Course Notes:

Algebraic Geometry – Schemes 1

Ben Gurion University
Fall Semester 2020-21

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Contents

Lecture 1, 21 Oct 2020

1. Introduction

First a few words on the format of the course.

The course will be held via the ZOOM video service, on Wednesdays at 12:10 - 14:00.

Formal registration is not required, unless you want a grade (pass/fail). Passing the course requires attending all lectures and submitting most of the HW (preferably typed).

However I expect all participants to attend all the course meetings (virtually, with cameras on), after the first two trial meetings.

The homework exercises are included in the course notes.

If you have not done so yet, please email me your name, academic status, and relevant mathematical knowledge.

Each week (usually on Tuesday) I will send an email with the zoom link for that week's lecture, and a draft of the typed lecture notes. A corrected version of the notes will be uploaded to the course web page after the lecture.

Participants are welcome to ask questions in real time (by unmuting your microphone), or by email to me or to Dr. Mattia Ornaghi (a postdoc who will assist me).

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Now a few words on prerequisites.

I expect all participants to have good knowledge of abstract topology.

We shall use *categories and functors* a lot. Since some students are not familiar with this topic (unfortunately), we will learn it quickly next week (with the help of homework).

Some *commutative algebra* is essential, like prime ideals, tensor products (of rings and modules), and localization.

Some understanding of *differential geometry* is needed, mostly for use in examples.

Galois theory will be used in examples. Knowing it is very useful, but not crucial, for this course.

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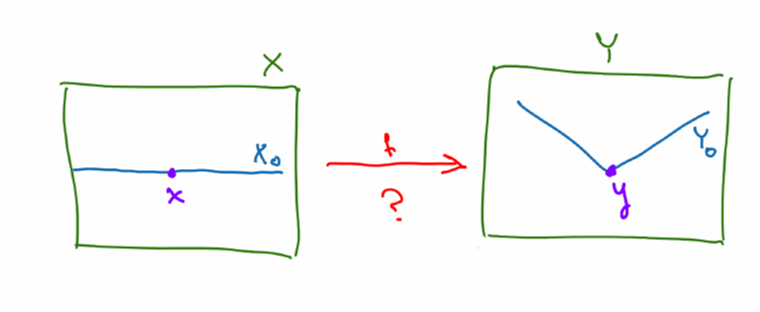
Let me say a few words on *geometry*.

By geometry I mean the mathematical study of spaces that have *more structure* than *topological spaces*. Usually (in modern mathematics) the geometric structure comes on top of a given underlying topological structure.

Here is an example from differential geometry. Let X and Y be the real plane $\mathbb{R}^2$, considered as a differentiable manifold (over $\mathbb{R}$ of type C^∞). Inside X we have a straight line X_0 and a point $x \in X_0$. Inside Y we have a broken line Y_0, and $y \in Y_0$ is the singularity (the breaking point). See picture:



Topologically the configurations $x \in X_0 \subseteq X$ and $y \in Y_0 \subseteq Y$ are indistinguishable. Namely we can find a homeomorphism $f : X \to Y$ such that $f(X_0) = Y_0$ and $f(x) = y$.

However, in differential geometry they are distinct geometric configurations: *there does not exist a diffeomorphism $f : X \to Y$ such that $f(X_0) = Y_0$ and $f(x) = y$.*

This can be seen by examining tangent directions at x and y. (Try to give full proofs of these assertions.)

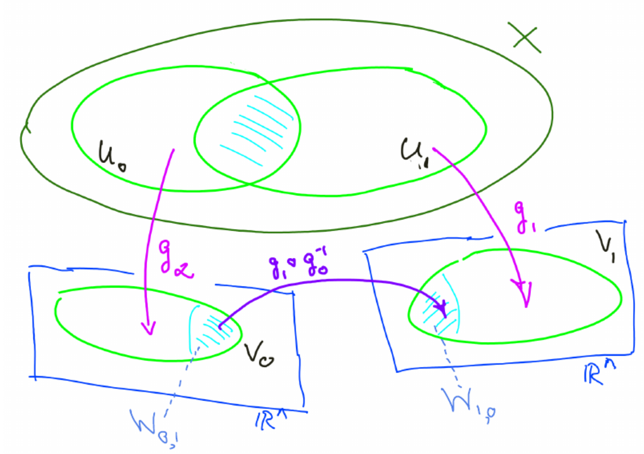
How do we describe the extra geometric structure on an n-dimensional differentiable manifold X, that comes on top of the given topology?

