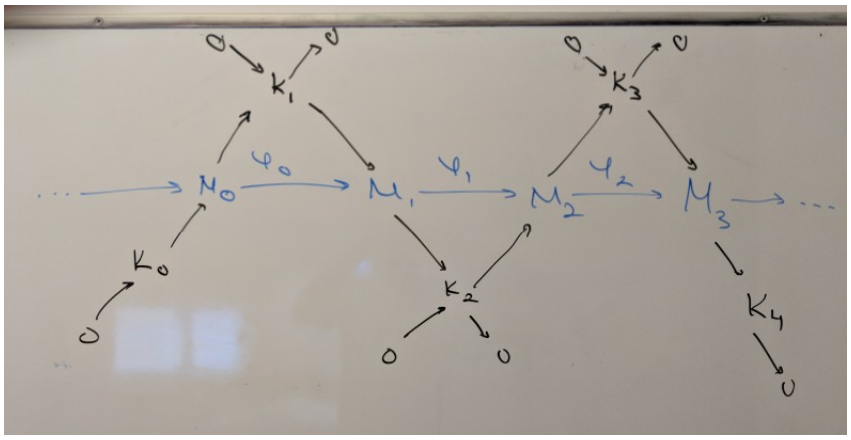
Now we define new modules

$$K_i := \operatorname{Ker}(\phi_i) \cong \operatorname{Im}(\phi_{i-1})$$

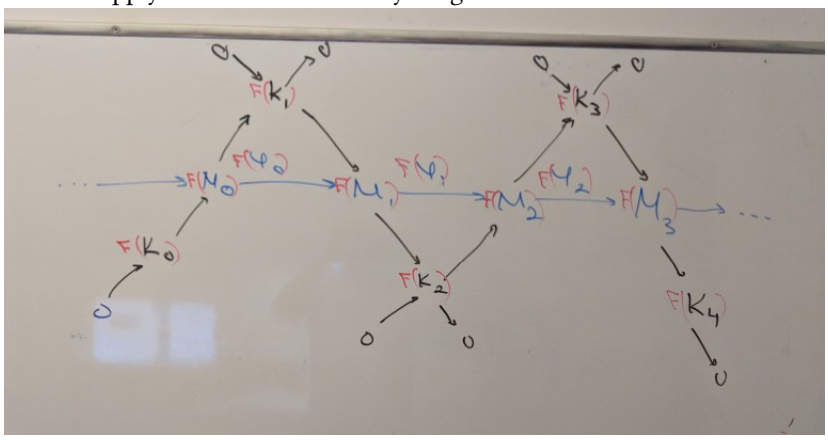
for all i. Then the long given exact sequence $\boldsymbol{S}$ can be broken into short exact sequences like this:

$$\boldsymbol{S}_i := \big(0 \to K_i \to M_i \to K_{i+1} \to 0\big)$$

See the picture below of the resulting big commutative diagram in $\operatorname{Mod} A$. The black diagonal short sequences are exact.



We now apply the functor F to everything:



Because F is exact, the sequences $F(\boldsymbol{S}_i)$ are all exact. Once we know that the homomorphisms

$$F(K_i) \to \operatorname{Ker}(F(\phi_i))$$

are all bijective, we are done. I leave this as an exercise. from □

Exercise 11.13. Finish the proof of the proposition.

Lecture 9, 16 May 2018.

Theorem 11.14. *Let $M \in \operatorname{Mod} A$. The functor*

$$F := \operatorname{Hom}_A(M, -) : \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}$$

is left exact.

Proof. Take a short exact sequence $\boldsymbol{S}$ in $\operatorname{Mod} A$, with notation (11.4). We need to check the exactness of the sequence

$$0 \to F(M_0) \xrightarrow{F(\phi_0)} F(M_1) \xrightarrow{F(\phi_1)} F(M_2).$$

First exactness at $F(M_0)$. Let

$$\chi \in \operatorname{Ker}(F(\phi_0)) \subseteq F(M_0) = \operatorname{Hom}_A(M, M_0).$$

This means that

$$\phi_0 \circ \chi = F(\phi_0)(\chi) = 0.$$

We know that ϕ_0 is a monomorphism. This implies that $\chi = 0$. So $F(\phi_0)$ is a monomorphism.

Now for exactness at $F(M_1)$. Since $\phi_1 \circ \phi_0 = 0$ it follows that $F(\phi_1) \circ F(\phi_0) = 0$. So

$$\operatorname{Im}(F(\phi_0)) \subseteq \operatorname{Ker}(F(\phi_1)).$$

For the opposite inclusion: let

$$\chi \in \operatorname{Ker}(F(\phi_1)) \subseteq F(M_1) = \operatorname{Hom}_A(M, M_1).$$

This means that

$$0 = F(\phi_1)(\chi) = \phi_1 \circ \chi.$$

So $\chi(M) \subseteq \operatorname{Ker}(\phi_1)$, and χ is in fact a homomorphism $\chi : M \to \operatorname{Ker}(\phi_1)$. By exactness at M_1 we know that

$$\operatorname{Ker}(\phi_1) = \operatorname{Im}(\phi_0).$$

And by exactness at M_0 we know that

$$\phi_0 : M_0 \to \operatorname{Im}(\phi_0)$$

is bijective. Thus we can find a homomorphism $\chi' : M \to M_0$ such that

$$\phi_0 \circ \chi' = \chi.$$

And then

$$\chi = F(\phi_0)(\chi') \in \operatorname{Im}(\phi_0).$$

□

Example 11.15. We can't expect more than left exactness of the functor $F = \operatorname{Hom}_A(M, -)$.

Take $A = \mathbb{Z}$ and $M = \mathbb{Z}/(39)$. Consider the module $N := \mathbb{Z}/(3)$ and the short exact sequence

$$\boldsymbol{S} = \big(0 \to \mathbb{Z} \xrightarrow{3 \cdot (-)} \mathbb{Z} \xrightarrow{\pi} N \to 0\big)$$

where π is the canonical epimorphism. The sequence $F(\boldsymbol{S})$ is

$$F(\boldsymbol{S}) = \big(0 \to 0 \to 0 \to N \to 0\big),$$

and it is not exact at N.

Definition 11.16. An A-module P is called *projective* if for every homomorphism $\phi : P \to N$ and every epimorphism $\pi : M \twoheadrightarrow N$, both in $\operatorname{Mod} A$, there is a homomorphism $\tilde{\phi} : P \to M$ such that

$$\phi = \pi \circ \tilde{\phi}.$$

In a commutative diagram:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\phi}"'] \arrow[d, "\phi"] \\ M \arrow[r, two heads, "\pi"] & N \end{tikzcd}$$

Theorem 11.17. *Let $P \in \operatorname{Mod} A$. The following four conditions are equivalent.*

(i) *P is a projective A-module.*

(ii) *The functor*

$$F := \operatorname{Hom}_A(P, -) : \operatorname{Mod} A \to \operatorname{Mod} K$$

is exact.

(iii) *Every short exact sequence*

$$0 \to M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} P \to 0$$

is split.

(iv) *P is a direct summand of a free A-module.*

Proof.

(i) $\Rightarrow$ (ii): Take a short exact sequence $\boldsymbol{S}$ in $\operatorname{Mod} A$, with notation (11.4). Theorem 11.14 says that there is exactness at $F(M_0)$ and at $F(M_1)$. It remains to show exactness at $F(M_2)$, namely that

$$F(\phi_1) : F(M_1) \to F(M_2)$$

is surjective. Consider an element

$$\psi \in F(M_2) = \operatorname{Hom}_A(P, M_2).$$

Because $\phi_1 : M_1 \to M_2$ is surjective and P is projective, there exists some $\tilde{\psi} : M \to M_1$ s.t. $\phi_1 \circ \tilde{\psi} = \psi$. So $F(\phi_1)(\tilde{\psi}) = \psi$.

(ii) $\Rightarrow$ (iii): Consider the identity $\mathrm{id}_P : P \to P$. Since the functor F is exact, the $\mathbb{K}$-linear homomorphism

$$F(\phi_1) : F(M_1) \to F(P)$$

is surjective. Let $\sigma \in F(M_1)$ be s.t. $F(\phi_1)(\sigma) = \mathrm{id}_P$. So $\phi_1 \circ \sigma = \mathrm{id}_P$, and this means that σ splits the exact sequence.

(iii) $\Rightarrow$ (iv): Choose a surjection $\phi_1 : Q \twoheadrightarrow P$ from a free A-module Q. We get a short exact sequence

$$0 \to M_0 \xrightarrow{\phi_0} Q \xrightarrow{\phi_1} P \to 0.$$

By assumption it is split. Thus P is a direct summand of Q.

(iv) $\Rightarrow$ (i): Say $P \oplus P' \cong Q$ for some free A-module Q. So there are homomorphisms $\gamma : P \to Q$ and $\delta : Q \to P$ s.t. $\delta \circ \gamma = \mathrm{id}_P$. We know that Q is projective (Example 11.7).

