Course Notes:

Algebraic Geometry – Schemes 2

Ben Gurion University
Spring Semester 2020-21

Amnon Yekutieli

Contents

1. Locally Ringed Spaces 3
1.1. Recalling some Material. 3
1.2. Locally Ringed Spaces and Their Maps 5
2. Affine Schemes 9
2.1. Definitions and Basic Properties 9
2.2. More Properties of Affine Schemes 13
2.3. Maps of Affine Schemes 19
References 25

Lecture 1, 3 March 2021

This course is a continuation of the course "Algebraic Geometry – Schemes 1" from the previous semester. The notes of that course are [Ye4].

1. Locally Ringed Spaces

1.1. Recalling some Material. We work over a base ring $\mathbb{K}$, which is some nonzero commutative ring. In particular examples $\mathbb{K}$ will be specified.

Recall that a ringed space over $\mathbb{K}$, or a $\mathbb{K}$-ringed space, is a pair $(X, \mathcal{O}_X)$, where X is a topological space X, and $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-rings on X.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be $\mathbb{K}$-ringed spaces. A *map of* $\mathbb{K}$*-ringed spaces*

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of topological spaces

$$f : Y \to X$$

together with a homomorphism of sheaves of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y).$$

We saw a few examples of ringed spaces.

Example 1.1.1. A not very interesting ringed space is $(X, \mathcal{O}_X)$, where X is some topological spaces, and $\mathcal{O}_X := \mathbb{K}_X$, the constant sheaf with values in $\mathbb{K}$.

A map $f : Y \to X$ in Top gives rise to a map of $\mathbb{K}$-ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

where $\psi : \mathbb{K}_X \to f_*(\mathbb{K}_Y)$ is gotten by sheafication from the obvious

The way to produce ψ is this. Recall that $\Gamma(U, \mathbb{K}_X)$ is the ring of continous functions $a : U \to \mathbb{K}$, for the discrete topology on $\mathbb{K}$. Let $V := f^{-1}(U) \subseteq Y$. Then

$$\psi(a) \in \Gamma(U, f_*(\mathbb{K}_Y)) = \Gamma(V, \mathbb{K}_X)$$

is is the continous function $\psi(a) := a \circ f|_V : V \to \mathbb{K}$.

We also had a list of more interesting ringed spaces, repeated in the next example.

Example 1.1.2. A category of *geometric spaces* is a category Sp belonging to the list below. For each type we indicate the structure sheaf $\mathcal{O}_X$ and the base ring $\mathbb{K}$.

(1) The category Top of topological spaces and continous maps between them. Here the base ring is $\mathbb{K} = \mathbb{R}$, the field of real numbers. The sheaf $\mathcal{O}_X$ is the sheaf of continous $\mathbb{R}$-valued functions.
(2) The category Mfld of real differentiable manifolds and differentiable maps between them, where by differentiable we mean of class C^∞. Here $\mathbb{K} = \mathbb{R}$. The sheaf $\mathcal{O}_X$ is the sheaf of differentiable $\mathbb{R}$-valued functions.
(3) The category Var of quasi-projective algebraic varieties over an algebraically closed field $\mathbb{K}$. The sheaf $\mathcal{O}_X$ is the sheaf of algebraic, or regular, $\mathbb{K}$-valued functions.

In each of these cases, a map $f : Y \to X$ in Sp induced a map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

of ringed spaces, which is by *pullback*. I.e. for $U \subseteq X$ open and $V := f^{-1}(U) \subseteq Y$ the ring homomorphism

$$\Gamma(U, \psi) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, f_*(\mathcal{O}_Y)) = \Gamma(V, \mathcal{O}_Y)$$

sends $a \in \Gamma(U, \mathcal{O}_X)$ to

(1.1.3) $$\Gamma(U, \psi)(a) := a \circ f|_V : V \to \mathbb{K}.$$

For this reason we shall often write $f^* := \psi$, so the map of ringed spaces is (f, f^*).

In this way we obtain a functor

(1.1.4) $$\mathrm{RS} : \mathsf{Sp} \to \mathsf{RSp}/\mathbb{K}, \quad X \mapsto (X, \mathcal{O}_X), \quad f \mapsto (f, f^*).$$

This functor is clearly faithful.

