Course Notes:

Homological Algebra

BGU, Spring 2019-20

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Available here:
https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/homol-alg/course_page.html

Contents

comment: Start of Lecture 1, 11 March 2020. Lecture via SKYPE.

0. Introduction

This course is a continuation of the course "Commutative Algebra" from the previous semester. We will use the notes [Yek1] a lot.

Here is a revised syllabus for this course. It is tentative: I will change material, and the order of presentation, as I go along.

Course Topics:

(1) Adjoint functors, equivalences and exactness.
(2) Bimodules and noncommutative tensor products.
(3) Projective modules, invertible bimodules and Morita Theory.
(4) Injective modules, including Matlis Theory for noetherian commutative rings.
(5) Complexes of modules, homotopies and homotopy equivalences, quasi-isomorphisms.
(6) The long exact cohomology sequence.
(7) Projective, flat and injective resolutions of modules.
(8) Left and right derived functors.
(9) Applications of derived functors to commutative algebra.
(10) Further applications of derived functors and cohomology, including non-abelian cohomology.

The notes from the course "Homological Algebra" [Yek2] from two years ago cover topics 5-8 roughly. (They also include the material on categories and functors, that we already covered in [Yek1].)

1. Adjoint Functors and Equivalences

The material we start with is very abstract.

Categories and functors were introduced in Sections 4, 7 and 9 of [Yek1].

The product category $\mathsf{C} \times \mathsf{D}$ and the opposite category C^{op} were also introduced in [Yek1].

Recall that we are ignoring set theoretical issues (see [Yek1, Remark 4.3] or [Yek3, Section 1.1]).

We shall use the expression *morphism of functors* instead of *natural transformation*.

comment: (200317) next def corrected

Definition 1.1 (Adjoints)**.** Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors between categories.

An *adjunction between F and G* is a bijection of sets

$$\alpha_{D,C} : \operatorname{Hom}_{\mathsf{D}}\big(D, F(C)\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}}\big(G(D), C\big),$$

which is functorial in $D \in \mathsf{D}$ and $C \in \mathsf{C}$.

Namely the collection of isomorphisms

$$\alpha := \{\alpha_{D,C}\}_{(D,C) \in \mathsf{D} \times \mathsf{C}}$$

is an isomorphism

$$\alpha : \operatorname{Hom}_{\mathsf{D}}\big(-, F(-)\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{C}}\big(G(-), -\big)$$

of functors

$$\mathsf{D}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}\,.$$

Given such an adjunction, we say that *G is a left adjoint of F*, and that *F is a right adjoint of G*.

Remark 1.2. Adjoint functors were invented by D. Kan in 1958.

The name is borrowed from functional analysis. Let V and W be Hilbert spaces with inner products $\langle -,- \rangle_V$ and $\langle -,- \rangle_W$. Continuous linear operators $F : V \to W$ and $G : W \to V$ are called adjoint to each other if

$$\langle G(v), w\rangle_W = \langle v, F(w)\rangle_V$$

for all $v \in V$ and $w \in W$. The analogy is clear – but, since the inner products are symmetric, one just talks about $G = F^*$ being the adjoint of F.

Let's recall *equivalences of categories* from [Yek1, Definition 13.28]. A functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence if there is a functor $G : \mathsf{D} \to \mathsf{C}$ that is a quasi-inverse of F. In detail, there is an isomorphism of functors

$$\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}} \tag{1.3}$$

from C to itself, and an isomorphism of functors

$$\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}} \tag{1.4}$$

from D to itself.

Theorem 1.5. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be functors. Assume F is an equivalence. TFAE*:

(i) *G is a quasi-inverse of F.*
(ii) *G is a left adjoint of F.*
(iii) *G is a right adjoint of F.*

A direct proof is very messy; see Remark 1.9 regarding an elegant (but not easy nor quick) proof.

I don't think we will need this theorem. In case it will be needed (for Morita Theory), I will give a proof of this theorem later.

