Course Notes:

Algebraic Geometry – Schemes 1

Ben Gurion University
Fall Semester 2020-21

Amnon Yekutieli

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Contents

[1]New from 9 Nov 2020: subsections, and matching change of enumeration.

Lecture 1, 21 Oct 2020

1. Introduction

1.1. **Administration and Prerequisites.** First a few words on the format of the course.

The course will be held via the ZOOM video service, on Wednesdays at 12:10 - 14:00.

Formal registration is not required, unless you want a grade (pass/fail). Passing the course requires attending all lectures and submitting most of the HW (preferably typed).

However I expect all participants to attend all the course meetings (virtually, with cameras on), after the first two trial meetings.

The homework exercises are included in the course notes.

If you have not done so yet, please email me your name, academic status, and relevant mathematical knowledge.

Each week (usually on Tuesday) I will send an email with the zoom link for that week's lecture, and a draft of the typed lecture notes. A corrected version of the notes will be uploaded to the course web page after the lecture.

Participants are welcome to ask questions in real time (by unmuting your microphone), or by email to me or to Dr. Mattia Ornaghi (a postdoc who will assist me).

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Now a few words on prerequisites.

I expect all participants to have good knowledge of abstract topology.

We shall use *categories and functors* a lot. Since some students are not familiar with this topic (unfortunately), we will learn it quickly next week (with the help of homework).

Some *commutative algebra* is essential, like prime ideals, tensor products (of rings and modules), and localization.

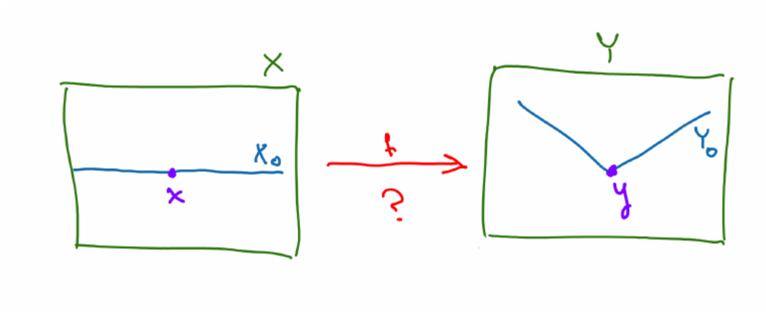
Some understanding of *differential geometry* is needed, mostly for use in examples.

Galois theory will be used in examples. Knowing it is very useful, but not crucial, for this course.

1.2. **On Geometry.** Let me say a few words on *geometry*.

By geometry I mean the mathematical study of spaces that have *more structure* than *topological spaces*. Usually (in modern mathematics) the geometric structure comes on top of a given underlying topological structure.

Here is an example from differential geometry. Let X and Y be the real plane $\mathbb{R}^2$, considered as a differentiable manifold (over $\mathbb{R}$ of type C^∞). Inside X we have a straight line X_0 and a point $x \in X_0$. Inside Y we have a broken line Y_0, and $y \in Y_0$ is the singularity (the breaking point). See picture:



Topologically the configurations $x \in X_0 \subseteq X$ and $y \in Y_0 \subseteq Y$ are indistinguishable. Namely we can find a homeomorphism $f : X \to Y$ such that $f(X_0) = Y_0$ and $f(x) = y$.

However, in differential geometry they are distinct geometric configurations: *there does not exist a diffeomorphism $f : X \to Y$ such that $f(X_0) = Y_0$ and $f(x) = y$.*

This can be seen by examining tangent directions at x and y. (Try to give full proofs of these assertions.)

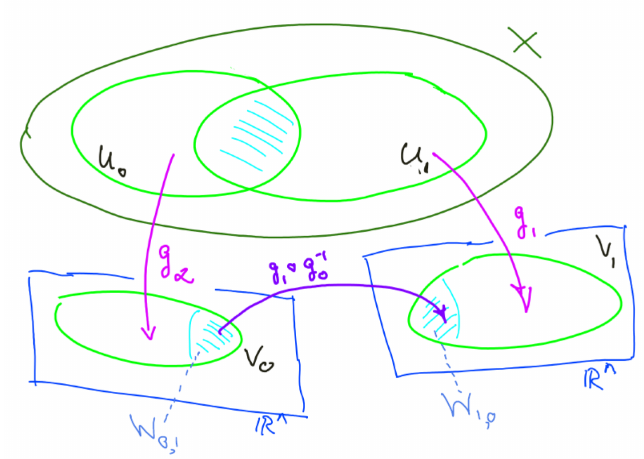
How do we describe the extra geometric structure on an n-dimensional differentiable manifold X, that comes on top of the given topology?

In earlier courses you learned that this is done by an *atlas*. Namely the space X has an open covering $X = \bigcup U_i$, and for each i there is a homeomorphism $g_i : U_i \to V_i$ to an open set $V_i \subseteq \mathbb{R}^n$. The condition is that the map

$$g_j \circ g_i^{-1} : W_{i,j} \to \mathbb{R}^n$$

where $W_{i,j} := g_i(U_i \cap U_j) \subseteq \mathbb{R}^n$, is differentiable for all i, j.

(1.2.1)



It turns out that instead of an atlas, the same geometric information can be encoded by declaring what are the differentiable functions $f : U \to \mathbb{R}$, for every open set $U \subseteq X$.

This means that for every open set $U \subseteq X$ we need to provide a ring of functions, let's denote it by $\Gamma(U, \mathcal{O}_X)$, and these rings must interact suitably w.r.t. inclusions $U' \subseteq U$.

This data $\mathcal{O}_X$ is called a *sheaf of rings* on X. The pair $(X, \mathcal{O}_X)$ is called a *locally ringed space.*

In the case of a differentiable manifold X, the ring $\Gamma(U, \mathcal{O}_X)$ is just the ring of differentiable functions $f : U \to \mathbb{R}$.

Moreover, we will see that given two differentiable manifolds $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a continuous map $f : Y \to X$ is differentiable iff a certain condition involving the sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, and the continuous function f, is satisfied.

For those fluent in the language of categories, the statement is that the category of differentiable manifolds embeds fully faithfully inside the category of locally $\mathbb{R}$-ringed spaces. This is something we are going to prove.

For purposes of differential geometry the sheaf approach is not essential; but for more complicated geometric settings, especially for algebraic geometry, the sheaf approach is crucial.

Indeed, a *scheme*, as defined by Grothendieck in the 1950's, is a locally ringed space $(X, \mathcal{O}_X)$, which locally is an *affine scheme*. Heuristically this means that there is an atlas like in (1.2.1), but the U_i are affine schemes.

An affine scheme is a geometric object that is totally controlled by its ring of functions. Indeed, for every (commutative) ring A there is an affine scheme $\mathrm{Spec}(A)$; and maps of schemes

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

are the same as ring homomorphisms $f^* : A \to B$.

If there is time I will say more on affine schemes today.

Note that the role of *calculus* in differential geometry is played in algebraic geometry by the theory of *commutative algebra*.

We shall spend the next few weeks learning about *sheaves on topological spaces*, before we introduce schemes.

Here are some aspects of scheme theory that make it superior to classical algebraic geometry:

- Algebraic geometry over fields that are *not algebraically closed*. We will see the affine real line $\mathbf{A}^1_{\mathbb{R}}$ soon.
- In *arithmetic geometry* the schemes are defined over the base ring $\mathbb{Z}$; there is no base field at all. The prototypical example is the curve $(X, \mathcal{O}_X) = \mathrm{Spec}(\mathbb{Z})$, whose points are the prime ideals of $\mathbb{Z}$.
- In a scheme $(X, \mathcal{O}_X)$ the sheaf of rings $\mathcal{O}_X$ can have *nilpotent elements*. These nilpotents enable *algebraic infinitesimal calculus*, including the detection of singularities.

1.3. **Examples.** The rest of this first lecture will consist of examples, without giving all the definitions – a kind of a preview of things to come.

Please let me know in real time if some concept I am talking about is not familiar to you, or if some assertion is not clear. I will either explain these, or defer them to later stages of the course.

In these examples we will work the the field of real numbers $\mathbb{R}$.

By an $\mathbb{R}$-*ring* we mean a commutative ring A, equipped with a ring homomorphism $\phi_A : \mathbb{R} \to A$, called the *structural homomorphism.* (Older books used the expression "$\mathbb{R}$-algebra".) If B is another $\mathbb{R}$-ring, then an *R-ring homomorphism* $\psi : A \to B$ is a ring homomorphism such that $\psi \circ \phi_A = \phi_B$.

All rings have unit elements, and these units must be preserved by ring homomorphisms, i.e. $\psi(1_A) = 1_B$.

We shall start with the ring $A := \mathbb{R}[t]$ of polynomials in one variable over the real numbers. It is an $\mathbb{R}$-ring in an obvious way.

Note that given an ideal $\mathfrak{a} \subseteq A$, the quotient ring $A/\mathfrak{a}$ is automatically an $\mathbb{R}$-ring, and the canonical surjection $\pi : A \to A/\mathfrak{a}$ is the unique $\mathbb{R}$-ring homomorphism from A to $A/\mathfrak{a}$.

For us A is the ring of global algebraic functions on the affine line $\mathbf{A}^1_{\mathbb{R}}$.

But what is the affine line $\mathbf{A}^1_{\mathbb{R}}$ as a geometric object?

The answer is this: $\mathbf{A}^1_{\mathbb{R}}$ is the *prime spectrum* of the ring $A = \mathbb{R}[t]$.

