Homological Algebra

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There a several ways to view homological algebra.

- An effective method to record equivalence classes of mathematical objects (e.g. extensions.)
- A method to detect and measure deviation from regularity (lack of e.g. exactness of a functor).
- New invariants of mathematical objects (algeb. topology, alg. geom.)
- A setting for New mathematical objects (dualizing complexes in derived categories).

HARD

Homological algebra evolved from algebraic topology.

Even though this will not be our focus of study, I will begin with some alg. top.

For $n \geq 1$ let $\mathbb{B}^n$ be the closed unit ball in $\mathbb{R}^n$.

Theorem (Brouwer Fixed Point).
Let $f: \mathbb{B}^n \to \mathbb{B}^n$ be a continuous function. Then f has a fixed point.

[do demonstration w. 2 pages]

For $n=0$ it's trivial.

For $n=1$ this is an easy exercise. (intermediate value thm.)

But for $n \geq 2$ it is an amazing and mysterious fact !!

I will explain the proof using singular homology. This will give us some idea of homological considerations.

For $n \geq 0$ let Δ^n be the n-dimensional simplex:

$$\boxed{\Delta^n := \{ x \in \mathbb{R}^{n+1} \mid t_i(x) \geq 0, \ \sum_{i=1}^{n+1} t_i(x) = 1 \}}$$

($t_1, \dots, t_{n+1}$ are the coordinate functions.)

$\underline{\Delta}^0$ $\cdot$ point

$\underline{\Delta}^1$ $0 \longrightarrow 1$ line segment

$\underline{\Delta}^2$ triangle

Let X be a topological space.

A singular n-simplex in X is a continuous function

$$\sigma : \underline{\Delta}^n \to X$$

The group of <u>n-chains</u> of X is the free abelian group $C_n(X)$ with basis the set of singular n-simplices.

There is a homomorphism

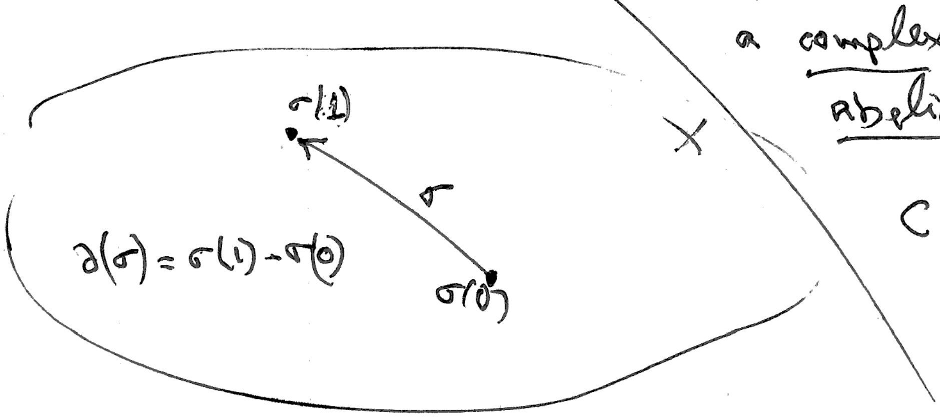
$$\boxed{\partial : C_n(X) \to C_{n-1}(X)}$$

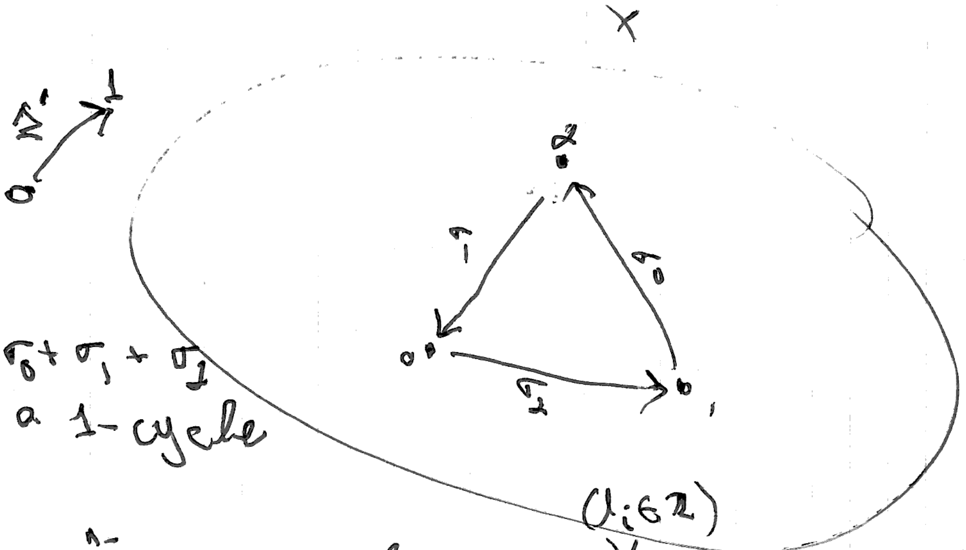
called the boundary operator.

