Course Homological Algebra

Lecture 5, 22 Apr. 2020

"Morita" in Alg. geometry
Derived Fourier-Mukai Transforms

$A, B, C \rightsquigarrow M \otimes_B N \in \underline{M}(A \otimes C^{op})$

$M \in \underline{M}(A \otimes B^{op})$, $N \in \underline{M}(B \otimes C^{op})$

$M, N \in \underline{M}(B) \rightsquigarrow \mathrm{Hom}_B(M, N)$ k-mod.

Prop 3.19 $M \in \underline{M}(B \otimes A^{op})$, $N \in \underline{M}(B \otimes C^{op})$

$$\Rightarrow \quad \mathrm{Hom}_B(M, N) \in \underline{M}(A \otimes C^{op})$$

$\varphi \in \mathrm{Hom}_B(M, N)$

$$\big((a \otimes op(c)) \cdot \varphi\big)(m) = \varphi(m \cdot a) \cdot c$$

Prop 3.20. The formula $(M, N) \mapsto \mathrm{Hom}_B(M, N)$ is a k-bilin functor

$$\underline{M}(B \otimes A^{op}) \times \underline{M}(B \otimes C^{op}) \to \underline{M}(A \otimes C^{op})$$

Exercises

Thm 3.22 Let P be an B-A-bimodule.

Define
$$G_P := P \otimes_A (-) : \underline{M}(A) \to \underline{M}(B)$$
$$F_P := \mathrm{Hom}_B(P, -) : \underline{M}(B) \to \underline{M}(A)$$

Then F_P & G_P are adjoint to each other.

F_P: right G_P: left

Proof. The pf is like the commutative case (Prop 1.28).

Need: isom of $\mathbb{Z}$-mods

$$(3.23)\quad \alpha_{M,N} : \mathrm{Hom}_{\underline{M}(A)}(M, F(N)) \xrightarrow{\sim} \mathrm{Hom}_{\underline{M}(B)}(G(M), N)$$

$$\varphi \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(P, N)) \qquad \mathrm{Hom}_B(P \otimes_A M, N) \ni \alpha(\varphi)$$

functorial in M & N.

$\varphi \in \mathrm{Hom}_A(M, \mathrm{Hom}_B(P, N))$

$\tilde{\alpha}_{M,N}(\varphi)(p, m) = \varphi(m)(p) \in N$

check! is A-bilinear.

Get $\alpha_{M,N}(\varphi) : P \otimes_A M \to N$ check: this is B-lin.

This is bifunctorial.

To prove that $\alpha_{M,N}$ is $\cong$, write the inverse — like Prop. 1.28. $\square$

4. Linear Equivalences & Exactness

$F : \underline{M}(A) \longrightarrow \underline{M}(B)$ linear functor.
F is exact if it takes short exact seq. to sh. ex. seq.

in $\underline{M}(A)$: s.e.s.

$$0 \longrightarrow M_0 \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_2 \longrightarrow 0$$

then $\Big\downarrow F$

$$0 \rightarrow F(M_0) \xrightarrow{F(\varphi_1)} F(M_1) \xrightarrow{F(\varphi_2)} F(M_2) \rightarrow 0$$

is a s.e.s.

Thm 4.5. Let A & B be rings, and let $F : \underline{M}(A) \to \underline{M}(B)$ be a linear functor, which is an equivalence of cats. ^instructures

Then:

(1) If $G : \underline{M}(B) \to \underline{M}(A)$ is a qu. inverse of F, then G is a linear functor.

(2) F is an exact functor.

Kernels in $M(A)$,

(K, k)

$$
\begin{tikzcd}
K' \arrow[d, "\exists!"'] \arrow[dr, "k'"] \arrow[drr, bend left, "0"] & & \\
K \arrow[r, "k"] \arrow[rr, bend right, "0"'] & M \arrow[r, "\varphi"] & N
\end{tikzcd}
$$

standard choice:

$K := \{ m \in M \mid \varphi(m) = 0 \} \xhookrightarrow{k} M$

$\boxed{\text{end } 1^{st} \text{ hour}}$

Cokernels

(C, c)

$\parallel$

cokernel of φ

$$
\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[rr, bend right, "0"'] \arrow[drr, bend right, "0"'] & N \arrow[r, "c"] \arrow[dr, "c'"] & C \arrow[d, "\exists!"] \\
& & C'
\end{tikzcd}
$$

standard:

$C := N/\varphi(M)$, c is can. proj.