The notation is

$$\mathbf{A}^1_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[t]).$$

The set of points of the scheme $\mathbf{A}^1_{\mathbb{R}}$ are the *prime ideals* of the ring $A = \mathbb{R}[t]$.

We know that all prime ideals of A are *principal*, and they are of three kinds:

(i) The maximal ideals $\mathfrak{m} = (t - \lambda)$ for $\lambda \in \mathbb{R}$.

(ii) The maximal ideals $\mathfrak{m}$ generated by irreducible quadratic monic polynomials, such as $\mathfrak{m} = (t^2 + 1)$.

(iii) The prime ideal (0).

Correspondingly, the affine line $\mathbf{A}^1_{\mathbb{R}}$ has three kinds of points, and each point has a *residue field*:

(i) A point $x = \mathfrak{m} = (t - \lambda)$ with $\lambda \in \mathbb{R}$ is called an $\mathbb{R}$-*valued point of* $\mathbf{A}^1_{\mathbb{R}}$. The residue field of x is

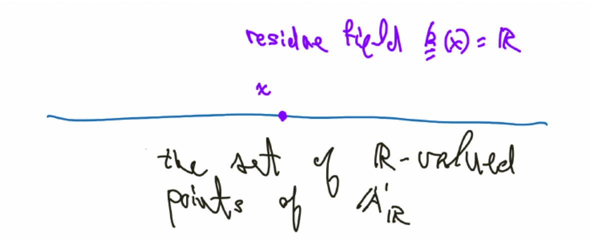
$$\boldsymbol{k}(x) = A/\mathfrak{m} \cong \mathbb{R}.$$

The "coordinate function" t has a value $t(x) \in \boldsymbol{k}(x)$, which by definition is the residue class of t modulo $\mathfrak{m}$.

But since there is a unique $\mathbb{R}$-ring isomorphism $\boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{R}$, we can say that the value is $t(x) = \lambda \in \mathbb{R}$.

The set of $\mathbb{R}$-valued points of $\mathbf{A}^1_{\mathbb{R}}$ is denoted by $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$. We see that there is a canonical bijection of sets $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R}) \xrightarrow{\simeq} \mathbb{R}$, $x \mapsto t(x)$.

Here is the picture:



For instance, the *origin* is the point $x \in \mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ s.t. $t(x) = 0$.

(ii) A point $x = \mathfrak{m}$ s.t. the maximal ideal $\mathfrak{m}$ is generated by a quadratic irreducible monic polynomial $p(t)$ has a residue field

$$\boldsymbol{k}(x) = A/\mathfrak{m} \cong \mathbb{C}.$$

There is no canonical $\mathbb{R}$*-ring isomorphism* $\boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$. Indeed, there are two equally good isomorphisms $\phi_i : \boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$, $i = 1, 2$, and they are related by $\phi_2 = \sigma \circ$

ϕ_1, where σ is complex conjugation. The isomorphism ϕ_i sends $t \mapsto \lambda_i$, where $\lambda_1, \lambda_2 \in \mathbb{C}$ are the two distinct roots of the polynomial $p(t)$.

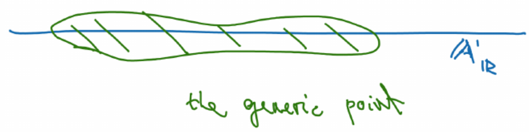
In other words, the value $t(x) \in \boldsymbol{k}(x)$ is well-defined: it is the residue class of t modulo $\mathfrak{m}$; but the value $t(x) = \lambda_i \in \mathbb{C}$ depends on the isomorphism $\phi_i : \boldsymbol{k}(x) \xrightarrow{\simeq} \mathbb{C}$ chosen.

To be specific, let's consider $p(t) = t^2 + 1$. Then the value $t(x) \in \boldsymbol{k}(x)$ can be $\pm \boldsymbol{i} \in \mathbb{C}$.

For this reason it is confusing to draw a picture of the $\mathbb{C}$-valued points of $\mathbf{A}^1_{\mathbb{R}}$, and we shall try to avoid it...

(iii) The point $x = (0) \in \mathbf{A}^1_{\mathbb{R}}$, i.e. the zero ideal. This is called the *generic point* of $\mathbf{A}^1_{\mathbb{R}}$, for a reason we shall see later. Its residue field is $\boldsymbol{k}(x) \cong \mathbb{R}(t)$, the fraction field of $A = \mathbb{R}[t]$. The value $t(x) \in \boldsymbol{k}(x) \cong \mathbb{R}(t)$ is $t(x) = t$.

It is even less obvious how to draw the generic point. Often it is drawn as a blurb:



For the reasons explained above, in illustrations we shall usually just draw the set $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ of $\mathbb{R}$-valued points, or subsets of it.

More generally, when drawing an arbitrary scheme X, we shall usually pretend it is defined over $\mathbb{R}$, and then we'll draw "the set $X(\mathbb{R})$ of $\mathbb{R}$-valued points of X". This approach is usually more instructive.

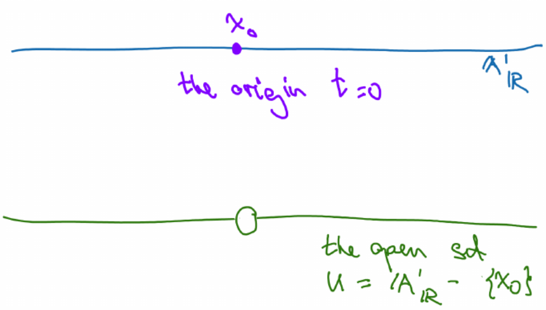
I will not talk about the *Zariski topology* of the space $X := \mathbf{A}^1_{\mathbb{R}}$ now, nor about the sheaf of "functions" $\mathcal{O}_X$ on X.

Let me only say that the points $x \in X = \mathbf{A}^1_{\mathbb{R}}$ corresponding to maximal ideals are *closed*, i.e. the subset $\{x\} \subseteq \mathbf{A}^1_{\mathbb{R}}$ is closed. Hence, removing any *finite* number of them gives an open subset $U \subseteq X$.

The point x corresponding to the ideal $(0) \subseteq \mathbb{R}[t]$ is *dense*, namely the closure of the set $\{x\}$ is the whole space $\mathbf{A}^1_{\mathbb{R}}$. This is why it is called the generic point.

If we remove the origin, i.e. the point $x_0 \in \mathbf{A}^1_{\mathbb{R}}$ s.t. $t(x_0) = 0$, then the "function" $t \in A = \mathbb{R}[t]$ becomes *invertible* on the open set $U := \mathbf{A}^1_{\mathbb{R}} - \{x_0\}$, and the ring of "functions" on U is the *localization*

(1.3.1) $$\Gamma(U, \mathcal{O}_X) = A_t = A[t^{-1}] = \mathbb{R}[t, t^{-1}].$$



Exercise 1.3.2. Let $(X, \mathcal{O}_X) := \mathbf{A}^1_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[t])$.

(1) Let $x \in X = \mathbf{A}^1_{\mathbb{R}}$ be a closed point, and let $U := X - \{x\}$.

Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$. (Hint: study the case $x = x_0$ above, but now the maximal ideal $\mathfrak{m} = x$ is generated by some irreducible monic polynomial $p(t)$.)

(2) Now $x_1, \ldots, x_l$ are finitely many distinct closed points in X, and $U := X - \{x_1, \ldots, x_l\}$. Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$.

Exercise 1.3.3. We know that the set of points of the affine scheme $(X, \mathcal{O}_X) := \operatorname{Spec}(\mathbb{Z})$ is the set of prime ideals in $\mathbb{Z}$.

Try to say what are the residue fields of the points $x \in X$. (Hint: look at the case of the affine line $\mathbf{A}^1_{\mathbb{R}}$, and make an analogy.)

Let $x \in X$ be a closed point, i.e. the ideal $x = \mathfrak{m} \subsetneq \mathbb{Z}$ is maximal, and let $U := X - \{x\}$. Try to say what is the ring of functions $\Gamma(U, \mathcal{O}_X)$. (Hint: make an analogy to formula (1.3.1).)

End of live Lecture 1

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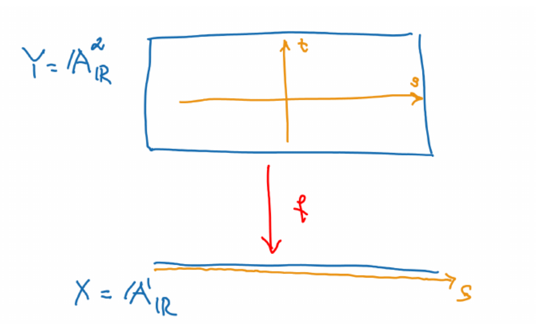
The material below is for self-reading before lecture 2. Some of it is pretty hard, and also the exercises are hard. This material is *optional* only, meant to make the introduction richer.

Consider the affine real line $X := \mathbf{A}^1_{\mathbb{R}}$ and the affine real plane $Y := \mathbf{A}^2_{\mathbb{R}}$. Writing $A := \mathbb{R}[s]$ and $B := \mathbb{R}[s, t]$, we have $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$.

Let $f : Y \to X$ be the projection on the first coordinate.

Here is a picture, showing only the sets of real points $X(\mathbb{R})$ and $Y(\mathbb{R})$.

(1.3.4)



The "coordinate axes" are meaningless in algebraic geometry, as is the limit $s \to \infty$ that the arrow tip on the s-axis indicates. I am drawing them only to help the imagination.