Given a homomorphism $\phi : P \to N$ and and epimorphism $\pi : M \twoheadrightarrow N$, consider $\phi \circ \delta : Q \to N$. There is a homomorphism $\tilde{\phi}' : Q \to M$ s.t.

$$\pi \circ \tilde{\phi}' = \phi \circ \delta.$$

Define

$$\tilde{\phi} := \tilde{\phi}' \circ \gamma : P \to N.$$

Then

$$\pi \circ \tilde{\phi} = \phi \circ \tilde{\phi}' \circ \gamma = \phi \circ \delta \circ \gamma = \phi.$$

In a commutative diagram:

$$\begin{tikzcd}
 & P \arrow[d, "\gamma"] \arrow[dd, bend left, "\mathrm{id}_P"] \arrow[ddl, dashed, "\tilde{\phi}"'] \\
 & Q \arrow[d, "\delta"] \arrow[dl, dashed, "\tilde{\phi}'"'] \\
 & P \arrow[d, "\phi"] \\
M \arrow[r, two heads, "\pi"] & N
\end{tikzcd}$$

□

Proposition 11.18. *Let*

$$F, F_1, F_2 : \operatorname{Mod} A \to \operatorname{Mod} B$$

be linear functors, and assume that

$$F \cong F_1 \oplus F_2.$$

Namely there are morphism of functors $\epsilon_i : F_i \to F$ such that for every $M \in \operatorname{Mod} A$ the homomorphism

$$\epsilon_{1,M} \oplus \epsilon_{2,M} : F_1(M) \oplus F_2(M) \to F(M)$$

is an isomorphism. The following conditions are equivalent:

(i) *F is exact* (*resp. left exact, resp. right exact*).
(ii) *F_1 and F_2 are exact* (*resp. left exact, resp. right exact*).

Exercise 11.19. Prove this proposition.

Corollary 11.20. *Let $P, P_1, P_2 \in \operatorname{Mod} A$, and assume that*

$$P \cong P_1 \oplus P_2.$$

The following conditions are equivalent:

(i) *P is flat* (*resp. projective*).
(ii) *P_1 and P_2 are flat* (*resp. projective*).

Corollary 11.21. *Let P be a projective A-module. Then P is a flat A-module.*

Exercise 11.22. Prove the last two corollaries. (Hint: use Proposition 11.18.)

Theorem 11.23. *Let $M \in \operatorname{Mod} A$. The functor*

$$F := (-) \otimes_A M : \operatorname{Mod} A^{\mathrm{op}} \to \operatorname{Mod} \mathbb{K}$$

is right exact.

Proof. This is the same as the commutative case. See the theorem and its proof on page 71 of lecture 7 in the notes [Ye2]. □

12. Exact Contravariant Functors

Here again is a short exact sequence in $\operatorname{Mod} A$.

$$S = \big(0 \to M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \to 0\big) \tag{12.1}$$

Definition 12.2. Let A and B be rings and let

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a contravariant $\mathbb{K}$-linear functor.

(1) The functor F is called a *left exact contravariant functor* if for every short exact sequence S in $\operatorname{Mod} A$, with notation (12.1), the sequence

$$0 \to F(M_2) \xrightarrow{F(\phi_1)} F(M_1) \xrightarrow{F(\phi_0)} F(M_0)$$

in $\operatorname{Mod} B$ is exact.

(2) The functor F is called a *right exact contravariant functor* if for every short exact sequence S in $\operatorname{Mod} A$, with notation (12.1), the sequence

$$F(M_2) \xrightarrow{F(\phi_1)} F(M_1) \xrightarrow{F(\phi_0)} F(M_0) \to 0$$

in $\operatorname{Mod} B$ is exact.

(3) The functor F is called *exact* if it is both left and right exact.

The moral is that the type of exactness is determined in the target category.

Exercise 12.3. State and prove the contravariant version of Proposition 11.12.

Theorem 12.4. *Let $M \in \operatorname{Mod} A$. The contravariant functor*

$$F := \operatorname{Hom}_A(-, M) : \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}$$

is left exact.

Proof. Take a short exact sequence S in $\operatorname{Mod} A$, with notation (12.1). We need to check the exactness of the sequence

$$0 \to F(M_2) \xrightarrow{F(\phi_1)} F(M_1) \xrightarrow{F(\phi_0)} F(M_0).$$

First exactness at $F(M_2)$. Let

$$\chi \in \operatorname{Ker}(F(\phi_1)) \subseteq F(M_2) = \operatorname{Hom}_A(M_2, M).$$

This means that

$$\chi \circ \phi_1 = F(\phi_1)(\chi) = 0.$$

We know that ϕ_1 is an epimorphism. This implies that $\chi = 0$. So $F(\phi_1)$ is a monomorphism.

Now for exactness at $F(M_1)$. Since $\phi_1 \circ \phi_0 = 0$ it follows that $F(\phi_0) \circ F(\phi_1) = 0$. So

$$\operatorname{Im}(F(\phi_1)) \subseteq \operatorname{Ker}(F(\phi_0)).$$

For the opposite inclusion: let

$$\chi \in \operatorname{Ker}(F(\phi_0)) \subseteq F(M_1) = \operatorname{Hom}_A(M_1, M).$$

This means that

$$0 = F(\phi_0)(\chi) = \chi \circ \phi_0.$$

So χ vanishes on $\operatorname{Im}(\phi_0) = \operatorname{Ker}(\phi_1)$. Thus there is an induced homomorphism

$$\bar{\chi} : M_2 \cong M_1/M_0 \to M$$

s.t. $\bar{\chi} \circ \phi_1 = \chi$. And $\bar{\chi} \in F(M_2)$ satisfies $F(\phi_1)(\bar{\chi}) = \chi$. We see that $\chi \in \operatorname{Im}(F(\phi_1))$. □

Definition 12.5. An A-module I is called *injective* if for every homomorphism $\phi : M \to I$ and every monomorphism $\epsilon : M \rightarrowtail N$, both in $\operatorname{Mod} A$, there is a homomorphism $\tilde{\phi} : N \to I$ such that

$$\phi = \tilde{\phi} \circ \epsilon.$$

In a commutative diagram:

$$\begin{tikzcd} M \arrow[r, tail, "\epsilon"] \arrow[d, "\phi"] & N \arrow[dl, dashed, "\tilde{\phi}"] \\ I & \end{tikzcd}$$

We won't use injective modules in our course. So I leave the proof of the next theorem as an exercise.

Theorem 12.6. *Let $I \in \operatorname{Mod} A$. The following three conditions are equivalent.*

(i) *I is an injective A-module.*

(ii) *The contravariant functor*

$$F := \operatorname{Hom}_A(-, I) : \operatorname{Mod} A \to \operatorname{Mod} K$$

is exact.

(iii) *Every short exact sequence*

$$0 \to I \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \to 0$$

in $\operatorname{Mod} A$ is split.

Exercise 12.7. Prove this theorem. (Hint: modify the proof of Theorem 11.17.)

Every A-module M admits a monomorphism (an injective homomorphism) $\epsilon : M \rightarrowtail I$ to an injective module I. This is analogous to the fact that every module M admits an epimorphism $\pi : P \twoheadrightarrow M$ from a projective module.

However, injective modules have a very complicated structure. They are almost never finitely generated. Despite that apparent symmetry between projective and injective modules, in actuality there is a tremendous lack of symmetry here.

Example 12.8. When A is a commutative noetherian ring, every injective A-module is a direct sum of *indecomposable injective A-modules*, with multiplicities. The indecomposable injective A-modules are classified by the prime spectrum $\operatorname{Spec}(A)$. This result is due to Matlis in 1958.

For $A = \mathbb{Z}$ the indecomposable injectives are:

- The $\mathbb{Z}$-module $\mathbb{Q}$, corresponding to the prime ideal (0).

- The $\mathbb{Z}$-module
$$\widehat{\mathbb{Q}}_p / \widehat{\mathbb{Z}}_p \cong \mathbb{Q}/\mathbb{Z}_{(p)} = \bigcup_{n \geq 1} \mathbb{Z}/(p^n)$$
corresponding to a prime number p.

Recall that our rings are $\mathbb{K}$-central.

Theorem 12.9 (Adjunction)**.** *Let $f : A \to B$ be a ring homomorphism, and let $M \in \operatorname{Mod} A$ and let $N \in \operatorname{Mod} B$. Then there is a bijection*
$$\Phi : \operatorname{Hom}_A(N, M) \xrightarrow{\simeq} \operatorname{Hom}_B\big(N, \operatorname{Hom}_A(B, M)\big)$$
in $\operatorname{Mod} \mathbb{K}$. It is functorial in M and N.

Exercise 12.10. Prove this theorem. Hint: for $\phi : N \to M$ define $\Phi(\phi)(n)(b) := \phi(b \cdot n) \in M$.