In earlier courses you learned that this is done by an *atlas*. Namely the space X has an open covering $X = \bigcup U_i$, and for each i there is a homeomorphism $g_i : U_i \to V_i$ to an open set $V_i \subseteq \mathbb{R}^n$. The condition is that the map

$$g_j \circ g_i^{-1} : W_{i,j} \to \mathbb{R}^n$$

where $W_{i,j} := g_i(U_i \cap U_j) \subseteq \mathbb{R}^n$, is differentiable for all i, j.

(1.1)



It turns out that instead of an atlas, the same geometric information can be encoded by declaring what are the differentiable functions $f : U \to \mathbb{R}$, for every open set $U \subseteq X$.

This means that for every open set $U \subseteq X$ we need to provide a ring of functions, let's denote it by $\Gamma(U, \mathcal{O}_X)$, and these rings must interact suitably w.r.t. inclusions $U' \subseteq U$.

This data $\mathcal{O}_X$ is called a *sheaf of rings* on X. The pair $(X, \mathcal{O}_X)$ is called a *locally ringed space.*

In the case of a differentiable manifold X, the ring $\Gamma(U, \mathcal{O}_X)$ is just the ring of differentiable functions $f : U \to \mathbb{R}$.

Moreover, we will see that given two differentiable manifolds $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a continuous map $f : Y \to X$ is differentiable iff a certain condition involving the sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, and the continuous function f, is satisfied.

For those fluent in the language of categories, the statement is that the category of differentiable manifolds embeds fully faithfully inside the category of locally $\mathbb{R}$-ringed spaces. This is something we are going to prove.

For purposes of differential geometry the sheaf approach is not essential; but for more complicated geometric settings, especially for algebraic geometry, the sheaf approach is crucial.

Indeed, a *scheme*, as defined by Grothendieck in the 1950's, is a locally ringed space $(X, \mathcal{O}_X)$, which locally is an *affine scheme*. Heuristically this means that there is an atlas like in (1.1), but the U_i are affine schemes.

An affine scheme is a geometric object that is totally controlled by its ring of functions. Indeed, for every (commutative) ring A there is an affine scheme $\operatorname{Spec}(A)$; and maps of schemes

$$f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

are the same as ring homomorphisms $f^* : A \to B$.

If there is time I will say more on affine schemes today.

Note that the role of *calculus* in differential geometry is played in algebraic geometry by the theory of *commutative algebra*.

We shall spend the next few weeks learning about *sheaves on topological spaces*, before we introduce schemes.

Here are some aspects of scheme theory that make it superior to classical algebraic geometry:

- Algebraic geometry over fields that are *not algebraically closed*. We will see the affine real line $\mathbf{A}^1_{\mathbb{R}}$ soon.
- In *arithmetic geometry* the schemes are defined over the base ring $\mathbb{Z}$; there is no base field at all. The prototypical example is the curve $(X, \mathcal{O}_X) = \operatorname{Spec}(\mathbb{Z})$, whose points are the prime ideals of $\mathbb{Z}$.
- In a scheme $(X, \mathcal{O}_X)$ the sheaf of rings $\mathcal{O}_X$ can have *nilpotent elements*. These nilpotents enable *algebraic infinitesimal calculus*, including the detection of singularities.

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The rest of this first lecture will consist of examples, without giving all the definitions – a kind of a preview of things to come.

Please let me know in real time if some concept I am talking about is not familiar to you, or if some assertion is not clear. I will either explain these, or defer them to later stages of the course.

In these examples we will work the the field of real numbers $\mathbb{R}$.

By an $\mathbb{R}$-*ring* we mean a commutative ring A, equipped with a ring homomorphism $\phi_A : \mathbb{R} \to A$, called the *structural homomorphism*. (Older books used the expression "$\mathbb{R}$-algebra".) If B is another $\mathbb{R}$-ring, then an *R-ring homomorphism* $\psi : A \to B$ is a ring homomorphism such that $\psi \circ \phi_A = \phi_B$.

All rings have unit elements, and these units must be preserved by ring homomorphisms, i.e. $\psi(1_A) = 1_B$.

We shall start with the ring $A := \mathbb{R}[t]$ of polynomials in one variable over the real numbers. It is an $\mathbb{R}$-ring in an obvious way.