The effect of the projection f on functions is by pullback:

$$f^* : A = \mathbb{R}[s] \to B = \mathbb{R}[s, t], \quad f^*(s) = s. \tag{1.3.5}$$

My picture seems to indicate that there is a well-defined function of sets $f : Y(\mathbb{R}) \to X(\mathbb{R})$. This would make sense only if $f(Y(\mathbb{R})) \subseteq X(\mathbb{R})$, and the next exercise shows that this is true.

Exercise 1.3.6. Here $X = \mathbf{A}^1_{\mathbb{R}}$ and $Y = \mathbf{A}^2_{\mathbb{R}}$, $f : Y \to X$ is the projection (1.3.4), and $\phi := f^* : A \to B$ is the ring homomorphism in formula (1.3.5).

(1) Suppose $\mathfrak{q} \subseteq B$ is a prime ideal, with preimage $\mathfrak{p} := \phi^{-1}(\mathfrak{q}) \subseteq A$. Prove that $\mathfrak{p}$ is a prime ideal of A.

We shall learn later that this is the way the map $f : Y \to X$ is recovered from the ring homomorphism ϕ. Namely, writing $x := \mathfrak{p}$ and $y := \mathfrak{q}$, we have $f(y) = x$.

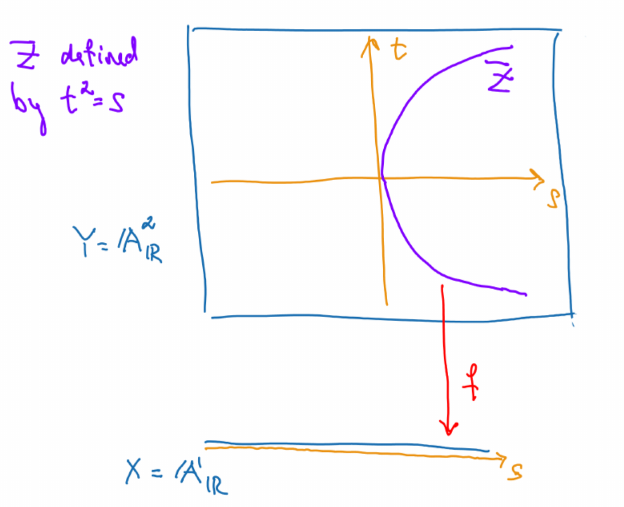
(2) Try to prove that if $y = \mathfrak{q}$ is in $Y(\mathbb{R})$ then $x = \mathfrak{p} = f(y) = \phi^{-1}(\mathfrak{q})$ is in $X(\mathbb{R})$. (Hint: the ring homomorphism $\phi : A \to B$ induces an *injective* $\mathbb{R}$-ring homomorphism $\bar{\phi} : A/\mathfrak{p} \to B/\mathfrak{q}$. The assumption $y \in X(\mathbb{R})$ says that $B/\mathfrak{q} \cong \mathbb{R}$ as $\mathbb{R}$-rings. Deduce that $\mathbb{R} \to A/\mathfrak{p}$ is also an isomorphism.)

As we will learn later, the rule $f \mapsto f^*$ gives a bijection between the set of maps of affine $\mathbb{R}$-schemes $f : Y \to X$ and the set of $\mathbb{R}$-ring homomorphisms $f^* : A \to B$.

For those familiar with categories, the precise statement is that Spec is a *duality*, namely a contravariant equivalence, between the category of $\mathbb{R}$-rings and the category of affine $\mathbb{R}$-schemes.

Next let Z be the parabola in the plane with equation $t^2 = s$.

Here is the set $Z(\mathbb{R})$ sitting inside the plane $Y(\mathbb{R})$:



The ring of algebraic functions of Z is

$$C := \mathbb{R}[s, t]/(t^2 - s).$$

Z is an affine scheme too: $Z = \mathrm{Spec}(C)$.

The projection $f : Y \to X$ restrict to a map of affine schemes $f : Z \to X$.

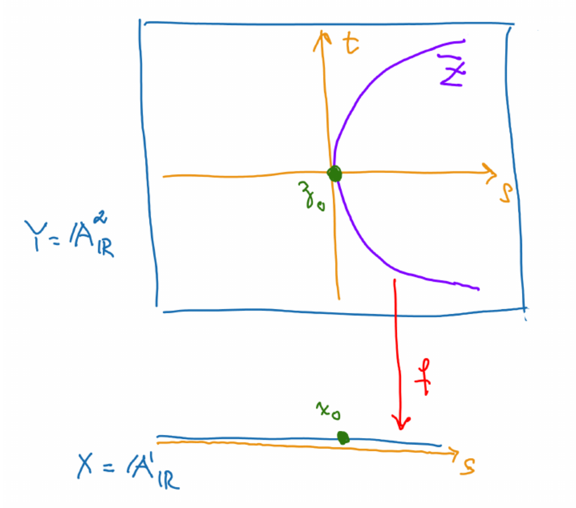
The ring homomorphism

$$f^* : A = \mathbb{R}[s] \to C = \mathbb{R}[s, t]/(t^2 - s)$$

is $f^*(s) = s$.

The projection $f : Z \to X$ is "not nice" at the origin $z_0 \in Z$, or above the origin $x_0 \in X$.

(1.3.7)



This is seen in several ways.

First, the size of the fibers: the fiber $f^{-1}(x_0)$ is a single point z_0, whereas all other fibers $f^{-1}(x)$ have size 2 or 0. (Warning: we are only looking at the function $f : Z(\mathbb{R}) \to X(\mathbb{R})$ between $\mathbb{R}$-valued points now.)

The second indication is that – when $f : Z(\mathbb{R}) \to X(\mathbb{R})$ is viewed as a map of differentiable manifolds – the map f is not a *local diffeomorphism* only at z_0. The induced linear map on tangent spaces degenerates at z_0. (This can also be stated in algebraic geometry).

Here is a third way to study the singularity of f at z_0.

In algebraic geometry the fiber of a map of schemes is a scheme. We will study this much later.

Here is the formula in our situation. Take a point $x \in X(\mathbb{R})$, and look at the ring

(1.3.8) $$C_x := C \otimes_A \boldsymbol{k}(x).$$

The fiber of f above x is the affine scheme

(1.3.9) $$Z_x := \operatorname{Spec}(C_x).$$

Exercise 1.3.10. Calculate the ring C_x for $x \in X(\mathbb{R})$ in the following cases.

(1) $s(x) > 0$. You should get $C_x \cong \mathbb{R} \times \mathbb{R} \cong \boldsymbol{k}(z_-) \times \boldsymbol{k}(z_+)$, where $f^{-1}(x) = \{z_-, z_+\}$.
(2) $x = x_0$ is the origin. Here you should get $C_x \cong \mathbb{R}[t]/(t^2)$. The element $t \in C_x$ is a nonzero nilpotent, and this is called *ramification*. It is the algebraic indication of singularity.
(3) $s(x) < 0$. You should get $C_x \cong \mathbb{C} \cong \boldsymbol{k}(z)$, where $f^{-1}(x) = \{z\}$. The point z is not in $Z(\mathbb{R})$, and this is why in our picture (1.3.7) the fiber $f^{-1}(x)$ looks empty.

Lecture 2, 28 Oct 2020

2. Category Theory

This concept is going to be extremely important for us, and some students are not familair with it.

I prepared too much material for one lecture, so most likely we will continue with it into lecture 3.

2.1. Categories.

Definition 2.1.1. A *category* C is a mathematical system consisting of:

- A set $\operatorname{Ob}(\mathsf{C})$, called the set of *objects* of C.
- For every pair $C_0, C_1 \in \operatorname{Ob}(\mathsf{C})$, there is a set $\operatorname{Hom}_{\mathsf{C}}(C_0, C_1)$, called the set of *morphisms* from C_0 to C_1.
- For every triple $C_0, C_1, C_2 \in \operatorname{Ob}(\mathsf{C})$, there is a function
$$\circ : \operatorname{Hom}_{\mathsf{C}}(C_1, C_2) \times \operatorname{Hom}_{\mathsf{C}}(C_0, C_1) \to \operatorname{Hom}_{\mathsf{C}}(C_0, C_2)$$
called *composition.*
- For every $C \in \operatorname{Ob}(\mathsf{C})$, there is a morphism
$$\mathrm{id}_C \in \operatorname{Hom}_{\mathsf{C}}(C, C),$$
called the *identity morphism.*

These are the axioms:

- ▹ Composition is associative: given objects $C_0, C_1, C_2, C_3 \in \operatorname{Ob}(\mathsf{C})$ and morphisms $f_i \in \operatorname{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$, there is equality
$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1 \in \operatorname{Hom}_{\mathsf{C}}(C_0, C_3).$$
- ▹ The identity morphisms are neutral for composition: for every $f \in \operatorname{Hom}_{\mathsf{C}}(C_0, C_1)$ there is equality
$$f \circ \mathrm{id}_{C_0} = f = \mathrm{id}_{C_1} \circ f.$$

We usually write $f : C \to D$ to mean that $f \in \operatorname{Hom}_{\mathsf{C}}(C, D)$.

Another notational convention is to write $C \in \mathsf{C}$ instead of $C \in \operatorname{Ob}(\mathsf{C})$.

Definition 2.1.2. A morphism $f : C \to D$ in a category C is called an *isomorphism* if there is a morphism $g : D \to C$ such that $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$.

Exercise 2.1.3. Let $C, D \in \mathsf{C}$.

(1) Prove that id_C is an isomorphism in C. (For this reason we call it the *identity automorphism* of the object C.)
(2) Suppose $f : C \to D$ is an isomorphism in C. Prove that there is exactly one morphism $g : D \to C$ in C satisfying $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$. (For this reason we call g *the inverse of* f, and denote t by f^{-1}.)

