Course Notes:

Algebraic Geometry – Schemes 1

Fall 2018-19

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Contents

1. Basics

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Before starting with the actual material, lets us go quickly over some basic ideas that we will need. I hope all these are familiar to all students; if not, then we will have to see how to close the gaps.

The first few weeks will be on geometry in general, but from the point of view of *locally ringed spaces*.

Everybody needs to know a sufficient amount of elementary topology. Some algebraic topology will be required (homology, cohomology and fundamental groups).

Categories, functors and natural transformations will be used a lot. I am assuming that all students have already been exposed to these notions. For instance, all should understand this statement:

- Let Top_* and Grp be the categories of pointed topological spaces and of groups, respectively. The fundamental group is a functor

$$\pi_1 : \mathsf{Top}_* \to \mathsf{Grp} .$$

If not, then we will have to see how to close this gap. (Maybe go over material from [Ye3].)

Differential geometry will serve as an introductory model for locally ringed spaces. (A preparation for the more complicated schemes.) Everybody should have some knowledge on this topic (C^∞ manifolds and maps between them, tangent bundles, etc.) Knowledge of complex analytic geometry will be very useful.

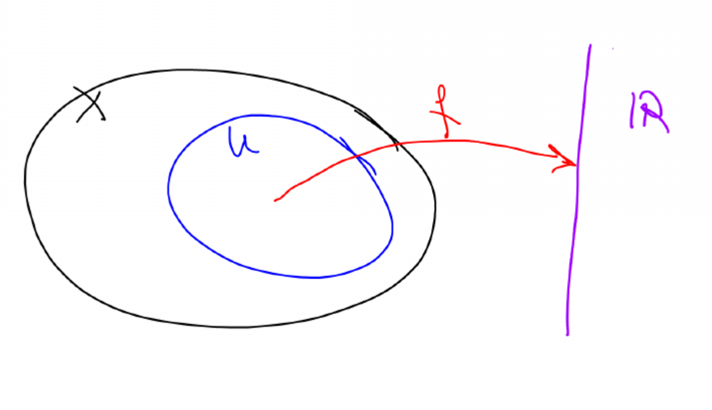
2. Sheaves of Functions on Topological Spaces

Consider a topological space X. We do not make any conditions on X, especially we don't assume X is Hausdorff. But at first you can pretend, to help intuition, that X is a topological subspace of $\mathbb{R}^n$ (with its usual topology).

Given an open subset $U \subseteq X$, consider the continous functions

$$f : U \to \mathbb{R}.$$

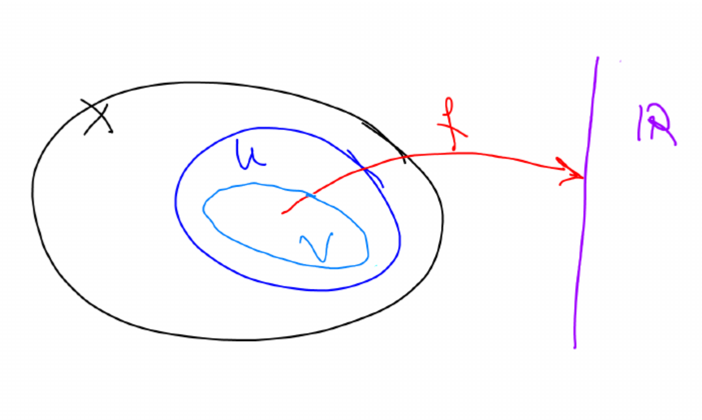
Let us denote this set of functions by $\Gamma(U, \mathcal{O}_X)$.



We know that $\Gamma(U, \mathcal{O}_X)$ is a commutative $\mathbb{R}$-ring.