$N \twoheadrightarrow N/\varphi(M)$

$F: \underline{M}(A) \longrightarrow \underline{M}(B)$ is lin. functor.

We say that $\boxed{F \text{ preserves kernels}}$ if for every $\varphi: M_1 \to M_2$ in $\underline{M}(A)$, with $\mathrm{Ker}(\varphi) = (K, k)$, the pair $\big(F(K), F(k)\big)$ is a kernel of $F(\varphi)$

in $\underline{M}(B)$

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[rr, bend left, "0"] & F(M) \arrow[r, "F(\varphi)"] & F(N) \\
(L, l) \arrow[u, dashed, "\exists !"] \arrow[ur, "l"'] \arrow[urr, "0"'] & &
\end{tikzcd}$$

Likewise "F preserves cokernels"

Lemma 4.3 Let $F: \underline{M}(A) \to \underline{M}(B)$ be some ~~linear~~ functor, TFAE:

(i) F is exact.

(ii) F preserves kernels and cokernels.

pf: Exercise.

Exer. Find $F: \underline{M}(A) \to \underline{M}(B)$ not linear

$A = B$ comm.

proof of Thm 4.5 (2) (partial)

Given that $F: \underline{M}(A) \to \underline{M}(B)$ is a lin. equiv. want to prove that F preserves kernels & cokernels.

Given

$$\begin{tikzcd} K \arrow[r, "k"] & M_0 \arrow[r, "\varphi"] & M_1 \\ K' \arrow[u, "\exists! \lambda"] \arrow[ur, "k'"] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

Ker(φ) in $\underline{M}(A)$

$\downarrow F$

is this a kernel of $F(\varphi)$?

$$\begin{tikzcd} F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(\varphi)"] & F(M_1) \\ L' = F(K') \arrow[u, dashed, "\exists! F(\lambda)"] \arrow[ur, "l'"] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

in $\underline{M}(B)$

F is ess. surj. on obj. $L' \cong F(K')$, $K' \in \underline{M}(A)$

Can assume $L' = F(K')$.

F is full, faithful: bij. on Hom.

[established "F pres. kernels".]

Exer Find $F: \underline{M}(A) \to \underline{M}(B)$ exact (linear) but not equivalence.

Morita Theory

Def **Projective Modules**: $P \in \underline{M}(A)$ is *projective*: if for every hom $\varphi: P \to N$ and every surjection $\pi: M \to N$ there exists a hom $\tilde{\varphi}: P \to M$ s.t.

$$\varphi = \pi \circ \tilde{\varphi}$$

$\underline{M}(A)$

$$\begin{tikzcd} & & P \arrow[d, "\varphi"] \arrow[dll, dashed, "\exists \tilde{\varphi}"'] \\ M \arrow[rr, two heads, "\pi"] & & N \end{tikzcd}$$

Prop 5.2. TFAE for an A-mod. P:

(i) P is proj,

(ii) every short exact seq.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & N \arrow[r, "\pi"] \arrow[l, bend left, "\tau"] & P \arrow[r] \arrow[l, bend left, "\sigma"] & 0 \end{tikzcd}$$

splits.

$N \cong P \oplus M$

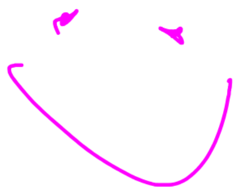
(iii) P is a direct summand of a free A-module; i.e. $\exists$ free mod. F & mod. P' s.t. $F \cong P \oplus P'$.

(iv) The functor $\mathrm{Hom}_A(P, -): \underline{M}(A) \to \underline{M}(K)$ is exact.

Explicit

$0 \neq A$ ring comm

nc ... $B = \mathrm{Mat}_r(A)$ $r \geq 2$



$$\underline{M}(A) \simeq \underline{M}(B)$$

$$\underline{M}(A) \underset{Q \otimes_B (-)}{\overset{P \otimes_A (-)}{\rightleftarrows}} \underline{M}(B)$$