Example 2.1.4. The category Set has all sets as its objects. The morphisms $f : X \to Y$ between $X, Y \in \mathrm{Ob}(\mathsf{Set})$ are the functions. The identity morphisms are the identity functions, and composition is the usual one.

Remark 2.1.5. The example above brings us to a variant of *Russell's Paradox*: is the set $X := \mathrm{Ob}(\mathsf{Set})$ an element of itself?

There are two main ways to avoid this set-theoretical difficulty. Both work by installing a size hierarchy among sets.

(1) Modify Definition 2.1.1, declaring that $\mathrm{Ob}(\mathsf{C})$ is a *class*. This requires defining "class", and this is done in the *von Neumann – Bernays – Gödel set theory*.

(2) Choose a *Grothendieck universe* U, which is some big set. Elements of U are called small sets. Then declare that $\mathrm{Ob}(\mathsf{C}) \subseteq \mathsf{U}$ and $\mathrm{Hom}_{\mathsf{C}}(C, D) \in \mathsf{U}$.

We will take the second way. Thus, for us (implicitly) Set is the category of small sets.

As usual with foundations, these issues are not really important or interesting. It is the algebraic properties of categories (and functors, etc.) that are interesting and useful.

Students who want to read more about these matters can look in [Mac2, Chapter I].

Example 2.1.6. Fix a ring A. The category $\mathsf{Mod}\, A$ of A-modules has all A-modules as its objects. (Regarding size: as in Remark 2.1.5: the ring A and the modules M are small sets.) The morphisms $\phi : M \to N$ between $M, N \in \mathrm{Ob}(\mathsf{Mod}\, A)$ are the A-module homomorphisms. The identity morphisms are the identity functions, and composition is the usual one.

Example 2.1.7. Let $\phi : M \to N$ be an A-module homomorphism, i.e. a morphism in $\mathsf{Mod}\, A$.

Consider this diagram

(2.1.8)
$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & N \\ {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle \epsilon} \\ M/\mathrm{Ker}(\phi) & \xrightarrow[\simeq]{\ \bar{\phi}\ } & \mathrm{Im}(\phi) \end{array}$$

in the category $\mathsf{Mod}\, A$.

It consists of four objects and four morphisms. The vertical morphisms are the canonical surjection π and the inclusion ϵ. The bottom isomorphism $\bar{\phi}$ is the canonical one.

This diagram is *commutative*. By this we mean that the two morphisms we get from M to N, namely ϕ and the composition of the other three morphisms, are equal.

Example 2.1.9. The category Rng of (commutative) rings has all rings as its objects. The morphisms $f : A \to B$ between $A, B \in \mathrm{Ob}(\mathsf{Rng})$ are the ring homomorphisms. The identity morphisms are the identity functions, and composition is the usual one.

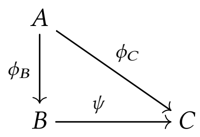
Example 2.1.10. Let A be a ring.

The category of *A-rings* is just like the category of $\mathbb{R}$-rings we mentioned earlier.

Namely, an A-ring a ring B, equipped with a ring homomorphism $\phi_B : A \to B$, called the structural homomorphism.

If C is another A-ring, then an *A-ring homomorphism* $\psi : B \to C$ is a ring homomorphism such that $\psi \circ \phi_B = \phi_C$.

In other words, the diagram



in Rng is commutative.

The category of A-rings is denoted by Rng/A.

Definition 2.1.11. Let C be a category.

(1) An object $C_0 \in \mathsf{C}$ is called an *initial object* if for every $C \in \mathsf{C}$ there is exactly one morphism $C_0 \to C$.
(2) An object $C_\infty \in \mathsf{C}$ is called a *terminal object* if for every $C \in \mathsf{C}$ there is exactly one morphism $C \to C_\infty C$.

Exercise 2.1.12. Prove that any two initial objects of a category C are uniquely isomorphic. Likewise for terminal objects.

Exercise 2.1.13.

(1) Let A be a ring. Show that there is exactly one ring homomorphism $\mathbb{Z} \to A$. Conclude that $\mathbb{Z}$ is the initial object of the category Rng.
(2) Find the terminal object of Rng.

Example 2.1.14. The category of *groups* Grp has groups as its objects and group homomorphisms as its morphisms.

Example 2.1.15. The category of *abelian groups* Ab has abelian groups as its objects, and group homomorphisms as its morpisms.

Example 2.1.16. The category of *topological spaces* Top has topological spaces as its objects, and continous maps homomorphisms as its morphisms.

Exercise 2.1.17. Find the initial and terminal objects of Grp, Ab and Top.

The categories in the previous examples have large sets of objects (in the sense that $\mathrm{Ob}(\mathsf{C}) \subseteq \mathsf{U}$, see Remark 2.1.5). But it very instructive to look at categories with very few objects.

Example 2.1.18. Take a group G. Define a category G as follows: there is a single object, say x; so that $\mathrm{Ob}(\mathsf{G}) = \{x\}$. The morphisms $g : x \to x$ are the elements of G, i.e. $\mathrm{Hom}_{\mathsf{G}}(x, x) := G$. The identity morphism is $\mathrm{id}_x := e \in G$, the unit element of G; and composition is the multiplication of G.

Likewise:

Example 2.1.19. Take a ring A. Define a category A as follows: there is a single object, say x; so that $\mathrm{Ob}(\mathsf{A}) = \{x\}$. The morphisms $a : x \to x$ are the elements of A, i.e. $\mathrm{Hom}_{\mathsf{A}}(x, x) := G$. The identity morphism is $\mathrm{id}_x := 1 \in A$, and composition is the multiplication of A.

2.2. **Functors.** If G and H are groups, we can talk about group homomorphisms $\phi : G \to H$. Likewise, if A and B are rings, we can talk about ring homomorphisms $f : A \to B$.

It raises the question: Is there an analogous concept of "homomorphism" between categories?

Definition 2.2.1. Let C and D be categories. A *functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of a function

$$F_{\mathrm{ob}} : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D}),$$

and for every pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C})$ a function

$$F_{C_0,C_1} : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F_{\mathrm{ob}}(C_0), F_{\mathrm{ob}}(C_1)\big).$$

There are two conditions:

(i) Identities: $F_{C,C}(\mathrm{id}_C) = \mathrm{id}_{F_{\mathrm{ob}}(C)}$.

(ii) Composition: For all $C_0, C_1, C_2 \in \mathrm{Ob}(\mathsf{C})$ and $\phi_i \in \mathrm{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$ there is equality

$$F_{C_1,C_2}(\phi_2) \circ F_{C_0,C_1}(\phi_1) = F_{C_0,C_2}(\phi_2 \circ \phi_1).$$

Usually we suppress the subscripts from F_{ob} and F_{C_0,C_1}.

Exercise 2.2.2. Let G and H be groups, with corresponding single-object categories G and H, as in Example 2.1.18. Show that there is a canonical bijection between group homomorphisms $\phi : G \to H$ and functors $F : \mathsf{G} \to \mathsf{H}$.

Many functors are "forgetful functors". Here is the protoypical example.

Example 2.2.3. The forgetful functor

$$F : \mathsf{Grp} \to \mathsf{Set}$$

sends a group G to its underlying set, and a group homomorphism ϕ to the function between the underlying sets.

Exercise 2.2.4. Functors can be *composed*. The exercise is to write the precise formulas for the composition $G \circ F$ of a functor $F : \mathsf{C} \to \mathsf{D}$ with a functor $G : \mathsf{D} \to \mathsf{E}$.

Definition 2.2.5. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between categories. The functor F is called *faithful* (resp. *full*) if for every pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C})$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_0), F(C_1)\big)$$

is injective (resp. surjective).

Forgetful functors (such as in Example 2.2.3) are faithful.

Definition 2.2.6. Let C be a category. A *subcategory* C' of C consists of a subset

$$\mathrm{Ob}(\mathsf{C}') \subseteq \mathrm{Ob}(\mathsf{C}),$$

and for every pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C}')$ a subset

$$\mathrm{Hom}_{\mathsf{C}'}(C_0, C_1) \subseteq \mathrm{Hom}_{\mathsf{C}}(C_0, C_1).$$

The conditions are:

(i) Identities: $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{C}'}(C, C)$ for every $C \in \mathrm{Ob}(\mathsf{C}')$.

(ii) Composition: For all $C_0, C_1, C_2 \in \mathrm{Ob}(\mathsf{C}')$ and $\phi_i \in \mathrm{Hom}_{\mathsf{C}'}(C_{i-1}, C_i)$, we have
$$\phi_2 \circ \phi_1 \in \mathrm{Hom}_{\mathsf{C}'}(C_0, C_2).$$

Clearly C' is itself a category, with the identity and composition of C; and the inclusion $F : \mathsf{C}' \to \mathsf{C}$ is a faithful functor.

Definition 2.2.7. Let C be a category and let $\mathsf{C}' \subseteq \mathsf{C}$ be a subcategory. We say that C' is a *full* subcategory of C if
$$\mathrm{Hom}_{\mathsf{C}'}(C_0, C_1) = \mathrm{Hom}_{\mathsf{C}}(C_0, C_1)$$
for all $C_0, C_1 \in \mathrm{Ob}(\mathsf{C}')$.

In other words, C' is a full subcategory of C if the inclusion functor $F : \mathsf{C}' \to \mathsf{C}$ is full.

Example 2.2.8. The category Ab is a full subcategory of Grp.

Exercise 2.2.9.

(1) Find a functor $F : \mathsf{C} \to \mathsf{D}$ that is full but not faithful.