The same goes for the next theorem:

Theorem 1.6. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between categories, and let $G, G' : \mathsf{D} \to \mathsf{C}$ be left adjoints of F. Then there is a unique isomorphism of functors $G \xrightarrow{\simeq} G'$ respecting the adjunction with F.*

Likewise for right adjoints.

Here is a variant of Theorem 1.6 that's easier:

Proposition 1.7. *If $F : \mathsf{C} \to \mathsf{D}$ is an equivalence, and if G and G' are quasi-inverses of F, then there is a unique isomorphism of functors $G \xrightarrow{\simeq} G'$ that respects the isomorphisms* (1.3) *and* (1.4).

Exercise 1.8. Prove this Proposition. (Hint: Study [Yek1, Example 13.29] and the results near it.)

Remark 1.9. The nice proofs of Theorems 1.5 and 1.6 require the *Yoneda Lemma*.

This is something I want to avoid doing in class (it is extremely abstract and confusing).

You can talk to the postdcos about this material – they will be glad to explain.

Here are a few useful examples of adjoints.

But first two conventions.

Convention 1.10. We fix a nonzero commutative base ring $\mathbb{K}$ (e.g. $\mathbb{Z}$ or a field). All rings are assumed by default to be central $\mathbb{K}$-rings, and all ring homomorphisms are assumed by default to be $\mathbb{K}$-ring homomorphisms. (If $\mathbb{K} = \mathbb{Z}$ then this is automatically satisfied.)

More generally, all linear categories are assumed by default to be $\mathbb{K}$-linear categories, and all linear functors between them as assumed by default to be $\mathbb{K}$-linear.

The expression $\otimes$ means $\otimes_{\mathbb{K}}$.

The base ring $\mathbb{K}$ will usually remain implicit.

Warning: unlike the previous course, here rings are not assume to be commutative!

Notation 1.11. The category of central $\mathbb{K}$-rings is $\mathsf{Rng}/_{\mathrm{c}}\,\mathbb{K}$.

The full subcategory of commutative $\mathbb{K}$-rings is $\mathsf{Rng}_{\mathrm{c}}\,/\mathbb{K}$.

Another caution: A homomorphism $f : A \to B$ in $\mathsf{Rng}/_{\mathrm{c}}\,\mathbb{K}$ need not itself be a central homomorphism. E.g. the inclusion of $A := \mathbb{K} \times \mathbb{K}$ into $B := \mathrm{Mat}_2(\mathbb{K})$, as the diagonal matrices, is not central.

On the other hand, a homomorphism $\mathbb{K} \to A$, when A is a commutative ring, is automatically central. This is why the notation $\mathsf{Rng}_{\mathrm{c}}\,/_{\mathrm{c}}\,\mathbb{K}$ is redundant.

Convention 1.12. For a ring A we write $\mathsf{M}(A) := \mathsf{Mod}\,A$, the category of left A-modules.

The the category of right A-modules is $\mathsf{M}(A^{\mathrm{op}})$.

These are $\mathbb{K}$-linear categories.

Example 1.13. Let A be a nonzero ring. We have the forgetful functor

$$F : \mathsf{M}(A) \to \mathsf{Set}\,.$$

There is the free module functor

$$G : \mathsf{Set} \to \mathsf{M}(A), \quad G(S) := A^{\oplus S} = A^{(S)} = \mathrm{F}_{\mathrm{fin}}(S, A) = \bigoplus_{s \in S} A.$$

As basis of $\mathrm{F}_{\mathrm{fin}}(S, A)$ we take the collection of delta functions $\{\delta_s\}_{s \in S}$.

We know that the free module has a universal property: Given an A-module N and a function of sets $f : S \to N$, there is a unique A-module homomorphism

$$G(f) : G(S) \to N, \quad G(f)(\delta_s) = f(s).$$

We can interpret this as an adjunction: G is a left adjoint of F, and F is a right adjoint of G.