Note that given an ideal $\mathfrak{a} \subseteq A$, the quotient ring $A/\mathfrak{a}$ is automatically an $\mathbb{R}$-ring, and the canonical surjection $\pi : A \to A/\mathfrak{a}$ is the unique $\mathbb{R}$-ring homomorphism from A to $A/\mathfrak{a}$.

For us A is the ring of global algebraic functions on the affine line $\mathbf{A}^1_{\mathbb{R}}$.

But what is the affine line $\mathbf{A}^1_{\mathbb{R}}$ as a geometric object?

The answer is this: $\mathbf{A}^1_{\mathbb{R}}$ is the *prime spectrum* of the ring $A = \mathbb{R}[t]$.

The notation is

$$\mathbf{A}^1_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[t]).$$

The set of points of the scheme $\mathbf{A}^1_{\mathbb{R}}$ are the *prime ideals* of the ring $A = \mathbb{R}[t]$.

We know that all prime ideals of A are *principal*, and they are of three kinds:

(i) The maximal ideals $\mathfrak{m} = (t - \lambda)$ for $\lambda \in \mathbb{R}$.

(ii) The maximal ideals $\mathfrak{m}$ generated by irreducible quadratic monic polynomials, such as $\mathfrak{m} = (t^2 + 1)$.

(iii) The prime ideal (0).

Correspondingly, the affine line $\mathbf{A}^1_{\mathbb{R}}$ has three kinds of points, and each point has a *residue field*:

(i) A point $x = \mathfrak{m} = (t - \lambda)$ with $\lambda \in \mathbb{R}$ is called an $\mathbb{R}$-*valued point of* $\mathbf{A}^1_{\mathbb{R}}$. The residue field of x is

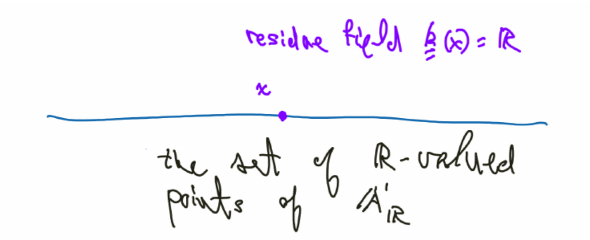
$$\boldsymbol{k}(x) = A/\mathfrak{m} \cong \mathbb{R}.$$

The "coordinate function" t has a value $t(x) \in \boldsymbol{k}(x)$, which by definition is the residue class of t modulo $\mathfrak{m}$.

But since there is a unique $\mathbb{R}$-ring isomorphism $\boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{R}$, we can say that the value is $t(x) = \lambda \in \mathbb{R}$.

The set of $\mathbb{R}$-valued points of $\mathbf{A}^1_{\mathbb{R}}$ is denoted by $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$. We see that there is a canonical bijection of sets $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R}) \xrightarrow{\simeq} \mathbb{R}$, $x \mapsto t(x)$.

Here is the picture:



For instance, the *origin* is the point $x \in \mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ s.t. $t(x) = 0$.

(ii) A point $x = \mathfrak{m}$ s.t. the maximal ideal $\mathfrak{m}$ is generated by a quadratic irreducible monic polynomial $p(t)$ has a residue field

$$\boldsymbol{k}(x) = A/\mathfrak{m} \cong \mathbb{C}.$$

There is no canonical $\mathbb{R}$*-ring isomorphism* $\boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$. Indeed, there are two equally good isomorphisms $\phi_i : \boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$, $i = 1, 2$, and they are related by $\phi_2 = \sigma \circ$

ϕ_1, where σ is complex conjugation. The isomorphism ϕ_i sends $t \mapsto \lambda_i$, where $\lambda_1, \lambda_2 \in \mathbb{C}$ are the two distinct roots of the polynomial $p(t)$.

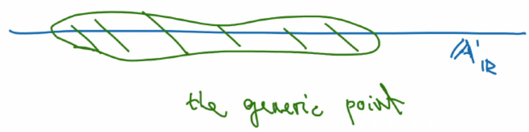
In other words, the value $t(x) \in \boldsymbol{k}(x)$ is well-defined: it is the residue class of t modulo $\mathfrak{m}$; but the value $t(x) = \lambda_i \in \mathbb{C}$ depends on the isomorphism $\phi_i : \boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$ chosen.