◇ ◇ ◇

To a ringed space $(X, \mathcal{O}_X)$ we attach the category $\mathsf{Mod}(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules.

There are several important internal operations on $\mathcal{O}_X$-modules.

Given a collection $\{\mathcal{M}_i\}_{i \in I}$ of $\mathcal{O}_X$-modules, we can consider the direct sum $\bigoplus_{i \in I} \mathcal{M}_i$ and the product $\prod_{i \in I} \mathcal{M}_i$.

For $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}(\mathcal{O}_X)$ we have the $\mathcal{O}_X$-modules $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. We know that

(1.1.5) $$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_U)}(\mathcal{M}|_U, \mathcal{N}|_U)).$$

◇ ◇ ◇

A sequence of homomorphisms

$$\cdots \to \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \to \cdots$$

in $\mathsf{Mod}(\mathcal{O}_X)$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \to \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \to \cdots$$

in $\mathsf{Mod}(\mathcal{O}_{X,x})$ is exact.

To a homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}(\mathcal{O}_X)$ we attach the kernel and the cokernel, and these make the sequence

$$0 \to \mathrm{Ker}(\phi) \to \mathcal{M} \xrightarrow{\phi} \mathcal{N} \to \mathrm{Coker}(\phi) \to 0$$

in $\mathsf{Mod}(\mathcal{O}_X)$ is exact.

◇ ◇ ◇

Suppose we are given a map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

There there are functors

$$f_* : \mathsf{Mod}(\mathcal{O}_Y) \to \mathsf{Mod}(\mathcal{O}_X)$$

and

$$f^* : \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}(\mathcal{O}_Y).$$

Recall that $f_*(\mathcal{N})$ is defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N}),$$

and

$$f^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{M}),$$

where $f^{-1}(\mathcal{M})$ is the sheaf associated to the presheaf

$$V \mapsto \lim_{\to} \Gamma(U, \mathcal{M})$$

where $U \subseteq X$ is open and $f(V) \subseteq U$.

These functors satisfy the adjunction formula

$$\mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_X)}(\mathcal{M}, f_*(\mathcal{N})) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_Y)}(f^*(\mathcal{M}), \mathcal{N}). \tag{1.1.6}$$

This isomorphism is functorial in $\mathcal{M}$ and $\mathcal{N}$.

1.2. **Locally Ringed Spaces and Their Maps.** Now to new material.

Let's recall that a local ring is a ring A that has exactly one maximal ideal, say $\mathfrak{m}$. This is often denoted by $(A, \mathfrak{m})$. If $(B, \mathfrak{n})$ is another local ring, then a local ring homomorphism $\psi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a ring homomorphism $\psi : A \to B$ s.t. $\psi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Definition 1.2.1. A ringed space $(X, \mathcal{O}_X) \in \mathsf{RSp}/\mathbb{K}$ is called a *locally ringed space* if for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

Definition 1.2.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space over $\mathbb{K}$. Given a point $x \in X$, the maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$, and the residue field is

$$\boldsymbol{k}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

is

Definition 1.2.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces over $\mathbb{K}$. A *map of locally ringed spaces over* $\mathbb{K}$ is a map of $\mathbb{K}$-ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

such that for every point $y \in Y$ the $\mathbb{K}$-ring homomorphism

$$\psi_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$$

between the stalks is a local homomorphism.

The category of locally ringed spaces over $\mathbb{K}$, with maps as above, is denoted by $\mathsf{LRSp}/\mathbb{K}$.

Definition 1.2.4. If $(X, \mathcal{O}_X)$ is a locally ringed space and $U \subseteq X$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is also a locally ringed space. It is called an *open subspace* of $(X, \mathcal{O}_X)$, and the inclusion map

$$(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

is called an *open embedding.*

Proposition 1.2.5. *Consider one of our three favorite categories of geometric spaces* Sp *from Example 1.1.2, with corresponding base field* $\mathbb{K}$*. For* $X \in \mathsf{Sp}$ *the sheaf of functions is* $\mathcal{O}_X$*.*

1. *The* $\mathbb{K}$*-ringed space* $(X, \mathcal{O}_X)$ *is a locally ringed space.*
2. *Let* $f : Y \to X$ *be a map in* Sp*, and let*

$$(f, f^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

be the corresponding map of ringed spaces. Then (f, f^*) *is a map of locally ringed spaces over* $\mathbb{K}$*.*

(3) *We obtain a functor*

$$\mathrm{LRSp} : \mathsf{Sp} \to \mathsf{LRSp}/\mathbb{K}, \quad X \mapsto (X, \mathcal{O}_X), \quad f \mapsto (f, f^*).$$

Exercise 1.2.6. Prove Proposition 1.2.5.