To be precise, given a set S and a module M we define the bijection

$$\alpha_{S,M} : \mathrm{Hom}_{\mathsf{Set}}\big(S, F(M)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(A)}\big(G(S), M\big)$$

to be

$$\alpha_{S,M}(f)(\delta_s) := f(s) \in M$$

for

$$f \in \mathrm{Hom}_{\mathsf{Set}}\big(S, F(M)\big)$$

and $s \in S$.

Something needs to be verified – see exercise.

Exercise 1.14. Prove that

$$\alpha := \{\alpha_{S,M}\}_{(S,M) \in \mathsf{Set} \times \mathsf{M}(A)}$$

is a functor

$$\alpha : \mathsf{Set}^{\mathrm{op}} \times \mathsf{M}(A) \to \mathsf{Set}\,.$$

Exercise 1.15. Let $\mathsf{Set}_{\mathrm{fin}}$ be the category of finite sets and let $\mathsf{Rng}_{\mathsf{c}}/\mathbb{K}$ be the category of commutative $\mathbb{K}$-rings.

We have the forgetful functor

$$F : \mathsf{Rng}_{\mathsf{c}}/\mathbb{K} \to \mathsf{Set},$$

and the polynomial ring functor

$$G : \mathsf{Set} \to \mathsf{Rng}_{\mathsf{c}}/\mathbb{K}, \quad G(S) = \mathbb{K}[S],$$

where $\mathbb{K}[S]$ is the ring of polymomials of the finite set of variables S.

Prove that G is a left adjoint of F.

Exercise 1.16. Let $f : A \to B$ be a homomorphism between commutative rings.

We have seen the restriction functor

$$\mathrm{Rest}_f : \mathsf{M}(B) \to \mathsf{M}(A),$$

which is just a forgetful functor.

We have also seen the induction functor

$$\mathrm{Ind}_f : \mathsf{M}(A) \to \mathsf{M}(B), \quad M \mapsto B \otimes_A M.$$

Prove that Ind_f is a left adjoint of Rest_f.

We end this section with an example. It will be made more general later, and that general *Hom-tensor adjunction* is a very important fact.

Example 1.17. Let A be a commutative ring. Given $L, M, N \in \mathsf{M}(A)$ there is an isomorphism

$$\mathrm{adj}_{L,M,N} : \mathrm{Hom}_A(L \otimes_A M, N) \xrightarrow{\simeq} \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N)) \tag{1.18}$$

in $\mathsf{M}(A)$, called Hom-tensor adjunction.

It is functorial in L, M, N – see Exercise 1.19.

The formula is this: given

$$\phi : L \otimes_A M \to N$$

we define $\mathrm{adj}_{L,M,N}(\phi)$ to be

$$\mathrm{adj}_{L,M,N}(\phi)(l)(m) := \phi(l \otimes M) \in N.$$

Conversely, given

$$\psi \in \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N))$$

we have a homomorphism

$$\chi : L \otimes_A M \to N$$

with formula

$$\chi(l \otimes m) := \psi(l)(m) \in N.$$

The homomorphism $\psi \mapsto \chi$ is the inverse of $\mathrm{adj}_{L,M,N}$.

Exercise 1.19. Prove that

$$\mathrm{adj} : \mathsf{M}(A)^{\mathrm{op}} \times \mathsf{M}(A)^{\mathrm{op}} \times \mathsf{M}(A) \to \mathsf{M}(A)$$

is a morphism of functors.

comment: Start of Lecture 2, 18 March 2020. Lecture via ZOOM.

It turns out that we shall need Theorems 1.5 and 1.6 for Morita theory.

Most likely I will prove these theorems using the Yoneda Lemma. This will be done next week.

I have a feeling that Exer 1.8, i.e. proving Proposition 1.7, was too difficult. But this will be taken care of when we prove Theorem 1.6.

Let me discuss [Yek1, Thm 13.15], whose proof was also an exercise, but I doubt many were able to solve it. Therefore I will give the proof now.

Consider a functor $F : \mathsf{C} \to \mathsf{D}$. For every pair of objects $C_0, C_1 \in \mathsf{C}$ there is a function of sets

$$F_{C_0,C_1} : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}(F(C_0), F(C_1)).$$

The functor F is called *full* (resp. *faithful*, resp. *fully faithful*) if the functions F_{C_0,C_1} are all surjective (resp. injective, resp. bijective).

