Corrected

Local Rings & Stalks

4 Jan 2020

Recall that a ring A is <u>local</u> if it has exactly one maximal ideal.

Here's a useful criterion.

<u>Lemma</u> Let A be a ring and $\mathfrak{m} \subseteq A$ an ideal. TFAE:

(i) A is local and $\mathfrak{m}$ is its only max. ideal.

(ii) Every element $a \in A - \mathfrak{m}$ is invertible.

~~Proof~~. Exercise.

We now consider a top. space X, and the sheaf $\mathcal{O}_X$ of continuous $\mathbb{R}$-valued functions. ($\mathbb{R}$ has the usual metric topology.)

Exercise. Let $x \in X$ be a point. Show that $\mathcal{O}_{X,x} := (\mathcal{O}_X)_x$, the stalk

of $\mathcal{O}_X$ at x, is a local ring.
Show that the residue field(*) of $\mathcal{O}_{X,x}$ is $\mathbb{R}$.

(*) Suppose A is a local ring with max ideal $\mathfrak{m}$. The residue field of A is the field $A/\mathfrak{m}$.

Def Let A and B be local rings, with max. ideals $\mathfrak{m} \subseteq A$ and $\mathfrak{n} \subseteq B$. A ring hom $\varphi : A \to B$ is called a local homomorphism if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Exercise. Let $f : Y \to X$ be a map of top. spaces, let $y \in Y$, and let $x := f(y) \in X$. Show that f induces a local hom. $f^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$.

(Hint: for every $U \subseteq X$ open, with preimage

$V := f^{-1}(U) \subseteq Y$, there is an induced

$\mathbb{R}$-ring hom. $f^* : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y)$

with formula $f^*(\varphi) := \varphi \circ f : V \to \mathbb{R}$.)