The next theorem is supposed to tell us how powerful the concept of locally ringed spaces is.

Theorem 1.2.7. *Consider one of the three categories of geometric spaces* Sp *from Example* 1.1.2*, with corresponding base field* $\mathbb{K}$*.*
Then the functor

$$\mathrm{LRSp} : \mathsf{Sp} \to \mathsf{LRSp}/\mathbb{K}$$

is fully faithful.

The proof will be given next week.

Remark 1.2.8. Schemes are locally ringed spaces. When we learn about affine schemes I will give an example of a map of ringed spaces $(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ between affine schemes that is not a map of locally ringed spaces.

Exercise 1.2.9. Try to find an example of real differentiable manifolds X and Y, and a map of ringed spaces $(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ that is not a map of locally ringed spaces. (Note that $f : Y \to X$ will not be a map of manifolds.) Try looking this up in [Jo] or mathexchange...

End of Lecture 1

Lecture 2, 10 March 2021

We will start with the proof of Theorem 1.2.7. Then we will start learning about affine schemes (a change of the program).

Observe that Theorem 1.2.7 means that the geometry Sp, namely the information added to the underlying topological space X, is completely encoded in the abstract notion of "locally ringed spaces and their maps".

Proof of Theorem 1.2.7. The faithfulness of the functor LRSp is easy to see: the forgetful functor $\mathsf{Sp} \to \mathsf{Set}$ is faithful ($f = g$ iff they are equal as functions between sets), and it factors through $\mathsf{LRSp}/\mathbb{K}$.

The challenge is to prove fullness. Let

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

be a morphism in $\mathsf{LRSp}/\mathbb{K}$ between $(X, \mathcal{O}_X) = \mathrm{LRSp}(X)$ and $(Y, \mathcal{O}_Y) = \mathrm{LRSp}(Y)$ for two spaces $X, Y \in \mathsf{Sp}$. We must prove that the continous map $f : Y \to X$ is a map in Sp, and that $\psi = f^*$.

Step 1. In this step we prove that $\psi = f^*$. Consider an open set $U \subseteq X$ and a function $a \in \Gamma(U, \mathcal{O}_X)$. We must prove that

$$\psi(a) = f^*(a) \in \Gamma(U, f_*(\mathcal{O}_Y)).$$

In other words, letting $V := f^{-1}(U) \subseteq Y$, we must prove that the functions

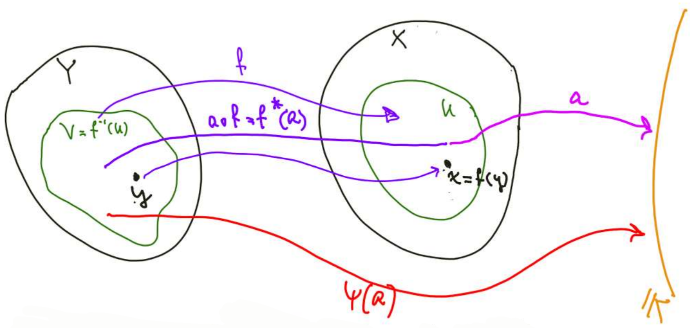
$$\psi(a),\ f^*(a) : V \to \mathbb{K}$$

are equal. Since $f^*(a) = a \circ f|_V$, this amounts to showing that for every point $y \in V$ there is equality

$$\psi(a)(y) = (a \circ f)(y) = a(x),$$

where $x := f(y) \in U$. See picture (1.2.10).

(1.2.10)



Let $\lambda := a(x) \in \mathbb{K}$. Becsue both ψ and f^* are $\mathbb{K}$-ring homomorphisms, it suffices to prove that the function $b := a - \lambda : U \to \mathbb{K}$ satisfies $\psi(b)(y) = f^*(b)(y)$. Now $f^*(b)(y) = b(x) = a(x) - \lambda = 0$. It remains to prove that $\psi(b)(y) = 0$.