(2) Find a functor $F : \mathsf{C} \to \mathsf{D}$ that is faithful but not full.

End of live Lecture 2

◇ ◇ ◇

The material below is for self-reading before lecture 3. (I just talked about it briefly during the live lecture.)

Exercise 2.2.10. Let A be a nonzero ring.

Show that there is a functor
$$F : \mathsf{Set} \to \mathsf{Mod}\, A$$
that sends an object X to the free module $F(X)$ with basis X,

and it sends a function $f : X \to Y$ to the unique A-module homomorphism $F(f)$ that makes the diagram of sets
$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\subseteq} \\ F(X) & \xrightarrow{\ F(f)\ } & F(Y) \end{array}$$
is commutative. Here the vertical arrows are the inclusions.

This is called the *free module functor.*

Exercise 2.2.11. Let A be a nonzero ring.

The category of finite sets is $\mathsf{Set}_{\mathrm{f}}$.

Show that there is a functor
$$F : \mathsf{Set}_{\mathrm{f}} \to \mathsf{Rng}/A$$
that sends an object X to the polynomial ring $A[X]$ in the set of variables X,

and it sends a function $f : X \to Y$ to the unique A-ring homomorphism $F(f)$ that makes the diagram of sets

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ F(X) & \xrightarrow{\ F(f)\ } & F(Y) \end{array}$$

is commutative. Here the vertical arrows are the inclusions.

This is called the *polynomial ring functor*.

Later we will explain how the functors F from the last two exercises are related to the corresponding forgetful functors.

Here are two examples of functors from $\mathsf{Mod}\, A$ to itself. (For those who need a reminder of tensor products, see Section 8 in [Ye2].)

Example 2.2.12. Fix $M \in \mathsf{Mod}\, A$. There is a functor

$$G : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

such that

$$G(N) = M \otimes_A N$$

on objects,

and on a morphism $\psi : N \to N'$ in $\mathsf{Mod}\, A$ it is

$$G(\psi) = \mathrm{id}_M \otimes \psi : G(N) = M \otimes_A N \to G(N') = M \otimes_A N'.$$

Example 2.2.13. Fix $M \in \mathsf{Mod}\, A$. There is a functor

$$F : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

such that

$$F(N) = \mathrm{Hom}_A(M, N)$$

on objects, and on a morphism $\psi : N \to N'$ it is

$$F(\psi) = \mathrm{Hom}(\mathrm{id}_M, \psi) : F(N) = \mathrm{Hom}_A(M, N) \to F(N') = \mathrm{Hom}_A(M, N').$$

Here

$$\mathrm{Hom}(\mathrm{id}_M, \psi)(\chi) := \psi \circ \chi : M \to N'$$

for $\chi \in \mathrm{Hom}_A(M, N)$.

Example 2.2.14. Suppose we try to change things around in the previous example; namely we fix $N \in \mathsf{Mod}\, A$, and we try to define a functor

$$H : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

such that

$$H(M) = \mathrm{Hom}_A(M, N)$$

on an object $M \in \mathsf{Mod}\, A$.

The only formula that seems to make sense for a morphism $\phi : M \to M'$ is

$$H(\phi) = \mathrm{Hom}(\phi, \mathrm{id}_N) : \mathrm{Hom}_A(M', N) \to \mathrm{Hom}_A(M, N),$$

where

$$\mathrm{Hom}(\phi, \mathrm{id}_N) := \chi \circ \phi : M \to N'$$

for $\chi \in \mathrm{Hom}_A(M', N)$.

But this is a homomorphism

$$H(\phi) : H(M') \to H(M).$$

It is in the wrong direction! What to do?

The answer is: a new definition.

Definition 2.2.15. Let C and D be categories. A *contravariant functor*

$$F : \mathsf{C} \to \mathsf{D}$$

consists of a function

$$F : \operatorname{Ob}(\mathsf{C}) \to \operatorname{Ob}(\mathsf{D}),$$

and for every pair of objects $C_0, C_1 \in \operatorname{Ob}(\mathsf{C})$ a function

$$F : \operatorname{Hom}_{\mathsf{C}}(C_0, C_1) \to \operatorname{Hom}_{\mathsf{D}}\big(F(C_1), F(C_0)\big).$$

There are two conditions:

(i) Identities: $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$.

(ii) Composition, reversed: For all $C_0, C_1, C_2 \in \operatorname{Ob}(\mathsf{C})$ and $\phi_i \in \operatorname{Hom}_{\mathsf{C}}(C_{i-1}, C_i)$ there is equality

$$F(\phi_1) \circ F(\phi_2) = F(\phi_2 \circ \phi_1)$$

in $\operatorname{Hom}_{\mathsf{D}}\big(F(C_2), F(C_0)\big)$.

An ordinary functor (Definition 2.2.1) is sometimes called a *covariant functor.*

Next week we will talk about *natural transformations.*

End of Lecture 2

Lecture 3, 4 Nov 2020

Today we will learn abouit *natural transformations*, *opposite categories* and *equivalences of categories*. If time permits we will start talking about *sheaves of rings of functions*. [In fact we did less than promised.]

2.3. **Natural Transformations.** Functors go from one category to another category, just like group homomorphism go from one group to another group.

But in the world of categories there is another level: something that goes from one functor to another functor; these are the *natural transformations* in the next definition.

Definition 2.3.1. Let C and D be categories and let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A *natural transformation* (sometimes called a *morphism of functors*)

$$\eta : F \to G$$

is a collection

$$\eta = \{\eta_C\}_{C \in \mathrm{Ob}(\mathsf{C})}$$

of morphisms

$$\eta_C : F(C) \to G(C)$$

in D, such that for every morphism $\psi : C_0 \to C_1$ in C the diagram

$$(\dagger) \qquad \begin{array}{ccc} F(C_0) & \xrightarrow{F(\psi)} & F(C_1) \\ \downarrow{\scriptstyle \eta_{C_0}} & & \downarrow{\scriptstyle \eta_{C_1}} \\ G(C_0) & \xrightarrow{G(\psi)} & G(C_1) \end{array}$$

in D is commutative.

Here are some examples / exercises.

Example 2.3.2. This is a variation of Example 2.2.12. The source and target categories are both $\mathsf{Mod}\, A$ here. Fix a homomorphism $\phi : L \to M$ in $\mathsf{Mod}\, A$. Consider the A-linear functors

$$G_L : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad G_L := L \otimes_A (-)$$

and

$$G_M : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad G_M := M \otimes_A (-)$$

as in that example.

For every module N there is a homomorphism

$$\eta_N : G_L(N) = L \otimes_A N \to G_M(N) = M \otimes_A N,$$

namely

$$\eta_N := \phi \otimes \mathrm{id}_N : L \otimes_A N \to M \otimes_A N.$$

I claim that $\eta = \{\eta_N\}_{N \in \mathsf{Mod}\, A}$ is a natural transformation $\eta : G_L \to G_M$. For this I must verify that diagram ($\dagger$) in Definition 2.3.1 is commutative, for every morphism $\psi : N_0 \to N_1$ in the source category.

Take an arbitrary homomorphism $\psi : N_0 \to N_1$. We need to prove that the diagram

$$\begin{array}{ccc} G_L(N_0) = L \otimes_A N_0 & \xrightarrow{G_L(\psi)} & G_L(N_1) = L \otimes_A N_1 \\ \downarrow{\scriptstyle \eta_{N_0}} & & \downarrow{\scriptstyle \eta_{N_1}} \\ G_M(N_0) = M \otimes_A N_0 & \xrightarrow{G_M(\psi)} & G_M(N_1) = M \otimes_A N_1 \end{array}$$

in $\mathsf{Mod}\, A$ is commutative.

This means that we have to prove that for every element $u \in L \otimes_A N_0$ there is equality

$$(\eta_{N_1} \circ G_L(\psi))(u) = (G_M(\psi) \circ \eta_{N_0})(u) \tag{2.3.3}$$

in $L \otimes_A N_1$.

Because both functions

$$\eta_{N_1} \circ G_L(\psi),\ G_M(\psi) \circ \eta_{N_0} \ :\ L \otimes_A N_0 \ \to\ M \otimes_A N_1$$

are A-linear, it is enough to verify (2.3.3) for a pure tensor $u = l \otimes n$, with $l \in L$ and $n \in N_0$.

Now

$$(\eta_{N_1} \circ G_L(\psi))(l \otimes n) = \eta_{N_1}(l \otimes \psi(n)) = \phi(l) \otimes \psi(n)$$

and

$$(G_M(\psi) \circ \eta_{N_0})(l \otimes n) = G_M(\psi)(\phi(l) \otimes n) = \phi(l) \otimes \psi(n).$$

These are equal, as claimed.

Definition 2.3.4. Let C and D be categories and let $F, G : \mathsf{C} \to \mathsf{D}$ be functors. A natural transformation $\eta : F \to G$ is called a *natural isomorphism* if for every object $C \in \mathsf{C}$ the morphism $\eta_C : F(C) \to G(C)$ in D is an isomorphism

Exercise 2.3.5. In the situation of Example 2.3.2, prove that the following two conditions are equivalent:

(i) $\phi : L \to M$ is an isomorphism of A-modules.
(ii) $\eta : G_L \to G_M$ is a natural isomorphism (an isomorphism of functors).

Exercise 2.3.6. Let C and D be categories, let $F, G : \mathsf{C} \to \mathsf{D}$ be functors, and let $\eta : F \to G$ be a natural transformation. Assume that every morphism in the category D is an isomorphism. (Such a category is called a *groupoid*.) Prove that η is a natural isomorphism.

