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PROOF OF YONEDA LEMMA

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Abstract. The purpose of these notes is to prove Yoneda Lemma stated as Theorem 1.7.4 in [1]. These notes and Section 1.7 of [1] are intended to be read and integrated in the Homological Algebra class by Prof. Amnon Yekutieli at BGU (cf [2]).

We recall Theorem 1.7.4 in [1]:

Theorem 1.7.4 (Yoneda Lemma)**.** *The Yoneda functor* $\mathrm{Y}_{\mathbf{C}}$ *is fully faithful.*

It means that given $C_0, C_1 \in \mathbf{C}$ what follows:

$$\begin{aligned} \mathrm{Y}_{\mathbf{C}} : \mathrm{Hom}_{\mathbf{C}}(C_0, C_1) &\to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathrm{Set})}(\mathrm{Y}_{\mathbf{C}}(C_0), \mathrm{Y}_{\mathbf{C}}(C_1)) \\ \phi &\mapsto \mathrm{Y}_{\mathbf{C}}(\phi). \end{aligned} \tag{1}$$

is a bijection.
We recall that $\mathrm{Y}_{\mathbf{C}}(\phi)$ is a natural transformation defined as:

$$\mathrm{Y}_{\mathbf{C}}(\phi) := \mathrm{Hom}_{\mathbf{C}}(-, \phi) : \mathrm{Y}_{\mathbf{C}}(C_0) \to \mathrm{Y}_{\mathbf{C}}(C_1).$$

For every morphism $\psi : D_0 \to D_1$ in $\mathbf{C}$, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Y}_{\mathbf{C}}(C_0)(D_1) \arrow[r, "\mathrm{Y}_{\mathbf{C}}(\phi)_{D_1}"] \arrow[d, "\mathrm{Y}_{\mathbf{C}}(C_0)(\psi)"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(D_1) \arrow[d, "\mathrm{Y}_{\mathbf{C}}(C_1)(\psi)"] \\ \mathrm{Y}_{\mathbf{C}}(C_0)(D_0) \arrow[r, "\mathrm{Y}_{\mathbf{C}}(\phi)_{D_0}"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(D_0). \end{tikzcd} \tag{2}$$

Where $\mathrm{Y}_{\mathbf{C}}(C_i)(\psi) := \mathrm{Hom}_{\mathbf{C}}(\psi, \mathrm{id}_{C_i})$ (for $i = 0, 1$).

Proof. Let us prove that (1) is bijective:

- (1) is surjective.
  Let $\eta : \mathrm{Y}_{\mathbf{C}}(C_0) \to \mathrm{Y}_{\mathbf{C}}(C_1)$ a natural transformation. We want to show that there exist a morphism $\phi : C_0 \to C_1$ such that $\eta = \mathrm{Y}_{\mathbf{C}}(\phi)$.
  A good candidate to be ϕ is given by $\eta_{C_0}(\mathrm{id}_{C_0})$, we recall that:

$$\begin{aligned} \eta_{C_0} : \mathrm{Y}_{\mathbf{C}}(C_0)(C_0) &\to \mathrm{Y}_{\mathbf{C}}(C_1)(C_0) \\ \mathrm{id}_{C_0} &\mapsto \eta_{C_0}(\mathrm{id}_{C_0}). \end{aligned} \tag{3}$$

  So $\eta_{C_0}(\mathrm{id}_{C_0})$ is a morphism $C_0 \to C_1$.
  Now we want to show that $\eta = \mathrm{Y}_{\mathbf{C}}(\eta_{C_0}(\mathrm{id}_{C_0}))$.
  Given D be an object in $\mathbf{C}$ and $g : D \to C_0$. We have:

$$\begin{aligned} \eta_D : \mathrm{Y}_{\mathbf{C}}(C_0)(D) &\to \mathrm{Y}_{\mathbf{C}}(C_1)(D) \\ g &\mapsto \eta_D(g). \end{aligned} \tag{4}$$

Let us prove that $\eta_D(g) = \mathrm{Y}_{\mathbf{C}}(\eta_{C_0}(\mathrm{id}_{C_0}))_D(g)$.
Since η is a natural transformation, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Y}_{\mathbf{C}}(C_0)(C_0) \arrow[r, "\eta_{C_0}"] \arrow[d, "\mathrm{Y}_{\mathbf{C}(C_0)(g)}"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(C_0) \arrow[d, "\mathrm{Y}_{\mathbf{C}(C_1)(g)}"] \\ \mathrm{Y}_{\mathbf{C}}(C_0)(D) \arrow[r, "\eta_D"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(D). \end{tikzcd} \tag{5}$$

We have $\eta_D(\mathrm{Y}_{\mathbf{C}}(C_0)(g)(-)) = \mathrm{Y}_{\mathbf{C}}(C_1)(g)(\eta_{C_0}(-))$.
In particular, taking id_{C_0}, we have:

$$\eta_D(\mathrm{Y}_{\mathbf{C}}(C_0)(g)(\mathrm{id}_{C_0})) = \mathrm{Y}_{\mathbf{C}}(C_1)(g)(\eta_{C_0}(\mathrm{id}_{C_0})). \tag{6}$$