Let $V \subseteq U$ be a smaller open set. We get a continous function

$$f|_V : V \to \mathbb{R}.$$

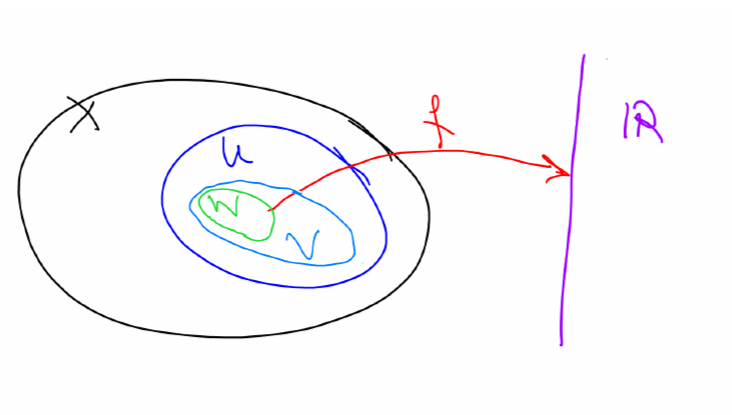


The opration $f \mapsto f|_V$ is a ring homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X).$$

If $W \subseteq V$ is another smaller open set, then of course

$$(f|_V)|_W = f|_W.$$



We see that the restriction homomorphisms satisfy

$$\mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U} = \mathrm{rest}_{W/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(W, \mathcal{O}_X).$$

This means that $\mathcal{O}_X$ *is a presheaf of* $\mathbb{R}$*-rings on* X.

Here is a categorical interpretation of this statement. Let $\mathsf{Rng}_c/\mathbb{R}$ be the category of commutative $\mathbb{R}$-rings.

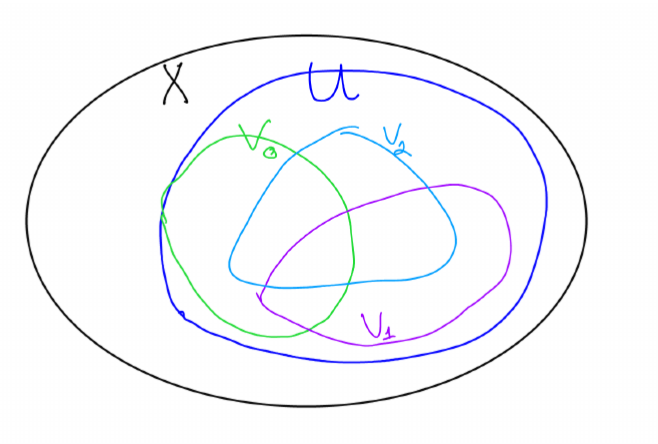
Let $\mathsf{Open}(X)$ be the category of open sets of X, where the morphisms are inclusions. Thus if $V \subseteq U$ then there is one arrow $V \to U$; and if $V \not\subseteq U$ then there are no arrows $V \to U$. The presheaf $\mathcal{O}_X$ is a functor

$$\Gamma(-, \mathcal{O}_X) : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Rng}_c/\mathbb{R}.$$

But in fact much more is true.

Suppose $U \subseteq X$ is an open set, and we are given an open covering

$$U = \bigcup_{i \in I} V_i.$$



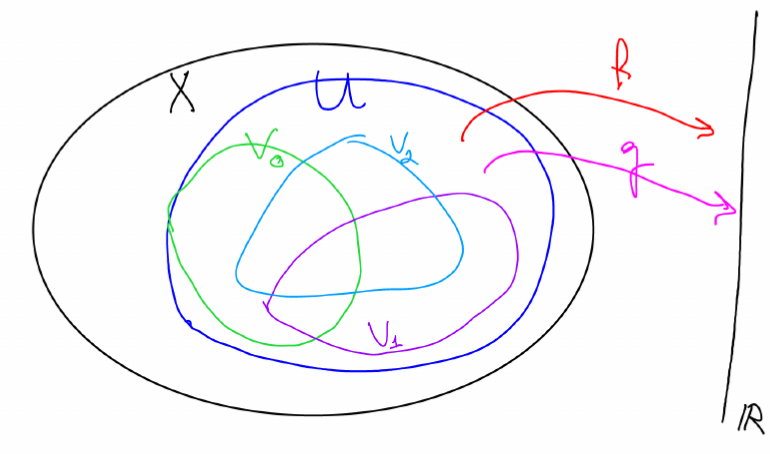
Let $f, g \in \Gamma(U, \mathcal{O}_X)$, i.e.

$$f, g : U \to \mathbb{R},$$

and assume that

$$f|_{V_i} = g|_{V_i}$$

for all i.