Example 2.3.7. Let G and H be groups, with corresponding single-object categories G and H, say with $\mathrm{Ob}(\mathsf{G}) = \{x\}$ and $\mathrm{Ob}(\mathsf{H}) = \{y\}$. See Example 2.1.18.

Let $\phi, \psi : G \to H$ be group homomorphisms, that we view (as in Exercise 2.2.2) as functors $F_\phi, F_\psi : \mathsf{G} \to \mathsf{H}$. Namely $F_\phi(x) = y$ and $F_\phi(g) = \phi(g)$, etc.

Given an element $h \in H$, let $\mathrm{Ad}(h)$ be the automorphism of H which is conjugation by h, namely $\mathrm{Ad}(h)(f) := h \cdot f \cdot h^{-1}$ for all $f \in H$.

Suppose there is a natural transformation $\eta : F_\phi \to F_\psi$.

Note that η is automatically a natural isomorphism, by Exercise 2.3.6 (since H is a groupoid).

For every morphism $g : x \to x$ in G, i.e. g is an element of G, we have a commutative diagram

$$\begin{array}{ccc} F_{\phi}(x) = y & \xrightarrow{F_{\phi}(g)} & F_{\phi}(x) = y \\ {\scriptstyle \eta_x} \downarrow & & \downarrow {\scriptstyle \eta_x} \\ F_{\psi}(x) = y & \xrightarrow{F_{\psi}(g)} & F_{\psi}(x) = y \end{array} \tag{2.3.8}$$

in H.

Now η_x is a morphisn $y \to y$ in the category H, so $\eta_x = h$ for some element $h \in H$. Also $F_{\phi}(g) = \phi(g) \in H$ and $F_{\psi}(g) = \psi(g) \in H$. So diagram (2.3.8) becomes

$$\begin{array}{ccc} F_{\phi}(x) = y & \xrightarrow{F_{\phi}(g) = \phi(g)} & F_{\phi}(x) = y \\ {\scriptstyle \eta_x = h} \downarrow & & \downarrow {\scriptstyle \eta_x = h} \\ F_{\psi}(x) = y & \xrightarrow{F_{\psi}(g) = \psi(g)} & F_{\psi}(x) = y \end{array} \tag{2.3.9}$$

Recall that the operation of composition in H is multiplication in the group H. We see that the condition of commutativity of diagram (2.3.8) in the category H is the same as equality

$$h \cdot \phi(g) = \psi(g) \cdot h \tag{2.3.10}$$

in H.

Multiplying both sides of (2.3.10) by h^{-1} on the right we get this equality

$$\psi(g) = h \cdot \phi(g) \cdot h^{-1} \tag{2.3.11}$$

in the group H. But

$$h \cdot \phi(g) \cdot h^{-1} = (\operatorname{Ad}(h) \circ \phi)(g).$$

Thus equality (2.3.11) for all $g \in G$ says that

$$\psi = \operatorname{Ad}(h) \circ \phi. \tag{2.3.12}$$

as group homomorphisms $G \to H$.

This argument can also be reversed.

The conclusion is that the functors F_{ϕ} and F_{ψ} are naturally isomorphic iff ϕ and ψ are related by an inner automorphism, as in (2.3.12).

Proposition 2.3.13. *Let C and D be categories, let $F, G, H : \mathsf{C} \to \mathsf{D}$ be functors, and let $\eta : F \to G$ and $\theta : G \to H$ be natural transformations. Then the collection*

$$\theta \circ \eta := \{\theta_C \circ \eta_C\}_{C \in \operatorname{Ob}(\mathsf{C})}$$

is a natural transformation $F \to H$.

Exercise 2.3.14. Prove this proposition.

A functor $F : \mathsf{C} \to \mathsf{D}$ has its *identity natural automorphism* $\operatorname{id}_F : F \to F$, defined a follows: for each object C of C it is

$$(\operatorname{id}_F)_C := \operatorname{id}_{F(C)} : F(C) \xrightarrow{\simeq} F(C),$$

the identity automorphism of the object $F(C)$ in D.

Proposition 2.3.15. *Let* C *and* D *be categories, let* $F, G : \mathsf{C} \to \mathsf{D}$ *be functors, and let* $\eta : F \to G$ *be a natural transformation. TFAE:*

(i) *η is a natural isomorphism (Definition 2.3.4).*

(ii) *There is a natural transformation* $\zeta : G \to F$ *such that* $\zeta \circ \eta = \mathrm{id}_F$ *and* $\eta \circ \zeta = \mathrm{id}_G$.

Moreover, if these equivalent conditions hold, then ζ is also a natural isomorphism, it is unique, and it is called the inverse of η.

Exercise 2.3.16. Prove this proposition. (Hint: it is easy, after you understand what it says.)

End of live Lecture 3

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The material below is for self-reading before lecture 4 (I just talked about it briefly during the live lecture). There are a few exercises, and some are hard.

Recall that a finitely generated abelian group M has as direct sum decomposition

$$M \cong F \oplus T, \tag{2.3.17}$$

where F is a free abelian group, i.e. $F \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, and T is a finite abelian group. (The letters "F" and "T" stand for "free" and "tosrsion", respectively.)

The next exercise gives meaning to the statement "the torsion subgroup of an abelian is not naturally a direct summand of the group".

Exercise 2.3.18. We work in the category $\mathsf{Ab}_{\mathrm{fg}} = \mathsf{Mod}_{\mathrm{f}}\, \mathbb{Z}$ of finitely generated abelian groups. It is a full subcategory of Ab.

Given a finitely generated abelian group M, let $T(M) \subseteq M$ be the subgroup of torsion elements (i.e. the elements of finite order).

(1) Prove that $T : \mathsf{Ab}_{\mathrm{fg}} \to \mathsf{Ab}_{\mathrm{fg}}$ is a functor.

(2) Let Id be the identity functor of the category $\mathsf{Ab}_{\mathrm{fg}}$. Prove that the inclusions

$$\epsilon_M : T(M) \to M,$$

for $M \in \mathsf{Ab}_{\mathrm{fg}}$, assemble to a natural transformation

$$\epsilon = \{\epsilon_M\}_{M \in \mathsf{Ab}_{\mathrm{fg}}} : T \to \mathrm{Id}$$

of functors from $\mathsf{Ab}_{\mathrm{fg}}$ to itself.

(3) (Hard) Prove that there *does not exist* a natural transformation $\sigma : \mathrm{Id} \to T$ of functors from $\mathsf{Ab}_{\mathrm{fg}}$ to itself, such that $\sigma \circ \epsilon = \mathrm{id}_T$ as natural transformations from the functor T to itself (i.e. for every M the homomorphism $\sigma_M \circ \epsilon_M : T(M) \to T(M)$ is the identity $\mathrm{id}_{T(M)}$.)

Hint for (3): Find a counterexample, as follows.

Given $M \in \mathsf{Ab}_{\mathrm{fg}}$, there is a short exact sequence

$$0 \to T(M) \xrightarrow{\epsilon_M} M \xrightarrow{\pi_M} F(M) \to 0 \tag{2.3.19}$$

in $\mathsf{Ab}_{\mathrm{fg}}$. Here $F(M) := M/T(M)$, and this is a free abelian group.

Suppose $\sigma_M : M \to T(M)$ is a homomorphism satisfying $\sigma_M \circ \epsilon_M = \mathrm{id}_{T(M)}$. Then there is an induced *splitting* of the short exact sequence (2.3.19). This means that there is an isomorphism

$$(\sigma_M, \pi_M) : M \xrightarrow{\simeq} T(M) \oplus F(M) \tag{2.3.20}$$

and it fits into this commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(M) & \xrightarrow{\epsilon_M} & M & \xrightarrow{\pi_M} & F(M) & \longrightarrow & 0 \\ & & \simeq \downarrow \mathrm{id} & & \simeq \downarrow (\sigma_M,\pi_M) & & \simeq \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & T(M) & \xrightarrow{(\mathrm{id},0)} & T(M) \oplus F(M) & \xrightarrow{(0,\mathrm{id})} & F(M) & \longrightarrow & 0 \end{array} \tag{2.3.21}$$

Let's assume that a natural transformation $\sigma : \mathrm{Id} \to T$ is given, satisfying $\sigma_M \circ \epsilon_M = \mathrm{id}_{T(M)}$ for every M. We want to produce a contradiction.

The way to do it is to find an abelian group M, and an automorphism ψ of M that does not respect the direct sum decomposition (2.3.20) induced by σ_M.

Try to do this for abelian group $M := \mathbb{Z} \oplus (\mathbb{Z}/(2))$. End of hint.

I will post a solution for item (3) before the next lecture (please remind me!).

The next exercise gives meaning to the statement "the center of a group is not a functor".

Exercise 2.3.22. (Hard) We work in the category Grp of groups. Given a group G, let $\mathrm{Cent}(G) \subseteq G$ be the center.

Prove that there *does not exist* a functor $C : \mathsf{Grp} \to \mathsf{Grp}$, together with a natural transformation $\gamma : C \to \mathrm{Id}$, such that for every group G we have $C(G) = \mathrm{Cent}(G)$, and the group homomorphism

$$\gamma_G : C(G) = \mathrm{Cent}(G) \to \mathrm{Id}(G) = G$$

is the inclusion. (Hint: look for an abelian group M that's a subgroup of a nonabelian group G, but M is not in the center of G. You can find such G of size 6.)

Opposite categories and *equivalences* will be discussed next week. Then we will introduce *sheaves of rings of functions*. *Adjoint functors* will be postponed until later (too much abstract material at once in confusing).