The functor F is said to be *essentially surjective on objects* if for every object $D \in \mathsf{D}$ there exists some object $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$ in D.

Theorem 1.20. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. TFAE:*

(i) *F is an equivalence.*
(ii) *F is fully faithful and essentially surjective on objects.*

Proof. The proof is in a few steps.

Step 1. Assume F is an equivalence. In this step we shall prove that F is essentially surjective on objects.

Let $G : \mathsf{D} \to \mathsf{C}$ be a quasi-inverse of F, equipped with isomorphisms of functors

$$\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}}$$

and

$$\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}};$$

see formula (1.3) and (1.4).

Given an object $D \in \mathsf{D}$, let $C := G(D) \in \mathsf{C}$. We then have an isomorphism $\zeta_D : F(C) \xrightarrow{\simeq} D$ in D.

This establishes that F is essentially surjective on objects.

Step 2. Again assume F is an equivalence, with G, η and ζ as in step 1. Here we prove that F is faithful.

Take a pair of objects $C_1, C_2 \in \mathsf{C}$. Consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \arrow[r, "F_{C_1,C_2}"] \arrow[dr, "\simeq", "\mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1})"'] & \mathrm{Hom}_{\mathsf{D}}\big(F(C_1), F(C_2)\big) \arrow[d, "G_{F(C_1),F(C_2)}"] \\ & \mathrm{Hom}_{\mathsf{C}}\big((G \circ F)(C_1), (G \circ F)(C_2)\big) \end{tikzcd} \tag{1.21}$$

in Set. We claim it is commutative.

To see this, take a morphism $\gamma : C_1 \to C_2$ in C.

The image of γ by going right and then down in diagram (1.21) is $(G \circ F)(\gamma)$. Note that I am omitting the subscripts from $G \circ F$.

Because η is an isomorphism of functors, the diagram

$$\begin{tikzcd} (G \circ F)(C_1) \arrow[r, "(G\circ F)(\gamma)"] \arrow[d, "\eta_{C_1}"', "\simeq"] & (G \circ F)(C_2) \arrow[d, "\eta_{C_2}", "\simeq"'] \\ C_1 \arrow[r, "\gamma"] & C_2 \end{tikzcd} \tag{1.22}$$

is commutative.

The commutativity of diagram (1.22) says that

$$(G \circ F)(\gamma) = \mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1})(\gamma).$$

So indeed diagram (1.21) is commutative.

The commutativity of diagram (1.21) implies that the function F_{C_1,C_2} is injective.

Thus F is a faithful functor.

Step 3. Again assume F is an equivalence, with G, η and ζ as in step 1. Here we prove that F is full.

Doing step 2, but with the roles of F and G reversed, we see that G is faithful. Thus for every $D_1, D_2 \in \mathsf{D}$ the function

$$G : \mathrm{Hom}_{\mathsf{D}}(D_1, D_2) \to \mathrm{Hom}_{\mathsf{D}}(G(D_1), G(D_2))$$

is injective.

Taking $D_i := F(C_i)$, it follows that the function $G_{F(C_1),F(C_2)}$ in diagram (1.21) is injective.

But

$$G_{F(C_1),F(C_2)} \circ F_{C_1,C_2} = \mathrm{Hom}(\eta_{C_1}, \eta_{C_2}^{-1}),$$

and this is a bijection. We see that F_{C_1,C_2} is surjective.

Thus F is a full functor.

Step 4. Now we assume that F is fully faithful and essentially surjective on objects, and we construct a quasi-inverse G.

For every object $D \in \mathsf{D}$ choose an object $C \in \mathsf{C}$ with an isomorphism $\zeta_D : F(C) \xrightarrow{\simeq} D$ in D.