Consider the commutative diagram

(1.2.11)
$$
\begin{tikzcd}
\Gamma(U, \mathcal{O}_X) \arrow[r, "\Gamma(U,\psi)"] \arrow[d, "\mathrm{rest}_{x/U}"] \arrow[dd, bend right=60, "\mathrm{ev}_x"'] & \Gamma(V, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{y/V}"'] \arrow[dd, bend left=60, "\mathrm{ev}_y"] \\
\mathcal{O}_{X,x} \arrow[r, "\psi_y"] \arrow[d, "\mathrm{ev}_x"] & \mathcal{O}_{Y,y} \arrow[d, "\mathrm{ev}_y"'] \\
\mathbb{K} \arrow[r, "\mathrm{id}"'] & \mathbb{K}
\end{tikzcd}
$$

of $\mathbb{K}$-rings. The function b is in the upper left corner. The germ $\tilde{b} := \mathrm{rest}_{x/U}(b) \in \mathcal{O}_{X,x}$ belongs to the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, because $\mathrm{ev}_x(\tilde{b}) = b(x) = 0$. We know that ψ_y is a *local homomorphism*, and therefore $\psi_i(\tilde{b}) \in \mathfrak{m}_y$. It follows that

$$\psi(b)(y) = \mathrm{ev}_y(\psi(b)) = \mathrm{ev}_y(\psi_y(\tilde{b})) = 0 \in \mathbb{K}.$$

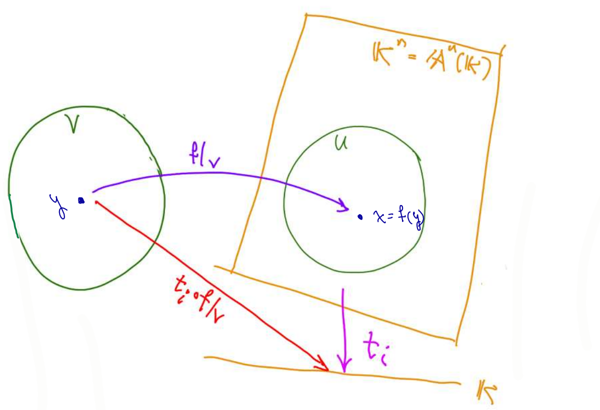
Step 2. Now we shall prove that $f : Y \to X$ is a map in Sp. For $\mathsf{SP} = \mathsf{Top}$ it is automatic. For the other two geometries this is a local question on Y. Take a point $y \in Y$, and let $x := f(y) \in X$. Choose an open neighborhood U of x in X that embeds into $\mathbf{A}^n(\mathbb{K})$, as an open subspace for $\mathsf{SP} = \mathsf{Mfld}$, and as a closed intersect open subvariety for $\mathsf{SP} = \mathsf{Var}$. Take $V := f^{-1}(U) \subseteq Y$, which is an open neighborhood of y. It suffices to prove that the map $f|_V : V \to U$ is a map in Sp.

Let
$$t_1, \ldots, t_n \in \Gamma\big(\mathbf{A}^n(\mathbb{K}), \mathcal{O}_{\mathbf{A}^n(\mathbb{K})}\big)$$
be the coordinate functions. The map $f|_V : V \to U$ is in Sp iff the functions

$$t_i \circ f|_V = f|_V^*(t_i) : V \to \mathbb{K}$$

are in Sp for all i; i.e. if
$$f|_V^*(t_i) \in \Gamma(V, \mathcal{O}_Y)$$
for all i. See diagrm (1.2.12).

(1.2.12)



But by step 1 we know that

$$f|_V^*(t_i) = \psi(t_i) \in \Gamma(V, \mathcal{O}_Y). \qquad \square$$

2. Affine Schemes

2.1. Definitions and Basic Properties.

Affine schemes were introduced by A. Grothendieck in the 1950's, following attempts by geometers such as Serre, Chevalley, Weil, Zariski and others.

Definition 2.1.1 (Affine Schemes as Sets)**.** Let A be a $\mathbb{K}$-ring. The *prime spectrum* of A is the set $\operatorname{Spec}(A)$ of prime ideals of A.