To be specific, let's consider $p(t) = t^2 + 1$. Then the value $t(x) \in \boldsymbol{k}(x)$ can be $\pm \boldsymbol{i} \in \mathbb{C}$.

For this reason it is confusing to draw a picture of the $\mathbb{C}$-valued points of $\mathbf{A}^1_{\mathbb{R}}$, and we shall try to avoid it...

(iii) The point $x = (0) \in \mathbf{A}^1_{\mathbb{R}}$, i.e. the zero ideal. This is called the *generic point* of $\mathbf{A}^1_{\mathbb{R}}$, for a reason we shall see later. Its residue field is $\boldsymbol{k}(x) \cong \mathbb{R}(t)$, the fraction field of $A = \mathbb{R}[t]$. The value $t(x) \in \boldsymbol{k}(x) \cong \mathbb{R}(t)$ is $t(x) = t$.

It is even less obvious how to draw the generic point. Often it is drawn as a blurb:



For the reasons explained above, in illustrations we shall usually just draw the set $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ of $\mathbb{R}$-valued points, or subsets of it.

More generally, when drawing an arbitrary scheme X, we shall usually pretend it is defined over $\mathbb{R}$, and then we'll draw "the set $X(\mathbb{R})$ of $\mathbb{R}$-valued points of X". This approach is usually more instructive.

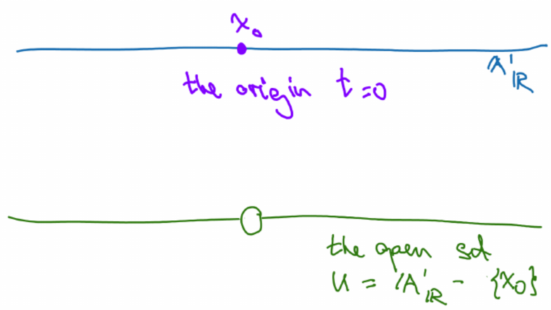
I will not talk about the *Zariski topology* of the space $X := \mathbf{A}^1_{\mathbb{R}}$ now, nor about the sheaf of "functions" $\mathcal{O}_X$ on X.

Let me only say that the points $x \in X = \mathbf{A}^1_{\mathbb{R}}$ corresponding to maximal ideals are *closed*, i.e. the subset $\{x\} \subseteq \mathbf{A}^1_{\mathbb{R}}$ is closed. Hence, removing any *finite* number of them gives an open subset $U \subseteq X$.

The point x corresponding to the ideal $(0) \subseteq \mathbb{R}[t]$ is *dense*, namely the closure of the set $\{x\}$ is the whole space $\mathbf{A}^1_{\mathbb{R}}$. This is why it is called the generic point.

If we remove the origin, i.e. the point $x_0 \in \mathbf{A}^1_{\mathbb{R}}$ s.t. $t(x_0) = 0$, then the "function" $t \in A = \mathbb{R}[t]$ becomes *invertible* on the open set $U := \mathbf{A}^1_{\mathbb{R}} - \{x_0\}$, and the ring of "functions" on U is the *localization*

$$\Gamma(U, \mathcal{O}_X) = A_t = A[t^{-1}] = \mathbb{R}[t, t^{-1}]. \tag{1.2}$$



Exercise 1.3. Let $(X, \mathcal{O}_X) := \mathbf{A}^1_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[t])$.

(1) Let $x \in X = \mathbf{A}^1_{\mathbb{R}}$ be a closed point, and let $U := X - \{x\}$.

Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$. (Hint: study the case $x = x_0$ above, but now the maximal ideal $\mathfrak{m} = x$ is generated by some irreducible monic polynomial $p(t)$.)

(2) Now $x_1, \ldots, x_l$ are finitely many distinct closed points in X, and $U := X - \{x_1, \ldots, x_l\}$. Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$.

Exercise 1.4. We know that the set of points of the affine scheme $(X, \mathcal{O}_X) := \operatorname{Spec}(\mathbb{Z})$ is the set of prime ideals in $\mathbb{Z}$.

Try to say what are the residue fields of the points $x \in X$. (Hint: look at the case of the affine line $\mathbf{A}^1_{\mathbb{R}}$, and make an analogy.)

Let $x \in X$ be a closed point, i.e. the ideal $x = \mathfrak{m} \subseteq \mathbb{Z}$ is maximal, and let $U := X - \{x\}$. Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$. (Hint: make an analogy to formula (1.2).)