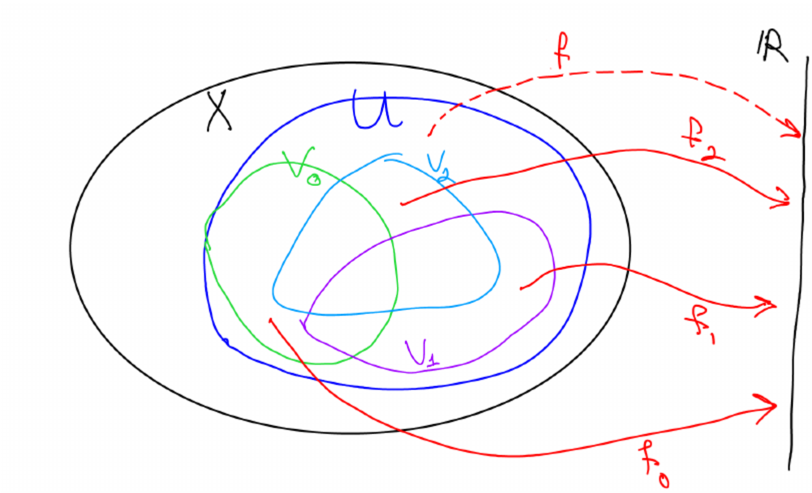
Then of course $f = g$.
Now assume that we are given

$$f_i \in \Gamma(V_i, \mathcal{O}_X)$$

such that

$$f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$$

for all i, j.



Because the various f_i agree on double intersections, there is a function

$$f : U \to \mathbb{R}$$

such that

$$f|_{U_i} = f_i.$$

Of course this function f is unique (by the previous discussion). But also f is continous. This is because continuity is a local property, and on each of the open sets U_i we know that f is continous.

Thus

$$f \in \Gamma(U, \mathcal{O}_X).$$

Let us summarize these two further properties of $\mathcal{O}_X$:

(a) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ an open covering, and let

$$f, g \in \Gamma(U, \mathcal{O}_X)$$

be such that $f|_{V_i} = g|_{V_i}$ for all i. Then $f = g$.

(b) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let

$$f_i \in \Gamma(V_i, \mathcal{O}_X)$$

be such that

$$f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$$

for all i, j. Then there exists

$$f \in \Gamma(U, \mathcal{O}_X)$$

such that $f|_{V_i} = f_i$ for all i.

These are the *sheaf axioms*. They tell us that $\mathcal{O}_X$ is a *sheaf of rings on* X.

Because rings have underlying abelian groups, axioms (a) and (b) can be stated in terms of *exact sequences*.

(∗) For every open set $U \subseteq X$ and every open covering $U = \bigcup_{i \in I} V_i$ the sequence of abelian groups

$$0 \to \Gamma(U, \mathcal{O}_X) \xrightarrow{\rho} \prod_{i \in I} \Gamma(V_i, \mathcal{O}_X) \xrightarrow{\delta^0 - \delta^1} \prod_{j,k \in I} \Gamma(V_j \cap V_k, \mathcal{O}_X)$$

is exact.

Here ρ is the product on all $i \in I$ of the restriction homomorphisms

$$\mathrm{rest}_{V_i/U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(V_i, \mathcal{O}_X).$$

The homomorphism δ^1 is the product on all $i = j \in I$ of the product on all $k \in I$ of

$$\mathrm{rest}_{V_j \cap V_k/V_j} : \Gamma(V_j, \mathcal{O}_X) \to \Gamma(V_j \cap V_k, \mathcal{O}_X).$$

And the homomorphism δ^0 is the product on all $i = k \in I$ of the product on all $j \in I$ of of

$$\mathrm{rest}_{V_j \cap V_k/V_k} : \Gamma(V_k, \mathcal{O}_X) \to \Gamma(V_j \cap V_k, \mathcal{O}_X).$$

Exercise 2.1. Prove that condition $(*)$ is equivalent to condition $\big($(a) and (b)$\big)$.

The next exercise gives a variation of what we did above.

Exercise 2.2. Let X be a differentiable manifold (of type C^∞). For every open set $U \subseteq X$ let $\Gamma(U, \mathcal{O}_X)$ be the set of differentiable functions $f : U \to \mathbb{R}$.

Prove that the assignment

$$U \mapsto \Gamma(U, \mathcal{O}_X)$$

is a sheaf of $\mathbb{R}$-rings on X. The sheaf $\mathcal{O}_X$ is called the *sheaf of differentiable functions on* X.