Define the function

$$G : \mathrm{Ob}(\mathsf{D}) \to \mathrm{Ob}(\mathsf{C})$$

by letting $G(D)$ be the object C chosen above. Thus we get an isomorphism

$$\zeta_D : (F \circ G)(D) \xrightarrow{\simeq} D$$

in D.

Step 5. Continuing from step 4, for a morphism $\psi : D_1 \to D_2$ in D define the morphism

$$G(\psi) : G(D_1) \to G(D_2)$$

as follows.

Let $\tilde{\psi}$ be the unique morphism in D for which the diagram

$$\begin{tikzcd} F(G(D_1)) \arrow[r, "\tilde{\psi}"] \arrow[d, "\zeta_{D_1}"', "\simeq"] & F(G(D_2)) \arrow[d, "\simeq"', "\zeta_{D_2}"] \\ D_1 \arrow[r, "\psi"] & D_2 \end{tikzcd} \tag{1.23}$$

is commutative.

Next let $G(\psi)$ be the unique morphism that goes to $\tilde{\psi}$ under the bijection

$$F : \mathrm{Hom}_{\mathsf{C}}\big(G(D_1), G(D_2)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}}\big(F(G(D_1)), F(G(D_2))\big).$$

Then G is a functor $\mathsf{D} \to \mathsf{C}$. (Exercise.)

By construction,

$$\zeta = \{\zeta_D\}_{D \in \mathsf{D}} : F \circ G \to \mathrm{Id}_{\mathsf{D}}$$

is an isomorphism of functors. (Exercise.)

Step 6. Finally we define the isomorphism

$$\eta_C : G(F(C)) \xrightarrow{\simeq} C,$$

for $C \in \mathsf{C}$, as follows.

From step 4 we already have an isomorphism

$$\zeta_{F(C)} : F(G(F(C))) \xrightarrow{\simeq} F(C)$$

in D.

Since F is fully faithful, there is a unique morphism η_C that goes to $\zeta_{F(C)}$ under the bijection

$$F : \mathrm{Hom}_{\mathsf{C}}\big(G(F(C), C\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}}\big(F(G(F(C))), F(C)\big).$$

Moreover, this η_C is an isomorphism, since $\zeta_{F(C)}$ is an isomorphism. (Exercise.)

By construction,

$$\eta = \{\eta_C\}_{C \in \mathsf{C}} : G \circ F \to \mathrm{Id}_{\mathsf{C}}$$

is an morphism of functors. (Exercise.) □

Exercise 1.24. Finish the arguments in the four places in the proof above marked "Exercise".

Let us examine a concrete situation of adjoint functors. This will be generalized later to the noncommutative setting.

Let $f : A \to B$ be a homomorphism of commutative rings. A B-module P is also an A-module by Rest_f.

Therefore it makes sense to look at the functors

$$G_P : \mathsf{M}(A) \to \mathsf{M}(B), \quad G_P(M) := P \otimes_A M. \tag{1.25}$$

and

$$F_P : \mathsf{M}(B) \to \mathsf{M}(A), \quad F_P(N) := \mathrm{Hom}_B(P, N). \tag{1.26}$$

These are both A-linear functors.

Example 1.27. If we take $P := B$, then $G_P = \mathrm{Ind}_f$ and $F_P = \mathrm{Rest}_f$.

Proposition 1.28. *Given a homomorphism $A \to B$ of commutative rings and a B-module P, the functor G_P is left adjoint to the functor F_P.*

Proof. We have to construct an A-linear isomorphism

$$\alpha_{M,N} : \mathrm{Hom}_{\mathsf{M}(A)}(M, F_P(N)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(B)}(G_P(M), N) \tag{1.29}$$

that's functorial in $M \in \mathsf{M}(A)$ and $N \in \mathsf{M}(B)$.

In other words, we are looking for a "canonical" A-linear homomorphism

$$\alpha_{M,N} : \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big) \to \mathrm{Hom}_B\big(P \otimes_A M, N\big).$$

"Canonical" will imply functorial. Als it should be bijective.