Corollary 12.11. *Suppose $\mathbb{K}$ is a field. Then $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is an injective A-module.*

Exercise 12.12. Prove this corollary. Hint: use the theorem, and the fact that for the field $\mathbb{K}$ we have an exact functor $\operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$.

Proposition 12.13. *If $\{I_x\}_{x \in X}$ is a collection of injective A-modules, then $I := \prod_{x \in X} I_x$ is an injective module.*

Exercise 12.14. Prove this proposition. Hint: immediate from the definition of a product and the definition of injective module.

Theorem 12.15 (Enough Injectives)**.** *Assume $\mathbb{K}$ is a field. Then every A-module M embeds into an injective A-module.*

Exercise 12.16. Prove this theorem. Hint: for an element $m \in M$ there is a homomorphism $\phi_m : M \to \mathbb{K}$ s.t. $\phi_m(m) \neq 0$. Use Thm 12.9 to find a homomorphism $\psi_m : M \to A^*$ s.t. $\psi_m(m) \neq 0$. Then use Prop 12.13.

Remark 12.17. When A does not contain a field, the proof of Thm 12.15 is different. We take $\mathbb{K} = \mathbb{Z}$. First one shows that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Then, by Thm 12.9,
$$A^* := \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$$
is an injective A-module. Finally a variation of the proof of Thm 12.15 is used.

Lecture 10, 23 May 2018.

13. Derived Functors – First Look

We saw that some linear functors are exact and some are not. And that properties of modules, such as being projective or flat, are determined by exactness of suitable functors.

The derived functors of a linear functor

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

determine to what extent, or "how much", the functor F is not left or right exact.

Sometimes the derived functors, or *cohomologies*, provide information about classification of certain mathematical objects. I hope to give a broad survey of such results in the last lecture of the course.

In our detailed study we will concentrate on the *left derived functors* $\mathrm{L}_i F$ of a covariant functor F, and on the *right derived functors* $\mathrm{R}^i F$ of a contravariant functor F. This is because we want to avoid the use of *injective resolutions*.

Of major importance will be the derived functors

$$\operatorname{Tor}_i^A(M, -) := \mathrm{L}_i G, \quad G := M \otimes_A (-)$$

and

$$\operatorname{Ext}_A^i(-, N) := \mathrm{R}^i F, \quad F := \operatorname{Hom}_A(-, N)$$

for modules $M \in \operatorname{Mod} A^{\mathrm{op}}$ and $N \in \operatorname{Mod} A$. These are in fact bilinear bifunctors

$$\operatorname{Tor}_i^A(-, -) : (\operatorname{Mod} A^{\mathrm{op}}) \times \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}$$

and

$$\operatorname{Ext}_A^i(-, -) : (\operatorname{Mod} A)^{\mathrm{op}} \times \operatorname{Mod} A \to \operatorname{Mod} \mathbb{K}.$$

Here are a few typical theorems:

Theorem 13.1. *The following conditions are equivalent for $M \in \operatorname{Mod} A$.*

(i) *M is projective.*
(ii) *$\operatorname{Ext}_A^i(M, N) = 0$ for all $i \geq 1$ and $N \in \operatorname{Mod} A$.*
(iii) *$\operatorname{Ext}_A^1(M, N) = 0$ for all $N \in \operatorname{Mod} A$.*

Theorem 13.2. *The following conditions are equivalent for $M \in \operatorname{Mod} A$.*

(i) *M is flat.*
(ii) *$\operatorname{Tor}_i^A(N, M) = 0$ for all $i \geq 1$ and $N \in \operatorname{Mod} A^{\mathrm{op}}$.*
(iii) *$\operatorname{Tor}_i^A(N, M) = 0$ for all $N \in \operatorname{Mod} A^{\mathrm{op}}$.*

Theorem 13.3. *Given a linear functor*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

and a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

in $\operatorname{Mod} A$*, there is an exact sequence*

$$\begin{aligned} \cdots &\xrightarrow{\delta_1} \mathrm{L}_1 F(M') \xrightarrow{\mathrm{L}_1 F(\phi)} \mathrm{L}_1 F(M) \xrightarrow{\mathrm{L}_1 F(\psi)} \mathrm{L}_1 F(M'') \\ &\xrightarrow{\delta_0} \mathrm{L}_0 F(M') \xrightarrow{\mathrm{L}_0 F(\phi)} \mathrm{L}_0 F(M) \xrightarrow{\mathrm{L}_0 F(\psi)} \mathrm{L}_0 F(M'') \to 0 \end{aligned}$$

in $\operatorname{Mod} B$*.*

The homomorphisms δ_i are called *connecting homomorphisms.*

Theorem 13.4. *Given a contravariant linear functor*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

and a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

in $\operatorname{Mod} A$*, there is an exact sequence*

$$\begin{aligned} 0 \to \mathrm{R}^0 F(M') &\xrightarrow{\mathrm{R}^0 F(\phi)} \mathrm{R}^0 F(M) \xrightarrow{\mathrm{R}^0 F(\psi)} \mathrm{R}^0 F(M'') \\ &\xrightarrow{\delta^0} \mathrm{R}^1 F(M') \xrightarrow{\mathrm{R}^0 F(\phi)} \mathrm{R}^0 F(M) \xrightarrow{\mathrm{R}^0 F(\psi)} \mathrm{R}^0 F(M'') \xrightarrow{\delta^1} \cdots \end{aligned}$$

in $\operatorname{Mod} B$*.*

As an application of Theorems 13.3 and 13.4 we will prove the following result in commutative ring theory.

Theorem 13.5. *Let A be a noetherian commutative ring, and let M be a finitely generated A-module. The following two conditions are equivalent*:

(i) *M is projective.*
(ii) *M is flat.*

Of course only the implication (ii) $\Rightarrow$ (i) is new to us. The key additional result is that Ext respects localization:

Theorem 13.6. *Let A be a noetherian commutative ring, and let M and N be A-modules, and let $S \subseteq A$ be a multiplicatively closed set. Assume that M is finitely generated. Then for every i there are isomorphisms*

$$\operatorname{Ext}^i_A(M, N) \otimes_A A_S \cong \operatorname{Ext}^i_A(M, N \otimes_A A_S) \cong \operatorname{Ext}^i_{A_S}(M_S, N_S).$$

Here A_S and $M_S \cong M \otimes_A A_S$ are the localizations.

14. Complexes of Modules

We fix a ring A.

Definition 14.1. A *complex of A-modules* is a diagram

$$M = \left(\cdots \to M^{-1} \xrightarrow{\mathrm{d}^{-1}_M} M^0 \xrightarrow{\mathrm{d}^0_M} M^1 \xrightarrow{\mathrm{d}^1_M} M^2 \to \cdots \right)$$

in $\operatorname{Mod} A$ such that

$$\mathrm{d}^i_M \circ \mathrm{d}^{i-1}_M = 0$$

for all i.

The collection $\mathrm{d}_M = \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}$ is called the *differential* of M, or the *coboundary operator*.

The module of *i-cocycles* of M is

$$\mathrm{Z}^i(M) := \mathrm{Ker}(\mathrm{d}^i) \subseteq M^i.$$

The module of *i-coboundaries* of M is

$$\mathrm{B}^i(M) := \mathrm{Im}(\mathrm{d}^{i-1}) \subseteq M^i.$$

The condition $\mathrm{d}_i \circ \mathrm{d}_{i-1} = 0$ implies that

$$\mathrm{B}^i(M) \subseteq \mathrm{Z}^i(M).$$

Definition 14.2. Let M be a complex of A-modules, with the notation of Definition 14.1. The *i-th cohomology* of M is the A-module

$$\mathrm{H}^i(M) := \mathrm{Z}^i(M) \,/\, \mathrm{B}^i(M).$$

Example 14.3. The complex M is an exact sequence iff $\mathrm{H}^i(M) = 0$ for all i. Such a complex is called *acyclic*.

Definition 14.4. Let M and N be complexes of A-modules. A *homomorphism of complexes* $\phi : M \to N$ is a collection $\{\phi^i\}_{i \in \mathbb{Z}}$ of homomorphisms

$$\phi^i : M^i \to N^i$$

such that

$$\mathrm{d}_N^i \circ \phi^i = \phi^{i+1} \circ \mathrm{d}_M^i.$$

A homomorphism of complexes $\phi : M \to N$ can be viewed as a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{-1} \arrow[r, "\mathrm{d}_M^{-1}"] \arrow[d, "\phi^{-1}"'] & M^0 \arrow[r, "\mathrm{d}_M^0"] \arrow[d, "\phi^0"'] & M^1 \arrow[r] \arrow[d, "\phi^1"'] & \cdots \\
\cdots \arrow[r] & N^{-1} \arrow[r, "\mathrm{d}_N^{-1}"] & N^0 \arrow[r, "\mathrm{d}_N^0"] & N^1 \arrow[r] & \cdots
\end{tikzcd}$$

The category of complexes of A-modules is denoted by $\mathsf{C}(\mathsf{Mod}\, A)$. It is a $\mathbb{K}$-linear category.