End of Lecture 3

Lecture 4, 11 Nov 2020

Today we are going to finish the first discussion of category theory. We will return to the concept of *adjoint functors* later, when it will be needed. By then we should be so fluent in category theory that this confusing concept will be digestible (I hope – frankly, I never really understood it well!).

2.4. **Opposite Categories.**

Definition 2.4.1. Given a category C, its *opposite category* C^{op} has the same objects, but its arrows and their compositions are reversed

Let me describe this in detail.

First, define the set of objects of the new category C^{op} to be

$$\mathrm{Ob}(\mathsf{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathsf{C}). \tag{2.4.2}$$

The identity automorphism of the set $\mathrm{Ob}(\mathsf{C})$ is now written as a bijection

$$\mathrm{Op} : \mathrm{Ob}(\mathsf{C}) \xrightarrow{\simeq} \mathrm{Ob}(\mathsf{C}^{\mathrm{op}}). \tag{2.4.3}$$

Now to morphisms. Given a pair of objects $C_0, C_1 \in \mathrm{Ob}(\mathsf{C})$, we let

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}\big(\mathrm{Op}(C_1), \mathrm{Op}(C_0)\big) := \mathrm{Hom}_{\mathsf{C}}\big(C_0, C_1\big). \tag{2.4.4}$$

There is a bijection of sets (the identity automorphism in disguise)

$$\mathrm{Op} : \mathrm{Hom}_{\mathsf{C}}\big(C_0, C_1\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}\big(\mathrm{Op}(C_1), \mathrm{Op}(C_0)\big). \tag{2.4.5}$$

This means that every morphism $\psi : D_1 \to D_0$ in C^{op} can be expressed as $\psi = \mathrm{Op}(\phi)$ for a unique morphism $\phi : C_0 \to C_1$ in C, with $D_i = \mathrm{Op}(C_i)$.

The composition $\circ^{\mathrm{op}}$ of C^{op} is as follows. Given morphisms $\psi_2 : D_2 \to D_1$ and $\psi_1 : D_1 \to D_0$ in C^{op}, let's express them as $\psi_i = \mathrm{Op}(\phi_i)$, for morphisms $\phi_i : C_{i-1} \to C_i$, with $D_i = \mathrm{Op}(C_i)$.

Then the composition in C^{op} is

$$\psi_1 \circ^{\mathrm{op}} \psi_2 = \mathrm{Op}(\phi_1) \circ^{\mathrm{op}} \mathrm{Op}(\phi_2) := \mathrm{Op}(\phi_2 \circ \phi_1). \tag{2.4.6}$$

In diagrams: first the commutative diagram of the composition in C.

$$C_0 \xrightarrow{\phi_1} C_1 \xrightarrow{\phi_2} C_2, \qquad C_0 \xrightarrow{\phi_2 \circ \phi_1} C_2$$

Now the commutative diagram of the composition in C^{op}.

$$D_2 = \mathrm{Op}(C_2) \xrightarrow{\psi_2 = \mathrm{Op}(\phi_2)} D_1 = \mathrm{Op}(C_1) \xrightarrow{\psi_1 = \mathrm{Op}(\phi_1)} D_0 = \mathrm{Op}(C_0), \qquad D_2 \xrightarrow{\psi_1 \circ \psi_2 = \mathrm{Op}(\phi_2 \circ \phi_1)} D_0$$

Lastly, the identity automorphism of an object $C \in \mathsf{C}^{\mathrm{op}}$ is the same as in C, but we can also write as

$$\mathrm{id}_C := \mathrm{Op}(\mathrm{id}_C). \tag{2.4.7}$$

Exercise 2.4.8. Let C be a category.

(1) Prove that C^{op} is indeed a category, with set of objects (2.4.2), sets of morphisms (2.4.4), composition (2.4.6) and identities (2.4.7).
(2) Prove that $\mathrm{Op} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ is a contravariant functor.

Proposition 2.4.9. *Let* C *and* D *be categories. The formula* $F \mapsto F \circ \mathrm{Op}$ *is a bijection from the set of contravariant functors* $F : \mathsf{C} \to \mathsf{D}$ *to the set of (ordinary, or covariant) functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Exercise 2.4.10. Prove Prop 2.4.9.

Since we often have several categories under discussion, we may decorate the opposite functor of C like this:

$$\mathrm{Op}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}} . \tag{2.4.11}$$

Exercise 2.4.12. Let C be a category. Prove that

$$\mathrm{Op}_{\mathsf{C}^{\mathrm{op}}} \circ \mathrm{Op}_{\mathsf{C}} = \mathrm{Id}_{\mathsf{C}} .$$

(These are functors from C to itself.)

From now on we will usually deal only with (covariant) functors. Whenever we encounter a contravariant functor, we make it covariant be replacing the source category with its opposite; this is justified by Proposition 2.4.9. For instance:

Example 2.4.13. Let C be a category and $C \in \mathsf{C}$. Then $\mathrm{Hom}_{\mathsf{C}}(-, C)$ is a contravariant functor $\mathsf{C} \to \mathsf{Set}$, but we prefer to see it a (covariant) functor

$$\mathrm{Hom}_{\mathsf{C}}(-, C) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set} .$$

Remark 2.4.14. Given a "concrete" category, such as $\mathsf{C} = \mathsf{Mod}(A)$ for a ring A, it is interesting to know "how concrete" the opposite category C^{op} is.

Here is a result, probably due to P. Freyd (it is an exercise in a book of his). Suppose A is a nonzero NC (i.e. not necessarily commutative) ring, and $\mathsf{Mod}(A)$ is the category of left A-modules. Then there does not exist a linear equivalence of categories between $\mathsf{Mod}(A)^{\mathrm{op}}$ and $\mathsf{Mod}(B)$, for any NC ring B. (An equivalence $F : \mathsf{Mod}(A)^{\mathrm{op}} \to \mathsf{Mod}(B)$ is linear if it respects the abelian group structures on morphisms in $\mathsf{Mod}(A)^{\mathrm{op}}$ and in $\mathsf{Mod}(B)$.) See [Ye1, Remark 2.7.20].

There is a complete (but hard) proof for the case of $A = \mathbb{Z}$ in [Ye1, Example 2.7.21].

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Definition 2.4.15. Given a noncommutative ring A, its *opposite ring* A^{op} has the same underlying $\mathbb{Z}$-module as A, the same unit element, but the multiplication $\cdot^{\mathrm{op}}$ is the reversed:

$$\mathrm{op}(a_1) \cdot^{\mathrm{op}} \mathrm{op}(a_0) := \mathrm{op}(a_0 \cdot a_1).$$

Here $\mathrm{op} : A \to A^{\mathrm{op}}$ is the identity automorphism of the $\mathbb{Z}$-module A.

The $\mathbb{Z}$-linear bjection $\mathrm{op} : A \to A^{\mathrm{op}}$ is a *ring anti-automorphism.* Of course $A = A^{\mathrm{op}}$ iff A is a commutative ring.

Example 2.4.16. Some noncommutative rings are isomorphic to their opposites.

Let $\mathbb{K}$ be a nonzero commutative ring, $n \geq 2$ and $A := \mathrm{Mat}_n(\mathbb{K})$. Then A is a NC ring. Transposition is a NC ring isomorphism $A \xrightarrow{\simeq} A^{\mathrm{op}}$.

Example 2.4.17. Let G be a nonabelian group. Then $g \mapsto g^{-1}$ is a group isomorphism $\phi : G \to G^{\mathrm{op}}$.

Example 2.4.18. Given a NC ring A, we can make it into a single-object category A, say $\mathrm{Ob}(\mathsf{A}) = \{x\}$, such that $\mathrm{End}_{\mathsf{A}}(x) = A$. Cf. Example 2.1.19.

Then the category corresponding to the opposite NC ring A^{op} is the opposite category A^{op}.

2.5. **Equivalences of Categories.**

Definition 2.5.1. A functor $F : \mathsf{C} \to \mathsf{D}$ is called an *equivalence* if there is a functor $G : \mathsf{D} \to \mathsf{C}$, and isomorphisms of functors $\eta : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}}$ and $\zeta : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}}$.

The functor G is called a *quasi-inverse* of F.

It is very important to note that an equivalence F does not induce a bijection $\mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$. We will give the precise statement about that later.

Example 2.5.2. Consider the category Set of sets, its full subcategory $\mathsf{Set}_{\mathrm{fin}}$ of finite sets, and the full subcategory $\mathsf{C} \subseteq \mathsf{Set}_{\mathrm{fin}}$ on the set of objects $\{S_i\}_{i \in \mathbb{N}}$, where

$$S_i := \{1, \dots, i\} \subseteq \mathbb{N}.$$

Thus $|S_i| = i$, and in particular $S_0 = \varnothing$.

The inclusion functor

$$F : \mathsf{C} \to \mathsf{Set}_{\mathrm{fin}}$$

turns out to be an equivalence.

To construct a quasi-inverse

$$G : \mathsf{Set}_{\mathrm{fin}} \to \mathsf{C}$$

we use the axiom of choice on the (big!) set $\mathrm{Ob}(\mathsf{Set}_{\mathrm{fin}})$. For every finite set T of cardinality $i := |T|$ we choose an isomorphism

$$\zeta_T : S_i \xrightarrow{\simeq} T$$

in $\mathsf{Set}_{\mathrm{fin}}$. In case $T = S_i$ we choose $\zeta_T := \mathrm{id}_T$.

The functor G is defined on objects by $G(T) := S_i$ where $i := |T|$.