It follows:

$$\begin{aligned} \eta_D(g) =& \eta_D(\mathrm{id}_{C_0} \circ g) \\ =& \eta_D(\mathrm{Y}_{\mathbf{C}}(C_0)(g)(\mathrm{id}_{C_0})) \\ \overset{(6)}{=}& \mathrm{Y}_{\mathbf{C}}(C_1)(g)(\eta_{C_0}(\mathrm{id}_{C_0})) \\ =& \eta_{C_0}(\mathrm{id}_{C_0}) \circ g \\ =& \mathrm{Y}_{\mathbf{C}}(\eta_{C_0}(\mathrm{id}_{C_0}))_D(g). \end{aligned}$$

It means $\eta_D(-) = \mathrm{Y}_{\mathbf{C}}(\eta_{C_0}(\mathrm{id}_{C_0}))_D(-)$ (for every $D \in \mathbf{C}$) and we are done.

- (1) is injective.
  Given two morphisms $\phi, \tilde{\phi} : C_0 \to C_1$ such that $\mathrm{Y}_{\mathbf{C}}(\phi) = \mathrm{Y}_{\mathbf{C}}(\tilde{\phi})$ we want to prove that $\phi = \tilde{\phi}$.
  Let us take the diagram (2) with $D_1 = D_0 = C_0$ and $\psi = \mathrm{id}_{C_0}$:

$$\begin{tikzcd} \mathrm{Y}_{\mathbf{C}}(C_0)(C_0) \arrow[r, "\mathrm{Y}_{\mathbf{C}}(\phi)_{C_0}"] \arrow[d, "\mathrm{Y}_{\mathbf{C}(C_0)(\mathrm{id}_{C_0})}"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(C_0) \arrow[d, "\mathrm{Y}_{\mathbf{C}(C_1)(\mathrm{id}_{C_0})}"] \\ \mathrm{Y}_{\mathbf{C}}(C_0)(C_0) \arrow[r, "\mathrm{Y}_{\mathbf{C}}(\tilde{\phi})_{C_0}"'] & \mathrm{Y}_{\mathbf{C}}(C_1)(C_0). \end{tikzcd} \tag{7}$$

The diagram above is commutative even if, in the bottom row, we have $\mathrm{Y}_{\mathbf{C}}(\tilde{\phi})_{C_0}$ instead of $\mathrm{Y}_{\mathbf{C}}(\phi)_{C_0}$ since by hypothesis $\mathrm{Y}_{\mathbf{C}}(\phi) = \mathrm{Y}_{\mathbf{C}}(\tilde{\phi})$.
Again, taking id_{C_0}, by the diagram (7) we have:

$$\mathrm{Y}_{\mathbf{C}}(C_1)(\mathrm{id}_{C_0})(\mathrm{Y}_{\mathbf{C}}(\phi)_{C_0}(\mathrm{id}_{C_0})) = \mathrm{Y}_{\mathbf{C}}(\tilde{\phi})_{C_0}(\mathrm{Y}_{\mathbf{C}}(C_0)(\mathrm{id}_{C_0})(\mathrm{id}_{C_0})). \tag{8}$$

It follows:

$$\begin{aligned} \phi =& \phi \circ \mathrm{id}_{C_0} \circ \mathrm{id}_{C_0} \\ =& (\mathrm{Y}_{\mathbf{C}}(\phi)_{C_0}(\mathrm{id}_{C_0})) \circ \mathrm{id}_{C_0} \\ =& \mathrm{Y}_{\mathbf{C}}(C_1)(\mathrm{id}_{C_0})(\mathrm{Y}_{\mathbf{C}}(\phi)_{C_0}(\mathrm{id}_{C_0})) \\ \overset{(8)}{=}& \mathrm{Y}_{\mathbf{C}}(\tilde{\phi})_{C_0}(\mathrm{Y}_{\mathbf{C}}(C_0)(\mathrm{id}_{C_0})(\mathrm{id}_{C_0})) \\ =& \tilde{\phi} \circ (\mathrm{Y}_{\mathbf{C}}(C_0)(\mathrm{id}_{C_0})(\mathrm{id}_{C_0})) \\ =& \tilde{\phi} \circ \mathrm{id}_{C_0} \circ \mathrm{id}_{C_0} \\ =& \tilde{\phi} \end{aligned}$$

  and we are done.

$\square$

References

[1] Yekutieli, A., *Derived Categories* (Cambridge Studies in Advanced Mathematics), Cambridge University Press., 2020. Available at https://arxiv.org/abs/1610.09640v4.

[2] Yekutieli, A., *Homological Algebra Notes*, https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/homol-alg/crs-notes_200318-d2.pdf.

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