Exercise 14.5. Show that the i-th cohomology is a linear functor

$$\mathrm{H}^i : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{Mod}\, A.$$

There are operations on complexes.

Definition 14.6. Let M and N be complexes of A-modules. We define the complex of $\mathbb{K}$-modules

$$\mathrm{Hom}_A(M, N)$$

as follows. In degree i we take the $\mathbb{K}$-module

$$\mathrm{Hom}_A(M, N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_A(M^j, N^{j+i}).$$

The differential

$$\mathrm{d}_{\mathrm{Hom}}^i : \mathrm{Hom}_A(M, N)^i \to \mathrm{Hom}_A(M, N)^{i+1}$$

is

$$\mathrm{d}_{\mathrm{Hom}}^i(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M.$$

Exercise 14.7. Show that

$$\operatorname{Hom}_{\mathsf{C}(\operatorname{Mod} A)}(M, N) = \mathrm{Z}^0\big(\operatorname{Hom}_A(M, N)\big).$$

Definition 14.8. Let $M \in \mathsf{C}(\operatorname{Mod} A^{\mathrm{op}})$ and $N \in \mathsf{C}(\operatorname{Mod} A)$. We define the complex of $\mathbb{K}$-modules

$$M \otimes_A N$$

as follows. In degree i we take the $\mathbb{K}$-module

$$(M \otimes_A N)^i := \bigoplus_{j \in \mathbb{Z}} (M^j \otimes_A N^{i-j}).$$

The differential

$$\mathrm{d}^i_{\otimes} : (M \otimes_A N)^i \to (M \otimes_A N)^{i+1}$$

is

$$\mathrm{d}^i_{\otimes}(m \otimes n) := \mathrm{d}_M(m) \otimes n + (-1)^j \cdot m \otimes \mathrm{d}_N(n)$$

for $m \in M^j$ and $n \in N^{i-j}$.

Exercise 14.9. Verify that $\mathrm{d}^{i+1}_{\mathrm{Hom}} \circ \mathrm{d}^i_{\mathrm{Hom}} = 0$ and $\mathrm{d}^{i+1}_{\otimes} \circ \mathrm{d}^i_{\otimes} = 0$.

When working with bimodules, the complexes become bimodules too. Thus, if $M \in \mathsf{C}(\operatorname{Mod} A \otimes B^{\mathrm{op}})$ and $N \in \mathsf{C}(\operatorname{Mod} A \otimes C^{\mathrm{op}})$, then

$$\operatorname{Hom}_A(M, N) \in \mathsf{C}(\operatorname{Mod} B \otimes C^{\mathrm{op}}).$$

If $M \in \mathsf{C}(\operatorname{Mod} A \otimes B^{\mathrm{op}})$ and $N \in \mathsf{C}(\operatorname{Mod} B \otimes C^{\mathrm{op}})$, then

$$M \otimes_B N \in \mathsf{C}(\operatorname{Mod} A \otimes C^{\mathrm{op}}).$$

Definition 14.10. Let

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a linear functor. For a complex $M \in \mathsf{C}(\operatorname{Mod} A)$ define

$$F(M)^i := F(M^i)$$

and

$$\mathrm{d}^i_{F(M)} := F(\mathrm{d}^i_M).$$

For a homomorphism

$$\phi = \{\phi^i\}_{i \in \mathbb{Z}} : M \to N$$

in $\mathsf{C}(\operatorname{Mod} A)$ define

$$F(\phi) := \{F(\phi^i)\}_{i \in \mathbb{Z}}.$$

In this way we obtain a functor

$$F : \mathsf{C}(\operatorname{Mod} A) \to \mathsf{C}(\operatorname{Mod} B).$$

Of course for contravariant functors there are complications.

Definition 14.11. Let

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a contravariant linear functor. For a complex $M \in \mathsf{C}(\operatorname{Mod} A)$ define

$$F(M)^i := F(M^{-i})$$

and

$$\mathrm{d}^i_{F(M)} := (-1)^{i+1} \cdot F(\mathrm{d}^{-i-1}_M) : F(M)^i = F(M^{-i}) \to F(M)^{i+1} = F(M^{-i-1}).$$

For a homomorphism

$$\phi = \{\phi^i\}_{i \in \mathbb{Z}} : M \to N$$

in $\mathsf{C}(\operatorname{Mod} A)$ define

$$F(\phi) : F(N) \to F(M)$$

by

$$F(\phi) = \{F(\phi)^i\}_{i \in \mathbb{Z}},$$

where

$$F(\phi)^i := F(\phi^{-i}) : F(N)^i = F(N^{-i}) \to F(M)^i = F(M^{-i}).$$

In this way we obtain a contravariant linear functor

$$F : \mathsf{C}(\operatorname{Mod} A) \to \mathsf{C}(\operatorname{Mod} B).$$

Exercise 14.12. Verify that Definition 14.11 is consistent, namely that the functor F on complexes defined there is really a contravariant linear functor.

15. Homotopies and Homotopy Equivalences

Definition 15.1. Let $M, N \in \mathsf{C}(\operatorname{Mod} A)$, and let

$$\phi_0, \phi_1 : M \to N$$

be homomorphisms in $\mathsf{C}(\operatorname{Mod} A)$. A *homotopy* $\gamma : \phi_0 \Rightarrow \phi_1$ is a collection $\gamma = \{\gamma^i\}_{i \in \mathbb{Z}}$ of homomorphisms

$$\gamma^i : M^i \to N^{i-1}$$

such that

$$\phi_1^i - \phi_0^i = \mathrm{d}_N^{i-1} \circ \gamma^i + \gamma^{i+1} \circ \mathrm{d}_M^i.$$

for all i.

Proposition 15.2.

(1) *Homotopy is an equivalence relation on the the* $\mathbb{K}$*-module* $\operatorname{Hom}_{\mathsf{C}(\operatorname{Mod} A)}(M, N)$.

(2) *Given homomorphisms* $L \xrightarrow{\theta} M \xrightarrow{\phi_i} N \xrightarrow{\psi} P$ *in* $\mathsf{C}(\operatorname{Mod} A)$*, and a homotopy* $\phi_0 \Rightarrow \phi_1$*, there exists a homotopy*

$$\psi \circ \phi_0 \circ \theta \Rightarrow \psi \circ \phi_1 \circ \theta.$$

(3) *There is a category* $\mathsf{K}(\operatorname{Mod} A)$*, with the same objects as* $\mathsf{C}(\operatorname{Mod} A)$*, whose morphisms are the homotopy classes of morphisms in* $\mathsf{C}(\operatorname{Mod} A)$*, and whose composition is induced from the composition in* $\mathsf{C}(\operatorname{Mod} A)$.

Exercise 15.3. Prove this proposition.

The category $\mathsf{K}(\operatorname{Mod} A)$ is called the *homotopy category of complexes.* There is a full linear functor

$$\mathrm{P} : \mathsf{C}(\operatorname{Mod} A) \to \mathsf{K}(\operatorname{Mod} A) \tag{15.4}$$

that is the identity on objects. Note that

$$\operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} A)}(M, N) = \mathrm{H}^0\big(\operatorname{Hom}_A(M, N)\big).$$

Proposition 15.5. *Let*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a linear functor (covariant or contravariant).

(1) *Let*
$$\phi_0, \phi_1 : M \to N$$
be homomorphisms in $\mathsf{C}(\mathsf{Mod}\, A)$*, and assume they are homotopic. Then the homomorphisms* $F(\phi_0)$ *and* $F(\phi_1)$ *in* $\mathsf{Mod}\, B$ *are homotopic.*

(2) *Given homomorphisms* $L \xrightarrow{\theta} M \xrightarrow{\phi_i} N \xrightarrow{\psi} P$ *in* $\mathsf{C}(\mathsf{Mod}\, A)$*, and a homotopy* $\phi_0 \Rightarrow \phi_1$*, there exists a homotopy*
$$\psi \circ \phi_0 \circ \theta \Rightarrow \psi \circ \phi_1 \circ \theta .$$

(3) *There is a functor (covariant or contravariant)*
$$F : \mathsf{K}(\mathsf{Mod}\, A) \to \mathsf{K}(\mathsf{Mod}\, B)$$
such that the diagram
$$\begin{tikzcd} \mathsf{C}(\mathsf{Mod}\, A) \arrow[r, "F"] \arrow[d, "\mathrm{P}"'] & \mathsf{C}(\mathsf{Mod}\, A) \arrow[d, "\mathrm{P}"'] \\ \mathsf{K}(\mathsf{Mod}\, A) \arrow[r, "F"] & \mathsf{K}(\mathsf{Mod}\, B) \end{tikzcd}$$
is commutative.