For a morphism $\phi : T \to U$ in $\mathsf{Set}_{\mathrm{fin}}$ we define

$$G(\phi) := \zeta_U^{-1} \circ \phi \circ \zeta_T.$$

This makes the diagram

$$\begin{array}{ccc} S_i = G(T) & \xrightarrow{G(\phi)} & S_j = G(U) \\ {\scriptstyle \zeta_T}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \zeta_U} \\ T = \mathrm{Id}_{\mathsf{Set}_{\mathrm{fin}}}(T) & \xrightarrow{\phi} & U = \mathrm{Id}_{\mathsf{Set}_{\mathrm{fin}}}(U) \end{array}$$

commutative.

We see that

$$\zeta := \{\zeta_T\}_{T \in \mathrm{Ob}(\mathsf{Set}_{\mathrm{fin}})} : F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{Set}_{\mathrm{fin}}}$$

is an isomorphism of functors.

And by construction we have

$$\eta := \mathrm{id}_{\mathrm{Id}_{\mathsf{C}}} : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}},$$

and of course this is also an isomorphism of functors.

Exercise 2.5.3. In the setting of Example 2.5.2, prove that G is a functor, and that it is a quasi-inverse of F.

Exercise 2.5.4. In the setting of Example 2.5.2, find two different quasi-inverses G_0, G_1 of F, and find an isomorphism of funtors $\chi : G_0 \xrightarrow{\simeq} G_1$.

Exercise 2.5.5. Let $\mathbb{K}$ be a field.

We shall use the notation $\mathsf{Mod}(\mathbb{K})$ for the category of $\mathbb{K}$-modules, and $\mathsf{Mod}_{\mathrm{f}}(\mathbb{K})$ for its full subcategory of finitely generated modules. (See Remark 2.5.6 below on terminology and notation).

State and prove the equivalence, analogous to that of Example 2.5.2 between $\mathsf{Mod}_{\mathrm{f}}(\mathbb{K})$, and its full subcategory M on the set of objects $\{M_i\}_{i \in \mathbb{N}}$, where $M_i := \mathbb{K}^i$.

Remark 2.5.6. Here is a side remark on terminology. Most people stick to an amusing tradition and refer to $\mathbb{K}$-modules as *vector spaces*; I think this is silly, and they (most people) think I am crazy.

I have a reason though: "vector space" indicates some kind of geometric quality, and this is almost always absent from $\mathbb{K}$-module. When we get to *vector bundles* there will indeed be geometry!

Simiarly, in the theory of *Lie algebras*, the object $\mathfrak{g}$ is both a $\mathbb{K}$-module (abstract) and a scheme (isomorphic to $\mathbf{A}^n_{\mathbb{K}}$ for some n). – Lie groups –

Remark 2.5.7. There is much stronger notion of *isomorphism of categories*. This is a functor $F : \mathsf{C} \to \mathsf{D}$ which is bijective on objects and on morphisms. An isomorphism F has an inverse isomorphism $G : \mathsf{D} \to \mathsf{C}$, and this G is unique.

If F is an isomorphism of categories, then it is an equivalence; but not vice verse, as Example 2.5.2 shows.

This is analogous to topology, where a homeomorphism $f : X \to Y$ is stronger than a homotopy equivalence.

The inclusion f of the origin in the plane $X = \{x_0\}$ into the closed unit disc Y is an example of a homotopy equivalence that is not a homeomorphism.

A functor $F : \mathsf{C} \to \mathsf{D}$ is called *fully faithful* if it is full and faithful; i.e. for every pair of objects $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}(F(C_0), F(C_1))$$

is bijective.

The functor F is said to be *essentially surjective on objects* if for every object $D \in \mathsf{D}$ there exists some object $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$ in D.

Theorem 2.5.8. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. TFAE:*

(i) *F is an equivalence.*
(ii) *F is fully faithful and essentially surjective on objects.*

Exercise 2.5.9. Prove Theorem 2.5.8. (Hint: Study Example 2.5.2.)

End of live Lecture 4

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Here are the solutions of the hard problems from last week.

Solution 2.5.10 (of Exercise 2.3.18(3))**.** The solution is much more difficult than I thought...

As hinted, we are going to assume that a natural transformation $\sigma : \mathrm{Id} \to T$ of functors from $\mathsf{Ab}_{\mathrm{fg}} = \mathsf{Mod}_{\mathrm{f}}(\mathbb{Z})$ to itself is given, and deduce a contradiction.

Let M be some object of $\mathsf{Mod}_{\mathrm{f}}(\mathbb{Z})$, and let ψ be some endomorphism of M. Because $\sigma : \mathrm{Id} \to T$ is a natural transformation, we have a commutative diagram

(2.5.11)
$$\begin{array}{ccc} M = \mathrm{Id}(M) & \xrightarrow{\sigma_M} & T(M) \\ {\scriptstyle \psi = \mathrm{Id}(\psi)} \downarrow & & \downarrow {\scriptstyle T(\psi)} \\ M = \mathrm{Id}(M) & \xrightarrow{\sigma_M} & T(M) \end{array}$$

This means that

(2.5.12)
$$T(\psi) \circ \sigma_M = \sigma_M \circ \psi$$

as homomorphisms $M \to T(M)$.

Likewise, Because $\pi : \mathrm{Id} \to F$ is a natural transformation, we have a commutative diagram

(2.5.13)
$$\begin{array}{ccc} M = \mathrm{Id}(M) & \xrightarrow{\pi_M} & F(M) \\ {\scriptstyle \psi = \mathrm{Id}(\psi)} \downarrow & & \downarrow {\scriptstyle F(\psi)} \\ M = \mathrm{Id}(M) & \xrightarrow{\pi_M} & F(M) \end{array}$$

This means that

(2.5.14)
$$F(\psi) \circ \pi_M = \pi_M \circ \psi$$

as homomorphisms $M \to F(M)$.

Putting (2.5.12) and (2.5.14) together we get the following commutative diagram

(2.5.15)
$$\begin{array}{ccc} M & \xrightarrow[\simeq]{(\sigma_M, \pi_M)} & T(M) \oplus F(M) \\ {\scriptstyle \psi} \downarrow & & \downarrow {\scriptstyle (T(\psi), F(\psi))} \\ M & \xrightarrow[\simeq]{(\sigma_M, \pi_M)} & T(M) \oplus F(M) \end{array}$$

This says that ψ must respect the direct sum decomposition (2.3.20), which we repeat here:

(2.5.16)
$$(\sigma_M, \pi_M) : M \xrightarrow{\simeq} T(M) \oplus F(M).$$

To finish, we will produce $\psi : M \to M$ that does not respect its direct sum decomposition (2.5.16).

Take

(2.5.17)
$$M := \mathbb{Z} \oplus (\mathbb{Z}/(2)).$$

Then $F(M) \cong \mathbb{Z}$ and $T(M) \cong \mathbb{Z}/(2)$.

Let $m \in T(M)$ be its unique generator, and let $n \in F(M)$ be one of its two generators.

Because $F(M)$ is a free $\mathbb{Z}$-module with basis n, there is a unique homomorphism $\psi_F : F(M) \to M$ s.t. $\psi_F(n) = m$. Let $\psi_T : T(M) \to M$ be the zero homomorphism.

Combined we get a homomorphism

$$\psi := (\psi_T, \psi_F) \circ (\sigma_M, \pi_M) : M \to M. \tag{2.5.18}$$

Here is how ψ fits into a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{(\sigma_M,\pi_M)} & T(M) \oplus F(M) \\ & \searrow^{\psi} & \downarrow{\scriptstyle (\psi_T,\psi_F)} \\ & & M \end{array} \tag{2.5.19}$$

But the endomorphism ψ satisfies $\psi(n) = m$, so it does not respect the direct sum decomposition (2.5.16). QED.

Solution 2.5.20 (of Exercise 2.3.22)**.** This is much easier!

Let G be the dihedral group D_3, which is also the symmetry group S_3 of the set $\{1, 2, 3\}$. This is (the only?) a nonabelian group of order 6. Let $H := C_2 = \mathbb{Z}/(2)$ and $K := C_3 = \mathbb{Z}/(3)$. There is an action of H on K by group automorphisms, and there are embeddings $\theta : H \rightarrowtail G$ and $K \rightarrowtail G$, such that $G \cong H \ltimes K$, a semi-direct product.

Since H is abelian, it center is $\operatorname{Cent}(H) = H$. On the other hand $\operatorname{Cent}(G) = \{e\}$.

If we had a functor $C : \mathsf{Grp} \to \mathsf{Grp}$, together with a natural transformation $\gamma : C \to \operatorname{Id}$, such that for every group G we have $C(G) = \operatorname{Cent}(G)$, and the group homomorphism

$$\gamma_G : C(G) = \operatorname{Cent}(G) \to \operatorname{Id}(G) = G$$

is the inclusion, there would be a commutative diagram

$$\begin{array}{ccc} C(H) = H & \xrightarrow{\gamma_H = \text{incl}} & H \\ {\scriptstyle C(\theta)}\downarrow & & \downarrow{\scriptstyle \theta = \text{incl}} \\ C(G) = \{e\} & \xrightarrow{\gamma_G = \text{incl}} & G \end{array} \tag{2.5.21}$$

in Grp, in which the arrows marked "incl" are the inclusions, so they are injective. This implies that $C(\theta)$ is an injective group homomorphism, from a group of order 2 to a group of order 1. QED.

End of Lecture 4

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Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel.
Email: amyekut@math.bgu.ac.il, *Web*: http://www.math.bgu.ac.il/~amyekut.