End of live Lecture 1

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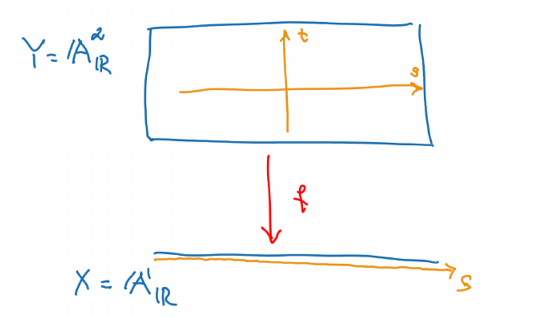
The material below is for self-reading before lecture 2. Some of it is pretty hard, and also the exercises are hard. This material is *optional* only, meant to make the introduction richer.

Consider the affine real line $X := \mathbf{A}^1_{\mathbb{R}}$ and the affine real plane $Y := \mathbf{A}^2_{\mathbb{R}}$. Writing $A := \mathbb{R}[s]$ and $B := \mathbb{R}[s, t]$, we have $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$.

Let $f : Y \to X$ be the projection on the first coordinate.

Here is a picture, showing only the sets of real points $X(\mathbb{R})$ and $Y(\mathbb{R})$.

(1.5)



The "coordinate axes" are meaningless in algebraic geometry, as is the limit $s \to \infty$ that the arrow tip on the s-axis indicates. I am drawing them only to help the imagination.

The effect of the projection f on functions is by pullback:

(1.6) $$f^* : A = \mathbb{R}[s] \to B = \mathbb{R}[s, t], \quad f^*(s) = s.$$

My picture seems to indicate that there is a well-defined function of sets $f : Y(\mathbb{R}) \to X(\mathbb{R})$. This would make sense only if $f(Y(\mathbb{R})) \subseteq X(\mathbb{R})$, and the next exercise shows that this is true.

Exercise 1.7. Here $X = \mathbf{A}^1_{\mathbb{R}}$ and $Y = \mathbf{A}^2_{\mathbb{R}}$, $f : Y \to X$ is the projection (1.5), and $\phi := f^* : A \to B$ is the ring homomorphism in formula (1.6).

(1) Suppose $\mathfrak{q} \subseteq B$ is a prime ideal, with preimage $\mathfrak{p} := \phi^{-1}(\mathfrak{q}) \subseteq A$. Prove that $\mathfrak{p}$ is a prime ideal of A.

We shall learn later that this is the way the map $f : Y \to X$ is recovered from the ring homomorphism ϕ. Namely, writing $x := \mathfrak{p}$ and $y := \mathfrak{q}$, we have $f(y) = x$.

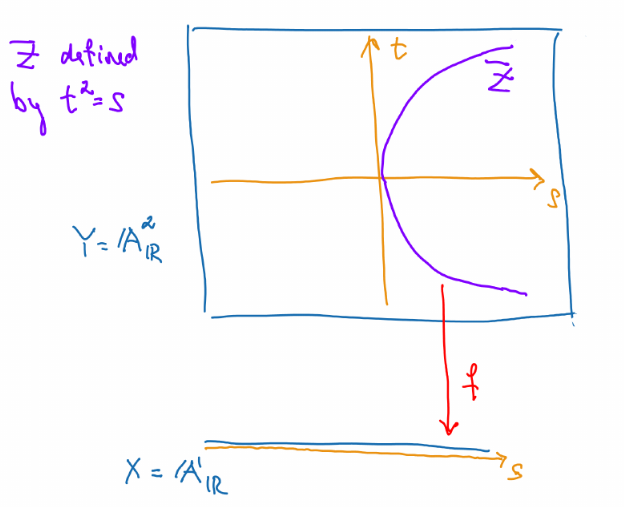
(2) Try to prove that if $y = \mathfrak{q}$ is in $Y(\mathbb{R})$ then $x = \mathfrak{p} = f(y) = \phi^{-1}(\mathfrak{q})$ is in $X(\mathbb{R})$. (Hint: the ring homomorphism $\phi : A \to B$ induces an *injective* $\mathbb{R}$-ring homomorphism $\bar{\phi} : A/\mathfrak{p} \to B/\mathfrak{q}$. The assumption $y \in X(\mathbb{R})$ says that $B/\mathfrak{q} \cong \mathbb{R}$ as $\mathbb{R}$-rings. Deduce that $\mathbb{R} \to A/\mathfrak{p}$ is also an isomorphism.)