We define $\alpha_{M,N}$ as follows: for

$$\phi \in \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big),$$

$p \in P$ and $m \in M$, we let

$$\alpha_{M,N}(\phi)(p \otimes m) := \phi(m)(p) \in N.$$

This is a well-defined A-linear homomorphism, and it is functorial in M and N.

Next we construct and A-linear homomorphism

$$\beta_{M,N} : \mathrm{Hom}_B\big(P \otimes_A M, N\big) \to \mathrm{Hom}_A\big(M, \mathrm{Hom}_B(P, N)\big).$$

Its formula is

$$\beta_{M,N}(\psi)(m)(p) := \psi(p \otimes m) \in N.$$

This is a well-defined A-linear homomorphism.

The homomorphisms $\alpha_{M,N}$ and $\beta_{M,N}$ are inverses of each other. Hence $\alpha_{M,N}$ is an isomorphism. □

Remark 1.30. Prop 1.28 will be generalized greatly next week – the rings A and B will be noncommutative central $\mathbb{K}$-rings, and there won't be a ring homomorphism $A \to N$. Instead of a module P, we will have a B-A-module P. And so on. Even in this generality the functors G_P and F_P will exists, and they will be $\mathbb{K}$-linear, and adjoints to each other (with basically the same proof!).

Remark 1.31. Still in the commutative setting of Prop 1.28, assume that F_P is an equivalence. By Theorem 1.5 the functor G_P is a quasi-inverse of G_P. (And vice versa).

For the sake of simplicity A and B are noetherian. We will prove that $A = B$, and that the A-module P is projective of rank 1; i.e. P is a finitely generated projective A-module, and for every prime ideal $\mathfrak{p} \subseteq A$ the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free of rank 1.

comment: End of Lecture 2, 18 March 2020. (Lecture via ZOOM)

References

[AlKl] A. Altman and S. Kleiman, "A Term of Commutative Algebra", free online at http://www.centerofmathematics.com/wwcomstore/index.php/commalg.html.

[Art] M. Artin, "Algebra", Prentice-Hall.

[Eis] D. Eisenbud, "Commutative Algebra", Springer, 1994.

[Har] R. Hartshorne, "Algebraic Geometry", Springer-Verlag, New-York, 1977.

[HiSt] P.J. Hilton and U. Stammbach, "A Course in Homological Algebra", Springer, 1971.

[Jac] N. Jacobson, "Basic Algebra I-II", Freeman.

[Lang] S. Lang, "Algebra", Addison-Wesley.

[Mac1] S. Maclane, "Homology", Springer, 1994.

[Mac2] S. Maclane, "Categories for the Working Mathematician", Springer, 1978.

[Mats] H. Matsumura, "Commutative Ring Theory", Cambridge University Press, 1986.

[Mi1] J.S. Milne, "Étale cohomology", Princeton Univ. Press, 1980.

[Mi2] J.S. Milne, "Algebric Groups", Cambridge Univ. Press, 2017.

[Rot] J. Rotman, "An Introduction to Homological Algebra", Academic Press, 1979.

[Row] L.R. Rowen, "Ring Theory" (Student Edition), Academic Press, 1991.

[SP] "The Stacks Project", an online reference, J.A. de Jong (Editor), http://stacks.math.columbia.edu.

[Wei] C. Weibel, "An introduction to homological algebra", Cambridge Univ. Press, 1994.

[Yek1] A. Yekutieli, Course Notes: Commutative Algebra (2019-20), https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/comm-alg/course_page.html

[Yek2] A. Yekutieli, Course Notes: Homological Algebra (2017-18), https://www.math.bgu.ac.il/~amyekut/teaching/2017-18/hom-alg/course_page.html.

[Yek3] A. Yekutieli, "Derived Categories", Cambridge University Press, 2019.

[Yek4] A. Yekutieli, Course Notes: Algebraic Geometry – Schemes (2018-19), https://www.math.bgu.ac.il/~amyekut/teaching/2018-19/schemes-1/notes-190201.pdf and https://www.math.bgu.ac.il/~amyekut/teaching/2018-19/schemes-2/notes-190618-d2.pdf.

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