Example 2.1.2. Before going any further, here is an attempt to explain the name "spectrum". Let $\mathbb{K}$ be an algebraically closed field, and let $a \in \operatorname{Mat}_{n\times n}(\mathbb{K})$ be a matrix of size $n \geq 1$. Define $A := \mathbb{K}[a] \subseteq \operatorname{Mat}_{n\times n}(\mathbb{K})$, so A is a commutative subring of the NC ring $\operatorname{Mat}_{n\times n}(\mathbb{K})$. What is the spectrum of A? It is the set of eigenvalues of a. Indeed, if $p(t) \in \mathbb{K}[t]$ is the minimal polynomial of a, then $A \cong \mathbb{K}[t]/(p(t))$ as $\mathbb{K}$-rings. If $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ are the eigenvalues, then $p(t) = \prod_i (t - \lambda_i)$, and the prime ideals of A are the maximal ideals $(a - \lambda_i) \subseteq A$.

For an ideal $\mathfrak{a} \subseteq A$ we define its *zero locus*

$$\operatorname{Zer}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A) \tag{2.1.3}$$

and *nonzero locus*

$$\operatorname{NZer}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A). \tag{2.1.4}$$

Of course

$$\operatorname{Spec}(A) = \operatorname{Zer}(\mathfrak{a}) \sqcup \operatorname{NZer}(\mathfrak{a}),$$

a disjoint union.

Proposition 2.1.5 (The Zariski Topology)**.** *Let A be a ring. The collection of subsets $\{\operatorname{NZer}(\mathfrak{a})\}$, indexed by the ideals $\mathfrak{a} \subseteq A$, is a topology on the set* $\operatorname{Spec}(A)$*. It is called the* Zariski topology.

Like many of the results here, you have seen them before in the course on classical algebraic geometry, so the proofs are left as exercises.

Exercise 2.1.6. Prove the proposition above.

Here is an explanation of the terminilogy. To a prime ideal $\mathfrak{p}$ we associate the local ring $A_{\mathfrak{p}}$ and the residue field $\boldsymbol{k}(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. An element $a \in A$ has a residue class $a(\mathfrak{p}) \in \boldsymbol{k}(\mathfrak{p})$, coming from the canonical ring homomorphism $A \to \boldsymbol{k}(\mathfrak{p})$. In this way the elements of the ring A are "functions", but the value $a(\mathfrak{p})$ of $a \in A$ at each point $\mathfrak{p}$ is in a field depending on $\mathfrak{p}$...

Anyhow:

$$\operatorname{Zer}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid a(\mathfrak{p}) = 0 \ \text{ for all } \ a \in \mathfrak{a}\}. \tag{2.1.7}$$

This formula interprets the set $\operatorname{Zer}(\mathfrak{a})$ as the "set of zeroes of all functions $a \in \mathfrak{a}$".

Exercise 2.1.8. Prove the formula above.

For an element $s \in A$ we define

$$\operatorname{NZer}(s) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \notin \mathfrak{p}\}. \tag{2.1.9}$$

This is an open set: it it the complement of the closed set $\operatorname{Zer}(\mathfrak{a})$, where $\mathfrak{a} := (s)$, the principal ideal generated by s. We call such an open set a *principal open set.* Analogously to (2.1.7) we have

$$\operatorname{NZer}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s(\mathfrak{p}) \neq 0\}. \tag{2.1.10}$$

Proposition 2.1.11. *The principal open sets are a basis of the topology of* $\operatorname{Spec}(A)$*. Namely every open set set* U *is a union* $U = \bigcup_i \operatorname{NZer}(s_i)$ *for a suitable collection* $\{s_i\}$ *of elements of* A*.*

Exercise 2.1.12. Prove the proposition above. (Hint: it is easy.)

[**comment**: (210405 AY) new definition next:]

Definition 2.1.13. Let A be a ring, and write $X := \operatorname{Spec}(A)$ for this topological space, with the Zariski topology. For an open set $U \subseteq X$ we let $\mathrm{S}(U) \subseteq A$ be the multiplicatively closed set

$$\mathrm{S}(U) := \{s \in A \mid s(\mathfrak{p}) \neq 0 \ \text{ for all } \ \mathfrak{p} \in U\}.$$

Let $A_{\mathrm{S}(U)}$ be the localization of A w.r.t. $\mathrm{S}(U)$.