It satisfies $\boxed{\partial \circ \partial = 0.}$ $(\dagger)$

(C, ∂) is called a complex of abelian groups

$C = \{C_n\}_{n \geq 0}$





$c = \sigma_0 + \sigma_1 + \sigma_2$
is a 1-cycle

an n-chain $\sum_i l_i \cdot \sigma_i = c$ $(l_i \in \mathbb{Z})$ is a <u>cycle</u> if $\partial(c) = 0$, i.e. $c \in \mathrm{Ker}(\partial)$.

c is a <u>boundary</u> if $c \in \mathrm{Im}(\partial)$.

n-th singular homology of X:

$$\boxed{H_n(X) = \frac{\{\text{group of } n\text{-cycles}\}}{\{\text{group of } n\text{-boundaries}\}}}$$

Formula (*) implies:

$$\{n\text{-boundaries}\} \subseteq \{n\text{-cycles}\} \subseteq \{n\text{-chains}\} = C_n(X)$$

Let <u>Top</u> be the category of topological spaces. (Will remind / define cats. & functors later.) Morphisms are continuous functions.

Let <u>Ab</u> be the category of abelian groups. Morphisms are group homs.

<u>Theorem</u> (1) For every $n \geq 0$,

$$H_n : \underline{\mathrm{Top}} \to \underline{\mathrm{Ab}} \text{ is a functor.}$$

<u>Theorem</u> (2) For every $n \geq 0$ and $i > 0$ we have

$$H_i(\mathbb{B}^n) = 0.$$

The n-sphere is

$$\mathbb{S}^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$$

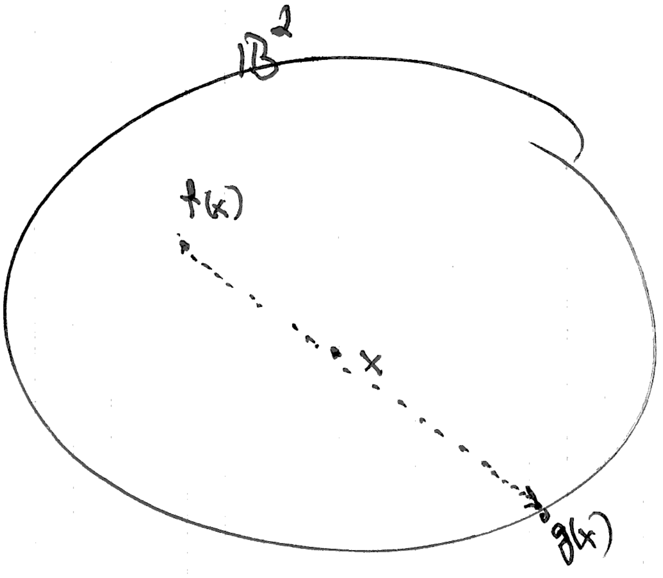
Observe that $\mathbb{S}^n$ is the boundary of $\mathbb{B}^{n+1}$.

<u>Theorem</u> (3). For $n \geq 1$ we have

$$H_i(\mathbb{S}^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

<u>Sketch of proof of Brouwer's Fixed Point Thm.</u>

Can assume $n \geq 2$. If $f : \mathbb{B}^n \to \mathbb{B}^n$ is continuous without fixed pt, then there's a cont. function $g : \mathbb{B}^n \to \mathbb{S}^{n-1}$ s.t. $g(x) = x$ for $x \in \mathbb{S}^{n-1}$.



Get comm. diagram in $\underline{\text{Top}}$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "inc"] \arrow[rr, bend right, "id_{S^{n-1}}"'] & \mathbb{B}^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

Apply functor H_{n-1} to get comm diagram in $\underline{\text{Ab}}$

$$\begin{tikzcd} & 0 \arrow[d, phantom, "\cong"] & \\ H^{n-1}(S^{n-1}) \arrow[r, "H^{n-1}(inc)"] \arrow[d, phantom, "\cong"] \arrow[rr, bend right, "id = H^{n-1}(id_{S^{n-1}})"'] & H^{n-1}(\mathbb{B}^n) \arrow[r, "H^{n-1}(g)"] & H^{n-1}(S^{n-1}) \arrow[d, phantom, "\cong"] \\ \mathbb{Z} & & \mathbb{Z} \end{tikzcd}$$

This is a $\underline{\text{contradiction}}$, since for an ab. grp. $M \cong \mathbb{Z}$, $id_M \neq 0$.

$\square$