Exercise 2.3. If you know about real or complex analytic manifolds, state and prove the corresponding analogue of Exercise 2.2.

Exercise 2.4. This exercise is for those who know the algebraic geometry of varieties. Let $\mathbb{K}$ be an algebraically closed field, and let X be an algebraic variety over $\mathbb{K}$. For every (Zariski) open set $U \subseteq X$ let $\Gamma(U, \mathcal{O}_X)$ be ring of algebraic functions on U. Prove that the assignment

$$U \mapsto \Gamma(U, \mathcal{O}_X)$$

is a sheaf of $\mathbb{K}$-rings on X. The sheaf $\mathcal{O}_X$ is called the *sheaf of algebraic functions on* X.

3. Sheaves on Topological Spaces

Until now we only saw ring valued sheaves. Here are some variations.

Definition 3.1. Let X be a topological space. A *presheaf of groups on* X is a functor

$$\mathcal{G} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Grp},$$

where Grp is the category of groups.

Concretely, the presheaf $\mathcal{G}$ is the data of a group $\Gamma(U, \mathcal{G})$ for every open set $U \subseteq X$, called the *group of sections of* $\mathcal{G}$ *over* U, and a group homomorphism

$$\mathrm{rest}_{V/U} : \Gamma(U, \mathcal{G}) \to \Gamma(V, \mathcal{G})$$

for every inclusion $V \subseteq U$, such that

$$\mathrm{rest}_{W/U} = \mathrm{rest}_{W/V} \circ \mathrm{rest}_{V/U}$$

for every triple inclusion $W \subseteq V \subseteq U$. And of course

$$\mathrm{rest}_{U/U} = \mathrm{id}_{\Gamma(U, \mathcal{G})}$$

for every U.

Definition 3.2. Let X be a topological space. A *sheaf of groups on* X is a presheaf of groups $\mathcal{G}$ on X that satisfies the two sheaf axioms:

(a) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let $g, g' \in \Gamma(U, \mathcal{G})$ be sections such that $g|_{V_i} = g'|_{V_i}$ for all i. Then $g = g'$.

(b) Let $U \subseteq X$ be an open set, let $U = \bigcup_{i \in I} V_i$ be an open covering, and let $g_i \in \Gamma(V_i, \mathcal{G})$ be sections such that
$$g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j}$$
for all i, j. Then there exists a section $g \in \Gamma(U, \mathcal{O}_X)$ such that
$$g|_{V_i} = g_i$$
for all i.

Recall that a *topological group* is a topological space G, that is also a group, such that the operations of multiplication and inversion are continous. Namely
$$\text{mult} : G \times G \to G$$
and
$$\text{inv} : G \to G$$
are continous functions.

Example 3.3. Let X be a topological space and G a topological group. For every open set $U \subseteq X$ define
$$\Gamma(U, \mathcal{G}) := \{\text{continous functions } g : U \to G\}.$$
I claim that $\mathcal{G}$ is a *sheaf of groups on* X.

That $\mathcal{G}$ is a presheaf is obvious. Sheaf axiom (a) is also clear, because for every point $x \in U$ we can find some i such that $x \in V_i$, and hence we have
$$g(x) = g|_{V_i}(x) = g'|_{V_i}(x) = g'(x).$$
Thus $g = g'$.

Axiom (b) is also easy to verify. The values $g_i(x)$ at a point $x \in U$ are equal, for all $i \in I$ such that $x \in V_i$. So there is a function $g : U \to G$. Because continuity is a local property, and $g|_{V_i} = g_i$, we see that g is continous. Thus $g \in \Gamma(U, \mathcal{G})$.

Exercise 3.4. Let X be a topological space. Let $G := \mathrm{GL}_n(\mathbb{R})$ for some positive integer n, with the usual topology (by the embedding $\mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$). So G is a topological group.

Let $\mathcal{G}$ be the sheaf of groups on X from Example 3.3 for this choice of G. And let $\mathcal{O}_X$ be the sheaf of continous real valued function on X. Prove that for every open set $U \subseteq X$ there is a group isomorphism
$$\Gamma(U, \mathcal{G}) \cong \mathrm{GL}_n(\Gamma(U, \mathcal{O}_X)),$$
that respects the restriction homomorphisms.