Exercise 15.6. Prove this proposition.

Proposition 15.7.

(1) *Let*
$$\phi_0, \phi_1 : M \to N$$
be homomorphisms in $\mathsf{C}(\mathsf{Mod}\, A)$*, and assume they are homotopic. Then for every* i *there is equality*
$$\mathrm{H}^i(\phi_0) = \mathrm{H}^i(\phi_1) : \mathrm{H}^i(M) \to \mathrm{H}^i(N).$$

(2) *The the* i*-th cohomology is a linear functor*
$$\mathrm{H}^i : \mathsf{K}(\mathsf{Mod}\, A) \to \mathsf{Mod}\, A.$$

Exercise 15.8. Prove this proposition.

Definition 15.9. Let $M, N \in \mathsf{C}(\mathsf{Mod}\, A)$, and let $\phi : M \to N$ be a homomorphism in $\mathsf{C}(\mathsf{Mod}\, A)$. We call ϕ a *homotopy equivalence* if there is a homomorphism $\psi : N \to M$ in $\mathsf{C}(\mathsf{Mod}\, A)$, and homotopies
$$\psi \circ \phi \Rightarrow \mathrm{id}_M$$
and
$$\phi \circ \psi \Rightarrow \mathrm{id}_N .$$

Proposition 15.10. *Let* $\phi : M \to N$ *be a homomorphism in* $\mathsf{C}(\mathsf{Mod}\, A)$*. The following are equivalent:*

(i) ϕ *is a homotopy equivalence.*

(ii) $\mathrm{P}(\phi)$ *is an isomorphism in* $\mathsf{K}(\mathsf{Mod}\, A)$*.*

Exercise 15.11. Prove this proposition.

Lecture 11, 6 June 2018.

16. Projective Resolutions

As before, A and B are central $\mathbb{K}$-rings.

Definition 16.1. Let $\phi : M \to N$ be homomorphisms in $\mathsf{C}(\operatorname{Mod} A)$. The homomorphism ϕ is called a *quasi-isomorphism* if for every i the homomorphism

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$$

is an isomorphism.

Proposition 16.2. *If $\phi : M \to N$ is a homotopy equivalence, then it is a quasi-isomorphism.*

Exercise 16.3. (1) Prove Proposition 16.2.
(2) Find a quasi-isomorphism $\phi : M \to N$ that is not a homotopy equivalence.

Definition 16.4. Let M be an A-module. A *projective resolution* of M is an exact sequence

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0$$

in $\operatorname{Mod} A$, in which the modules P^i are all projective.

Proposition 16.5. *Every A-module M has a projective resolution.*

Proof. The only fact we need is that every A-module N admits a surjection $Q \twoheadrightarrow N$ from a projective A-module. This is easy: choose a collection of generators $\{n_x\}_{x \in X}$ for N; then we get a surjection

$$Q := \mathrm{F}_{\mathrm{fin}}(X, A) \twoheadrightarrow M,$$

and the free module Q is projective.

Given M, we start with a surjection $\rho : P^0 \twoheadrightarrow M$ from some projective A-module P^0. Next we choose a surjection

$$\mathrm{d}_P^{-1} : P^{-1} \twoheadrightarrow \operatorname{Ker}\big(\rho : P^0 \to M\big)$$

from some projective A-module P^{-1}. So the sequence

$$P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0$$

is exact.

We proceed recursively: given $i \geq 1$ and a partial projective resolution

$$P^{-i} \xrightarrow{\mathrm{d}_P^{-i}} \cdots \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0$$

we choose a surjection

$$\mathrm{d}_P^{-i-1} : P^{-i-1} \twoheadrightarrow \operatorname{Ker}\big(\mathrm{d}_P^{-i} : P^{-i} \to P^{-i+1}\big)$$

from some projective A-module P^{-i-1}. This gives a longer partial projective resolution

$$P^{-i-1} \xrightarrow{\mathrm{d}_P^{-i-1}} P^{-i} \xrightarrow{\mathrm{d}_P^{-i}} \cdots \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0.$$

□

Exercise 16.6. Assume A is a left noetherian ring and M is a finitely generated A-module. Prove that M has a projective resolution $\rho : P \to M$ such that every P^i is a finitely generated free A-module.

Given a projective resolution

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\rho} M \to 0$$

of a module M, we can view it slightly differently. Consider the complex

$$P := \big(\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots\big).$$

The module M can also be viewed as a complex concentrated in degree 0. We get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r, "\mathrm{d}_P^{-1}"] \arrow[d, "0"'] & P^0 \arrow[r, "\mathrm{d}_P^{0}"] \arrow[d, "\rho"'] & 0 \arrow[r] \arrow[d, "0"'] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r, "0"] & M \arrow[r, "0"] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in $\operatorname{Mod} A$, and this can be viewed as a homomorphism

$$\rho : P \to M \tag{16.7}$$

in $\mathsf{C}(\operatorname{Mod} A)$.

Exercise 16.8. Prove that $\rho : P \to M$ is a quasi-isomorphism of complexes.

Thus a projective resolution of M is a quasi-isomorphism of complexes $\rho : P \to M$ from a nonpositive complex of projectives P.

Definition 16.9. Let $\phi : M \to N$ be a homomorphism in $\operatorname{Mod} A$, let $\rho : P \to M$ be projective resolution of M, and let $\sigma : Q \to N$ be projective resolution of N. A *lifting* of ϕ to these resolutions is a homomorphism

$$\tilde{\phi} : P \to Q$$

in $\mathsf{C}(\operatorname{Mod} A)$, such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-1} \arrow[r, "\mathrm{d}_P^{-1}"] \arrow[d, "\tilde{\phi}^{-1}"'] & P^0 \arrow[r, "\rho"] \arrow[d, "\tilde{\phi}^{0}"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \\
\cdots \arrow[r] & Q^{-1} \arrow[r, "\mathrm{d}_Q^{-1}"] & Q^0 \arrow[r, "\sigma"] & N \arrow[r] & 0
\end{tikzcd}$$

is commutative.

In other words, if we view the resolutions $\rho : P \to M$ and $\sigma : Q \to N$, and the homomorphism $\phi : M \to N$, as homomorphisms in $\mathsf{C}(\operatorname{Mod} A)$, then $\tilde{\phi}$ makes the diagram

$$\begin{tikzcd}
P \arrow[r, "\rho"] \arrow[d, "\tilde{\phi}"'] & M \arrow[d, "\phi"] \\
Q \arrow[r, "\sigma"] & N
\end{tikzcd}$$

in $\mathsf{C}(\operatorname{Mod} A)$ commutative.

Theorem 16.10. *Let $\phi : M \to N$ be a homomorphism in $\operatorname{Mod} A$, let $\rho : P \to M$ be projective resolution of M, and let $\sigma : Q \to N$ be projective resolution of N. Then there exists a lifting $\tilde{\phi} : P \to Q$ of ϕ.*

Proof. The homomorphisms $\tilde{\phi}^{-i}$ are constructed recursively for $i \in \mathbb{N}$. We start with $i = 0$. We have this solid commutative diagram

$$\begin{tikzcd} P^0 \arrow[r, "\rho"] \arrow[d, dashed, "\tilde{\phi}^0"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \\ Q^0 \arrow[r, "\sigma"] & N \arrow[r] & 0 \end{tikzcd} \tag{16.11}$$

with exact rows. Because P^0 is a projective module and $\sigma : Q^0 \to M$ is a surjective homomorphism, there exists a homomorphism $\tilde{\phi}^0 : P^0 \to Q^0$ that lifts $\phi \circ \rho$. This gives the dashed arrow above.

Now we construct $\tilde{\phi}^{-1}$. Define

$$M^{-1} := \operatorname{Ker}(\rho) = \operatorname{Im}(\mathrm{d}_P^{-1})$$

and

$$N^{-1} := \operatorname{Ker}(\sigma) = \operatorname{Im}(\mathrm{d}_Q^{-1}). \tag{16.12}$$

Then diagram (16.11) can be enlarged to the solid commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & M^{-1} \arrow[r] \arrow[d, dashed, "\phi^{-1}"'] & P^0 \arrow[r, "\rho"] \arrow[d, "\tilde{\phi}^0"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \\ 0 \arrow[r] & N^{-1} \arrow[r] & Q^0 \arrow[r, "\sigma"] & N \arrow[r] & 0 \end{tikzcd} \tag{16.13}$$

Since

$$(\sigma \circ \tilde{\phi}^0)(M^{-1}) = (\phi \circ \rho)(M^{-1}) = 0,$$

we get

$$\tilde{\phi}^0(M^{-1}) \subseteq N^{-1},$$

so there is a homomorphism ϕ^{-1} on the dashed arrow that makes the whole diagram commutative.