As we will learn later, the rule $f \mapsto f^*$ gives a bijection between the set of maps of affine $\mathbb{R}$-schemes $f : Y \to X$ and the set of $\mathbb{R}$-ring homomorphisms $f^* : A \to B$.

For those familiar with categories, the precise statement is that Spec is a *duality*, namely a contravariant equivalence, between the category of $\mathbb{R}$-rings and the category of affine $\mathbb{R}$-schemes.

Next let Z be the parabola in the plane with equation $t^2 = s$.

Here is the set $Z(\mathbb{R})$ sitting inside the plane $Y(\mathbb{R})$:



The ring of algebraic functions of Z is

$$C := \mathbb{R}[s,t]/(t^2 - s).$$

Z is an affine scheme too: $Z = \mathrm{Spec}(C)$.

The projection $f : Y \to X$ restrict to a map of affine schemes $f : Z \to X$.

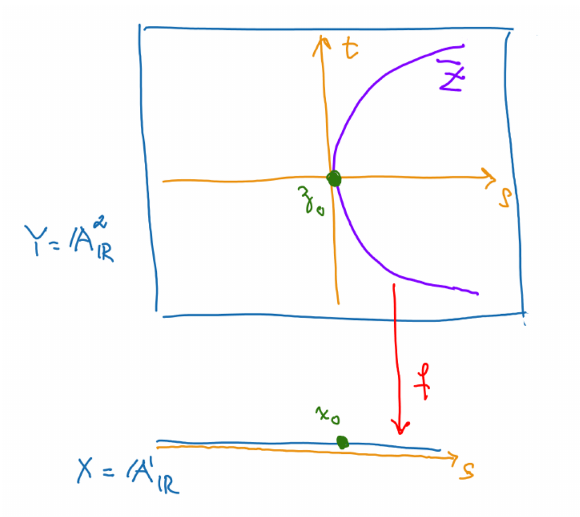
The ring homomorphism

$$f^* : A = \mathbb{R}[s] \to C = \mathbb{R}[s,t]/(t^2 - s)$$

is $f^*(s) = s$.

The projection $f : Z \to X$ is "not nice" at the origin $z_0 \in Z$, or above the origin $x_0 \in X$.

(1.8)



This is seen in several ways.

First, the size of the fibers: the fiber $f^{-1}(x_0)$ is a single point z_0, whereas all other fibers $f^{-1}(x)$ have size 2 or 0. (Warning: we are only looking at the function $f : Z(\mathbb{R}) \to X(\mathbb{R})$ between $\mathbb{R}$-valued points now.)

The second indication is that – when $f : Z(\mathbb{R}) \to X(\mathbb{R})$ is viewed as a map of differentiable manifolds – the map f is not a *local diffeomorphism* only at z_0. The induced linear map on tangent spaces degenerates at z_0. (This can also be stated in algebraic geometry).

Here is a third way to study the singularity of f at z_0.

In algebraic geometry the fiber of a map of schemes is a scheme. We will study this much later.

Here is the formula in our situation. Take a point $x \in X(\mathbb{R})$, and look at the ring

(1.9) $$C_x := C \otimes_A \boldsymbol{k}(x).$$

The fiber of f above x is the affine scheme

(1.10) $$Z_x := \operatorname{Spec}(C_x).$$

Exercise 1.11. Calculate the ring C_x for $x \in X(\mathbb{R})$ in the following cases.

(1) $s(x) > 0$. You should get $C_x \cong \mathbb{R} \times \mathbb{R} \cong \boldsymbol{k}(z_-) \times \boldsymbol{k}(z_+)$, where $f^{-1}(x) = \{z_-, z_+\}$.
(2) $x = x_0$ is the origin. Here you should get $C_x \cong \mathbb{R}[t]/(t^2)$. The element $t \in C_x$ is a nonzero nilpotent, and this is called *ramification*. It is the algebraic indication of singularity.
(3) $s(x) < 0$. You should get $C_x \cong \mathbb{C} \cong \boldsymbol{k}(z)$, where $f^{-1}(x) = \{z\}$. The point z is not in $Z(\mathbb{R})$, and this is why in our picture (1.8) the fiber $f^{-1}(x)$ looks empty.

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