Definition 3.5. Let X be a topological space.

(1) A *presheaf of abelian groups on* X is a functor
$$\mathcal{G} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Ab},$$
where Ab is the category of abelian groups.

(2) A *sheaf of abelian groups on* X is a presheaf of abelian groups $\mathcal{G}$ that satisfies the sheaf axioms (a) and (b) from Definition 3.2.

It is not hard to see that a sheaf of abelian groups $\mathcal{G}$ is the same as a sheaf of groups $\mathcal{G}$ such that each $\Gamma(U, \mathcal{G})$ is abelian.

Definition 3.6. Let X be a topological space.

(1) A *presheaf of commutative rings on* X is a functor
$$\mathcal{A} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{Rng}_{\mathrm{c}},$$
where $\mathsf{Rng}_{\mathrm{c}}$ is the category of commutative rings.

(2) A *sheaf of commutative rings on* X is a presheaf of commutative rings $\mathcal{A}$ that satisfies the sheaf axioms (a) and (b) from Definition 3.2.

If $\mathcal{A}$ is a sheaf of commutative rings, and we forget the multiplication of $\mathcal{A}$, then we obtain a sheaf of abelian groups.

Example 3.7. Let X be a topological space and let A be a commutative ring. The *constant sheaf of rings on* X *with values in* A is the sheaf A_X defined as follows. Put on A the discrete topology. Then for every $U \subseteq X$ open we let

$$\Gamma(U, A_X) := \{\text{continous functions } g : U \to A\}.$$

Exercise 3.8. Take a nonzero commutative ring A, say $A := \mathbb{Z}$. Calculate the ring $\Gamma(X, A_X)$ for these choices of X :

(1) $X := \mathbb{R}$ with the classical topology.
(2) $X := \mathbb{N}$ with the discrete topology.

4. Stalks

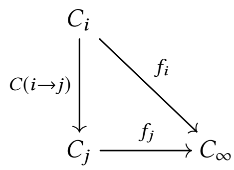
A *directed set* is a partially ordered set I such that for every $i, j \in I$ there exists some $k \in I$ with $i, j \leq k$. We can view the directed set I as a category, with a single arrow $i \to j$ if $i \leq j$, and no arrows otherwise.

A *direct system* in a category C, indexed by a directed set I, is a functor

$$C : I \to \mathrm{C}, \quad i \mapsto C_i.$$

We usually denote such a direct system by $\{C_i\}_{i \in I}$.

A *direct limit* of a direct system $\{C_i\}_{i \in I}$ is an object $C_\infty \in \mathrm{C}$, together with a collection of morphisms $f_i : C_i \to C_\infty$, such that the diagram



is commutative for every $i \to j$, and such that

$$\big(C_\infty, \{f_i\}_{i \in I}\big)$$

is universal for this property. We write

$$\lim_{i \to} C_i := C_\infty.$$

The categories Grp, Ab and $\mathrm{Rng_c}$ have direct limits. Here is the construction:

$$\lim_{i \to} C_i = \Big(\coprod\nolimits_{i \in I} C_i \Big) \, / \sim ,$$

where $\sim$ is the relation $c_i \sim c_j$ for $c_i \in C_i$ and $c_j \in C_j$ whenever there are arrows $i, j \to k$ such that

$$C(i \to k)(c_i) = C(j \to k)(c_j) \in C_k.$$

Let X be a topological space. For a point $x \in X$ let $\mathrm{Open}(X, x)$ be the set of open neighborhoods of x, made into a category by inclusion. Then $\mathrm{Open}(X, x)^{\mathrm{op}}$ is a directed set.

Definition 4.1. Let $\mathcal{G}$ be a sheaf of abelian groups on a topological space X. Let $x \in X$ be a point. The *stalk of* $\mathcal{G}$ *at* x is the abelian group

$$\mathcal{G}_x := \lim_{U \to} \Gamma(U, \mathcal{G}),$$

where the direct limit is on $U \in \mathrm{Open}(X, x)^{\mathrm{op}}$.

Likewise for a sheaf of groups and for a sheaf of commutative rings.

Exercise 4.2. With the assumptions of Exercise 3.8(1, 2), calculate the stalks $(A_X)_x$ for a point $x \in X$.

5. Morphisms of Sheaves

6. The Sheaf Associated to a Presheaf

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