But there is another commutative diagram with exact rows

$$\begin{tikzcd} P^{-1} \arrow[r, "\mathrm{d}_P^{-1}"] \arrow[d, dashed, "\tilde{\phi}^{-1}"'] & M^{-1} \arrow[r] \arrow[d, "\phi^{-1}"'] & 0 \\ Q^0 \arrow[r, "\mathrm{d}_Q^{-1}"] & N^{-1} \arrow[r] & 0 \end{tikzcd} \tag{16.14}$$

It looks just like diagram (16.11); and for the same reason we can find a homomorphism $\tilde{\phi}^{-1}$ on the dashed arrow that makes the whole diagram commutative.

And so on. □

Exercise 16.15. Write the full induction argument for the proof above.

Theorem 16.16. *Let $\phi : M \to N$ be a homomorphism in $\operatorname{Mod} A$, let $\rho : P \to M$ be projective resolution of M, and let $\sigma : Q \to N$ be projective resolution of N. Suppose*

$$\tilde{\phi}_0, \tilde{\phi}_1 : P \to Q$$

are two liftings of ϕ. Then there is a homotopy $\tilde{\phi}_0 \Rightarrow \tilde{\phi}_1$.

Proof. Step 1. Define

$$\tilde{\phi} := \tilde{\phi}_1 - \tilde{\phi}_0 : P \to Q.$$

This homomorphism lifts

$$0 = \phi - \phi : M \to N.$$

We will find a homotopy $\gamma : 0 \Rightarrow \tilde{\phi}$, namely a collection $\gamma = \{\gamma^i\}_{i \in \mathbb{Z}}$ of homomorphisms

$$\gamma^i : P^i \to Q^{i-1}$$

such that equation

$$\tilde{\phi}^i = (\mathrm{d}_Q^{i-1} \circ \gamma^i) + (\gamma^{i+1} \circ \mathrm{d}_P^i) \tag{$\diamond_i$}$$

holds for every i. Then

$$\gamma : \tilde{\phi}_0 \Rightarrow \tilde{\phi}_1$$

is a homotopy.

For $i > 0$ we take the homomorphisms $\gamma^i := 0$. Since $\tilde{\phi}^i = 0$ and $\mathrm{d}_Q^{i-1} = 0$ for all $i > 0$, it follows that equation $(\diamond_i)$ holds for all $i > 0$.

For $i \leq 0$ we construct the homomorphisms γ^i recursively, starting from $i = 0$ and going down.

Step 2. Here $i = 0$. We have this solid commutative diagram with exact rows:

$$\begin{tikzcd}
 & P^0 \arrow[r, "\rho"] \arrow[d, "\tilde{\phi}^0"] \arrow[dl, dashed, "\gamma^0"'] & M \arrow[r] \arrow[d, "0"] & 0 \\
Q^{-1} \arrow[r, "\mathrm{d}_Q^{-1}"] & Q^0 \arrow[r, "\sigma"] & N \arrow[r] & 0
\end{tikzcd} \tag{16.17}$$

Let

$$N^{-1} := \operatorname{Ker}(\sigma) = \operatorname{Im}(\mathrm{d}_Q^{-1}).$$

Because

$$\sigma \circ \tilde{\phi}^0 = 0 \circ \rho = 0,$$

we see that $\tilde{\phi}^0(P^0) \subseteq N^{-1}$, and we get another solid commutative diagram with exact rows:

$$\begin{tikzcd}
 & P^0 \arrow[d, "\tilde{\phi}^0"] \arrow[dl, dashed, "\gamma^0"'] & \\
Q^{-1} \arrow[r, "\mathrm{d}_Q^{-1}"] & N^{-1} \arrow[r] & 0
\end{tikzcd} \tag{16.18}$$

Now P^0 is a projective module, so we can find a homomorphism γ^0 on the dashed arrow making the whole diagram commutative.

But $\mathrm{d}_P^0 = 0$, so

$$\tilde{\phi}^0 = \mathrm{d}_Q^{i-1} \circ \gamma^0 = \mathrm{d}_Q^{i-1} \circ \gamma^0 + \gamma^1 \circ \mathrm{d}_P^0.$$

This is equations $(\diamond_0)$.

Step 3. Now suppose that $i \leq -1$, and we have homomorphisms γ^{i+1} and γ^{i+2} s.t. equation $(\diamond_{i+1})$ holds, i.e.

$$\tilde{\phi}^{i+1} = \mathrm{d}_Q^i \circ \gamma^{i+1} + \gamma^{i+2} \circ \mathrm{d}_P^{i+1}. \tag{†}$$

Then define

$$\tilde{\psi}^i := \tilde{\phi}^i - (\gamma^{i+1} \circ \mathrm{d}_P^i) : P^i \to Q^i.$$

Since $\mathrm{d}_P^{i+1} \circ \mathrm{d}_P^i = 0$ and

$$\mathrm{d}_Q^i \circ \tilde{\phi}^i = \tilde{\phi}^{i+1} \circ \mathrm{d}_P^i \tag{‡}$$

we have

$$\begin{aligned}
\mathrm{d}_Q^i \circ \tilde{\psi}^i &= (\mathrm{d}_Q^i \circ \tilde{\psi}^i) - (\gamma^{i+2} \circ \mathrm{d}_P^{i+1} \circ \mathrm{d}_P^i) \\
&= (\mathrm{d}_Q^i \circ \tilde{\phi}^i) - (\mathrm{d}_Q^i \circ \gamma^{i+1} \circ \mathrm{d}_P^i) - (\gamma^{i+2} \circ \mathrm{d}_P^{i+1} \circ \mathrm{d}_P^i) \\
&=^{\ddagger} (\tilde{\phi}^{i+1} \circ \mathrm{d}_P^i) - (\mathrm{d}_Q^i \circ \gamma^{i+1} \circ \mathrm{d}_P^i) - (\gamma^{i+2} \circ \mathrm{d}_P^{i+1} \circ \mathrm{d}_P^i) \\
&=^{\dagger} \big(\tilde{\phi}^{i+1} - (\mathrm{d}_Q^i \circ \gamma^{i+1}) - (\gamma^{i+2} \circ \mathrm{d}_P^{i+1})\big) \circ \mathrm{d}_P^i = 0.
\end{aligned}$$

Letting

$$N^{i-1} := \mathrm{Ker}(\mathrm{d}_Q^i) = \mathrm{Im}(\mathrm{d}_Q^{i-1})$$

we see that $\tilde{\psi}^i(P^i) \subseteq N^{i-1}$, and obtain the solid commutative diagram with exact row:

(16.19)
$$\begin{tikzcd}
 & P^i \arrow[dl, dashed, "\gamma^i"'] \arrow[d, "\tilde{\psi}^i"] & \\
Q^{-1} \arrow[r, "\mathrm{d}_Q^{i-1}"] & N^{i-1} \arrow[r] & 0
\end{tikzcd}$$

Because P^i is projective we can find a homomorphism γ^i on the dashed arrow. But then

$$\tilde{\phi}^i = \tilde{\psi}^i + (\gamma^{i+1} \circ \mathrm{d}_P^i) = (\mathrm{d}_Q^{i-1} \circ \gamma^i) + (\gamma^{i+1} \circ \mathrm{d}_P^i),$$

which is equation $(\diamond_i)$. □

Definition 16.20. A *system of projective resolutions* in $\mathsf{Mod}\, A$ is a collection

$$\boldsymbol{P} = \big\{(P_M, \rho_M)\big\}_{M \in \mathsf{Mod}\, A}$$

of projective resolutions $\rho_M : P_M \to M$ in $\mathsf{Mod}\, A$.

By Proposition 16.5 there exists a system of projective resolutions. Of course there are a lot of them! The next exercises shows that they are unique up to homotopy. (I don't think we will need this fact in our course – but it is good to know for future reference!)

Exercise 16.21. Suppose $\rho : P \to M$ and $\sigma : Q \to M$ are both projective resolutions of M. Prove that P and Q are homotopy equivalent. More precisely, if $\tilde{\phi} : P \to Q$ is a lifting of id_M to these resolutions, then $\tilde{\phi} : P \to Q$ is a homotopy equivalence. (Hint: use Theorems 16.10 and 16.16.)

Exercise 16.22. (Hard) Let us choose a system of projective resolutions $\boldsymbol{P} = \{P_M\}$ in $\operatorname{Mod} A$. Show that there is a functor

$$\operatorname{PrRes} : \operatorname{Mod} A \to \mathsf{K}(\operatorname{Mod} A),$$

such that on objects

$$\operatorname{PrRes}(M) = P_M,$$

and on a morphism $\phi : M \to N$ in $\operatorname{Mod} A$ it is the morphism

$$\operatorname{PrRes}(\phi) = \mathrm{P}(\tilde{\phi})$$

in $\mathsf{K}(\operatorname{Mod} A)$, where $\tilde{\phi} : P_M \to P_N$ is a lifting of ϕ to these resolutions, and

$$\mathrm{P} : \operatorname{Mod} A \to \mathsf{K}(\operatorname{Mod} A)$$

is the functor from (15.4). (Hint: use Theorems 16.10 and 16.16.)

Lecture 12, 13 June 2018.

17. Left Derived Functors

In this section we talk about the i-th left derived functor $\mathrm{L}_i F$ of a $\mathbb{K}$-linear functor F. Here i is a fixed natural number. This idea was developed by several people in the 1950's, culminating in A. Grothendieck's seminal paper [Gro] from 1957.

Very soon afterwards Grothendieck, with his student at the time J.L. Verdier, generalized this "classical" homological algebra to the much more sophisticated (but hard!) theory of *derived categories*. This first appeared in print in the book [RD]. I recommend reading about it in the Introduction of the new book [Ye1].

Recall from Definition 16.20 that a system of projective resolutions in $\operatorname{Mod} A$ is a collection

$$\boldsymbol{P} = \big\{(P_M, \rho_M)\big\}_{M \in \operatorname{Ob}(\operatorname{Mod} A)}$$

of projective resolutions $\rho_M : P_M \to M$.

By Theorem 16.10, given a homomorphism $\phi : M \to N$ in $\operatorname{Mod} A$, there is a lifting $\tilde{\phi} : P_M \to P_N$ of ϕ, in the sense of Definition 16.9.

Theorem 17.1. *Let*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a $\mathbb{K}$-linear functor, and let $\boldsymbol{P} = \{P_M\}$ be a system of projective resolutions in $\operatorname{Mod} A$. There is a unique $\mathbb{K}$-linear functor

$$\mathrm{L}_i^{\boldsymbol{P}}(F) : \operatorname{Mod} A \to \operatorname{Mod} B$$

with these two properties:

(i) *On objects it is*

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(M) = \mathrm{H}^{-i}(F(P_M)) \in \operatorname{Mod} B.$$

(ii) *On a homomorphism $\phi : M \to N$ in $\operatorname{Mod} A$ it is the homomorphism*

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi) = \mathrm{H}^{-i}(F(\tilde{\phi}))$$

in $\operatorname{Mod} B$, where $\tilde{\phi} : P_M \to P_N$ is some lifting of ϕ to these resolutions.

Proof. Uniqueness is clear: we specified the value of $\mathrm{L}_i^{\boldsymbol{P}}(F)$ on objects and morphisms.

For existence, we choose a lifting $\tilde{\phi} : P_M \to P_N$ of each homomorphism $\phi : M \to N$ in $\operatorname{Mod} A$. Then we define

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(M) := \mathrm{H}^{-i}(F(P_M))$$

and

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi) = \mathrm{H}^{-i}(F(\tilde{\phi})).$$

We must prove this is a functor, and that property (ii) holds.

Take a homomorphism $\phi : M \to N$ in $\operatorname{Mod} A$, and suppose $\tilde{\phi}' : M \to N$ is another lifting of ϕ, besides the lifting $\tilde{\phi} : M \to N$ that we already chose. We obtain these equalities in $\operatorname{Mod} B$:

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi) =^1 \mathrm{H}^{-i}(F(\tilde{\phi})) =^2 \mathrm{H}^{-i}(F(\tilde{\phi}')). \tag{17.2}$$

Here are the explanations:

- The equality $=^1$: this is the definition of $\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi)$.
- The equality $=^2$: By Theorem 16.16 there is a homotopy $\tilde{\phi} \Rightarrow \tilde{\phi}'$. So by Proposition 15.5(1) there is a homotopy $F(\tilde{\phi}) \Rightarrow F(\tilde{\phi}')$. The equality follows from Proposition 15.7(1).

Thus property (ii) is proven.

Now we prove that $\mathrm{L}_i^{\boldsymbol{P}}(F)$ respects identity automorphisms. Consider the identity automorphism $\mathrm{id}_M : M \to M$ of a module M. Let $\tilde{\phi} : P_M \to P_M$ be the chosen lifting of id_M. We obtain these equalities in $\operatorname{Mod} B$:

$$\begin{aligned} \mathrm{L}_i^{\boldsymbol{P}}(F)(\mathrm{id}_M) &=^{\mathrm{a}} \mathrm{H}^{-i}(F(\tilde{\phi})) =^{\mathrm{b}} \mathrm{H}^{-i}(F(\mathrm{id}_{P_M})) \\ &=^{\mathrm{c}} \mathrm{H}^{-i}(\mathrm{id}_{F(P_M)}) =^{\mathrm{d}} \mathrm{id}_{\mathrm{H}^{-i}(F(P_M))} = \mathrm{id}_{\mathrm{L}_i^{\boldsymbol{P}}(F)(M)} . \end{aligned}$$

Here are the explanations:

- The equality $=^{\mathrm{a}}$: this is the definition of $\mathrm{L}_i^{\boldsymbol{P}}(F)(\mathrm{id}_M)$.
- The equality $=^{\mathrm{b}}$: because the identity automorphism $\mathrm{id}_{P_M} : P_M \to P_M$ of the complex P_M is also a lifting of id_M, we can use property (ii).
- The equality $=^{\mathrm{c}}$: because F is a functor.
- The equality $=^{\mathrm{d}}$: because H^{-i} is a functor.

Next we prove that $\mathrm{L}_i^{\boldsymbol{P}}(F)$ respects compositions. Given homomorphisms $\phi_k : M_k \to M_{k+1}$ in $\operatorname{Mod} A$, for $k \in \{0, 1\}$, let $\tilde{\phi}_k : P_{M_k} \to P_{M_{k+1}}$ be their chosen liftings. We have these equalities in $\operatorname{Mod} B$:

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi_1 \circ \phi_0) =^{\dagger} \mathrm{H}^{-i}(\tilde{\phi}_1 \circ \tilde{\phi}_0) =^{\ddagger} \mathrm{H}^{-i}(\tilde{\phi}_0) \circ \mathrm{H}^{-i}(\tilde{\phi}_1) =^{\star} \mathrm{L}_i^{\boldsymbol{P}}(F)(\phi_0) \circ \mathrm{L}_i^{\boldsymbol{P}}(F)(\phi_1).$$

Here are the explanations:

- The equality $=^{\dagger}$: the homomorphism
$$\tilde{\phi}_1 \circ \tilde{\phi}_0 : P_{M_0} \to P_{M_2}$$
is a lifting of $\phi_1 \circ \phi_0$, and we use property (ii).
- The equality $=^{\ddagger}$: H^{-i} is a functor.
- The equality $=^{\star}$: by definition of $\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi_0)$ and $\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi_1)$.

Finally we prove that $\mathrm{L}_i^{\boldsymbol{P}}(F)$ is a $\mathbb{K}$-linear functor. Suppose $\phi, \psi : M \to N$ are homomorphisms in $\operatorname{Mod} A$. Let $\tilde{\phi}, \tilde{\psi} : P_M \to P_N$ be their chosen liftings. Since $\tilde{\phi} + \tilde{\psi} : P_M \to P_N$ is a lifting of $\phi + \psi : M \to N$, by property (ii) we have

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(\phi + \psi) = \mathrm{H}^{-i}(F(\tilde{\phi} + \tilde{\psi})) = \mathrm{H}^{-i}(F(\tilde{\phi})) + \mathrm{H}^{-i}(F(\tilde{\psi})) = \mathrm{L}_i^{\boldsymbol{P}}(F)(\phi) + \mathrm{L}_i^{\boldsymbol{P}}(F)(\psi).$$

Likewise we show that

$$\mathrm{L}_i^{\boldsymbol{P}}(F)(c \cdot \phi) = c \cdot \mathrm{L}_i^{\boldsymbol{P}}(F)(\phi)$$

for $c \in \mathbb{K}$. □

The next lemma was missing from my prior notes, and I added it in class in "real time".

Lemma 17.3. *Let*

$$F, G : \operatorname{Mod} A \to \operatorname{Mod} B$$

be $\mathbb{K}$-linear functors, and let

$$\zeta : F \to G$$

be a morphism of functors. Denote, temporarily, by

$$\tilde{F}, \tilde{G} : \mathsf{C}(\operatorname{Mod} A) \to \mathsf{C}(\operatorname{Mod} A)$$

the extended functors to complexes. For a complex $M \in \mathsf{C}(\operatorname{Mod} A)$ there is a unique morphism

$$\tilde{\zeta}_M : \tilde{F}(M) \to \tilde{G}(M)$$

in $\mathsf{C}(\operatorname{Mod} B)$, that in each degree j is

$$\zeta_{M^j} : F(M^j) \to G(M^j).$$

Moreover,

$$\tilde{\zeta} : \tilde{F} \to \tilde{G}$$

is a morphism of functors.

Proof. We need to check that for every j the diagram

$$\begin{tikzcd} F(M^{j+1}) \arrow[r, "\zeta_{M^{j+1}}"] & G(M^{j+1}) \\ F(M^j) \arrow[u, "F(\mathrm{d}^j_M)"] \arrow[r, "\zeta_{M^j}"] & G(M^j) \arrow[u, "G(\mathrm{d}^j_M)"'] \end{tikzcd}$$

in $\operatorname{Mod} B$ is commutative. This is true because $\zeta : F \to G$ is a morphism of functors, and we apply it to the morphism $\mathrm{d}^j_M : M^j \to M^{j+1}$ in $\operatorname{Mod} A$.

The last assertion (that $\tilde{\zeta}$ is itself a morphism of functors) is clear. □

We now drop the tilde from $\tilde{\zeta}$ and just write ζ.

Lemma 17.4. *Let*

$$F, G : \operatorname{Mod} A \to \operatorname{Mod} B$$

be $\mathbb{K}$-linear functors, let

$$\zeta : F \to G$$

be a morphism of functors, and let $\boldsymbol{P} = \{P_M\}$ and $\boldsymbol{Q} = \{Q_M\}$ be two systems of projective resolutions in $\operatorname{Mod} A$. There is a unique morphism of functors

$$\mathrm{L}^{\boldsymbol{P},\boldsymbol{Q}}_i(\zeta) : \mathrm{L}^{\boldsymbol{P}}_i(F) \to \mathrm{L}^{\boldsymbol{Q}}_i(G)$$

such that for every A-module M, and every lifting $\tilde{\phi} : P_M \to Q_M$ of $\mathrm{id}_M : M \to M$, there is equality

$$\mathrm{L}^{\boldsymbol{P},\boldsymbol{Q}}_i(\zeta)_M = \mathrm{H}^{-i}(F(\tilde{\phi})) \circ \mathrm{H}^{-i}(\zeta_{P_M})$$

of morphisms

$$\mathrm{L}^{\boldsymbol{P}}_i(F)(M) = \mathrm{H}^{-i}(F(P_M)) \to \mathrm{H}^{-i}(G(Q_M)) = \mathrm{L}^{\boldsymbol{Q}}_i(G)(M)$$

in $\operatorname{Mod} B$. The morphism $\zeta_{P_M} : F(P_M) \to G(P_M)$ is the one from Lemma 17.3.

The lemma says that the diagram

$$\begin{tikzcd} \mathrm{H}^{-i}(F(P_M)) \arrow[d, "\mathrm{H}^{-i}(\zeta_{P_M})"'] \arrow[dr, "\mathrm{L}^{\boldsymbol{P},\boldsymbol{Q}}_i(\zeta)_M"] & \\ \mathrm{H}^{-i}(G(P_M)) \arrow[r, "\mathrm{H}^{-i}(F(\tilde{\phi}))"] & \mathrm{H}^{-i}(G(Q_M)) \end{tikzcd}$$

in $\operatorname{Mod} B$ is commutative.

Exercise 17.5. Prove this lemma. (Hint: study the proof of Theorem 17.1.)

Lemma 17.6. *For $k \in \{0, 1, 2\}$ let*

$$F_k : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a $\mathbb{K}$-linear functor, and let $\boldsymbol{P}_k$ be a system of projective resolutions in $\operatorname{Mod} A$. For $k \in \{0, 1\}$ let

$$\zeta_k : F_k \to F_{k+1}$$

be a morphism of functors. Then there is equality

$$\mathrm{L}_i^{\boldsymbol{P}_1, \boldsymbol{P}_2}(\zeta_1) \circ \mathrm{L}_i^{\boldsymbol{P}_0, \boldsymbol{P}_1}(\zeta_0) = \mathrm{L}_i^{\boldsymbol{P}_0, \boldsymbol{P}_2}(\zeta_1 \circ \zeta_0)$$

of morphisms of functors

$$\mathrm{L}_i^{\boldsymbol{P}_0}(F_0) \to \mathrm{L}_i^{\boldsymbol{P}_2}(F_2).$$

Exercise 17.7. Prove this lemma.

Lemma 17.8. *In the situation of Lemma 17.4, if ζ is an isomorphism of functors, then*

$$\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{Q}}(\zeta) : \mathrm{L}_i^{\boldsymbol{P}}(F) \to \mathrm{L}_i^{\boldsymbol{Q}}(G)$$

is an isomorphism of functors.

Proof. Let $\theta : G \to F$ be the inverse of ζ. By direct calculation we have

$$\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{P}}(\mathrm{id}_F) = \mathrm{id}_{\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{P}}(F)}.$$

Since

$$\theta \circ \zeta = \mathrm{id}_F,$$

by Lemma 17.6 we get

$$\mathrm{L}_i^{\boldsymbol{Q}, \boldsymbol{P}}(\theta) \circ \mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{Q}}(\zeta) = \mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{P}}(\theta \circ \zeta) = \mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{P}}(\mathrm{id}_F) = \mathrm{id}_{\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{P}}(F)}.$$

A similar argument gives

$$\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{Q}}(\zeta) \circ \mathrm{L}_i^{\boldsymbol{Q}, \boldsymbol{P}}(\theta) = \mathrm{id}_{\mathrm{L}_i^{\boldsymbol{Q}, \boldsymbol{Q}}(G)}.$$

We see that $\mathrm{L}_i^{\boldsymbol{P}, \boldsymbol{Q}}(\zeta)$ is an isomorphism of functors, with inverse $\mathrm{L}_i^{\boldsymbol{Q}, \boldsymbol{P}}(\theta)$. □

Definition 17.9. Consider a $\mathbb{K}$-linear functor

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

and $i \in \mathbb{N}$. An *i-th left derived functor of F* is a pair

$$\big(\mathrm{L}_i(F), \eta_i^{\boldsymbol{P}}\big),$$

where

$$\mathrm{L}_i(F) : \operatorname{Mod} A \to \operatorname{Mod} B$$

is a $\mathbb{K}$-linear functor, $\boldsymbol{P}$ is a system of projective resolutions in $\operatorname{Mod} A$, and

$$\eta_i^{\boldsymbol{P}} : \mathrm{L}_i(F) \xrightarrow{\simeq} \mathrm{L}_i^{\boldsymbol{P}}(F)$$

is an isomorphism of functors, called a *presentation* of $\mathrm{L}_i(F)$.

Theorem 17.10. *Let*

$$F : \operatorname{Mod} A \to \operatorname{Mod} B$$

be a $\mathbb{K}$-linear functor, and let i be a natural number.

(1) *There exists an i-th left derived functor $\big(\mathrm{L}_i(F), \eta_i^{\boldsymbol{P}}\big)$ of F.*

(2) *If $\big(\mathrm{L}_i^{\dagger}(F), \eta_i^{\boldsymbol{Q}}\big)$ is another i-th left derived functor of F, then there is a unique isomorphism of functors*

$$\eta : \mathrm{L}_i(F) \xrightarrow{\simeq} \mathrm{L}_i^{\dagger}(F)$$

such that the diagram

$$\begin{tikzcd} \mathrm{L}_i(F) \arrow[r, "\eta", "\cong"'] \arrow[d, "\eta_i^{\boldsymbol{P}}"', "\cong"] & \mathrm{L}_i^{\dagger}(F) \arrow[d, "\cong"', "\eta_i^{\boldsymbol{Q}}"] \\ \mathrm{L}_i^{\boldsymbol{P}}(F) \arrow[r, "\mathrm{L}_i^{\boldsymbol{P},\boldsymbol{Q}}(\mathrm{id}_F)", "\cong"'] & \mathrm{L}_i^{\boldsymbol{Q}}(F) \end{tikzcd}$$

of isomorphisms of functors is commutative.

Proof. (1) Choose a system of projective resolutions $\boldsymbol{P} = \{P_M\}$ in $\operatorname{Mod} A$. Define

$$\mathrm{L}_i(F) := \mathrm{L}_i^{\boldsymbol{P}}(F)$$

and

$$\eta_i^{\boldsymbol{P}} := \mathrm{id}_{\mathrm{L}_i^{\boldsymbol{P}}(F)} .$$

This establishes existence.

(2) As for uniqueness, given the other i-th left derived functor, the unique isomorphism between them is

$$\eta := (\eta_i^{\boldsymbol{Q}})^{-1} \circ \mathrm{L}_i^{\boldsymbol{P},\boldsymbol{Q}}(\mathrm{id}_F) \circ \eta_i^{\boldsymbol{P}} .$$

□

By abuse of notation we usually refer to the i-th left derived functor of F as $\mathrm{L}_i(F)$ or $\mathrm{L}_i F$, leaving the presentation $\eta_i^{\boldsymbol{P}}$ implicit. But when we need to calculate something we must mention $\eta_i^{\boldsymbol{P}}$.

comment: Next week we will say a bit more on $\mathrm{L}_i(F)$. Then we will introduce $\mathrm{Tor}_i^A(-,-)$, and long exact cohomology sequence ... Hopefully!

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