Lemma 2.1.14. *The assignment* $U \mapsto \mathrm{S}(U)$ *is a presheaf of* $\mathbb{K}$*-rings on* $X = \operatorname{Spec}(A)$*, which we denote by* $\mathcal{O}_X^{\mathrm{pre}}$*.*

Proof. This is easy: if $V \subseteq U$ is a smaller open set, then $\mathrm{S}(U) \subseteq \mathrm{S}(V)$, so by the universal property of localization there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(V)}$. □

Definition 2.1.15. Let A be a ring. The *structure sheaf* of $X := \operatorname{Spec}(A)$ is the sheaf of rings $\mathcal{O}_X := \operatorname{Sh}(\mathcal{O}_X^{\mathrm{pre}})$.

The *affine scheme* $\operatorname{Spec}(A)$ is the $\mathbb{K}$-ringed space $(X, \mathcal{O}_X)$.

Proposition 2.1.16. *Let* A *be a ring and write* $X := \operatorname{Spec}(A)$*. For every point* $x = \mathfrak{p} \in X$ *the stalk of* $\mathcal{O}_X$ *at* x *is* $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$*, the local ring at* $\mathfrak{p}$*. More precisely, there is a unique* A*-ring isomorphism* $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$*.*

The proof was not done in the lecture. Please read it.

Proof. Let $\mathrm{S}(x) := A - \mathfrak{p}$, the complement of $\mathfrak{p}$. By definition we have $A_{\mathfrak{p}} = A_{\mathrm{S}(x)}$. The universal property of localization says that there is at most one A-ring isomorphism $A_{\mathfrak{p}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$. We will produce it.

Let us denote by $\boldsymbol{U}_x$ the set of principal open neighborhoods of x. So

$$\boldsymbol{U}_x = \{\operatorname{NZer}(s)\}_{s \in \mathrm{S}(x)}.$$

By Proposition 2.1.11, $\boldsymbol{U}_x$ is a basis of open neighborhoods of x. Because the stalks of the presheaf $\mathcal{O}_X^{\mathrm{pre}}$ and its associated sheaf $\mathcal{O}_X$ are the same, we have

$$\mathcal{O}_{X,x} = \lim_{U \to} \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = \lim_{U \to} A_{\mathrm{S}(U)} \tag{2.1.17}$$

where U runs over the poset $\boldsymbol{U}_x$.

For each $U = \operatorname{NZer}(s) \in \boldsymbol{U}_x$ we have $\mathrm{S}(U) \subseteq \mathrm{S}(x)$, so there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(x)} = A_{\mathfrak{p}}$. Going to the direct limit in U we get an A-ring homomorphism

$$\psi : \mathcal{O}_{X,x} \to A_{\mathfrak{p}}.$$

In the other direction, every element $s \in \mathrm{S}(x)$ belongs to $\mathrm{S}(U)$ for $U = \operatorname{NZer}(s) \in \boldsymbol{U}_x$, and hence s is invertible in $A_{\mathrm{S}(U)}$. This means that every $s \in \mathrm{S}(x)$ is invertible in the A-ring $\mathcal{O}_{X,x} = \lim_{\to} A_{\mathrm{S}(U)}$. Hence there is a unique A-ring homomorphism

$$\phi : A_{\mathfrak{p}} = A_{\mathrm{S}(x)} \to \mathcal{O}_{X,x}.$$

The homomorphism ϕ is surjective, because every $f \in \mathcal{O}_{X,x}$ is the image of some fraction $a/s \in A_{\mathrm{S}(U)}$, see (2.1.17). But $s \in \mathrm{S}(x)$, so $a/s \in A_{\mathfrak{p}}$ and $f = \phi(a/s)$.

Finally consider the A-ring homomorphism $\psi \circ \phi$ from $A_{\mathfrak{p}}$ to itself. By uniqueness there is equality $\psi \circ \phi = \mathrm{id}_{A_{\mathfrak{p}}}$. This shows that ϕ is injective. In conclusion, ϕ is an isomorphism of A-rings. □

Corollary 2.1.18. *Let A be a $\mathbb{K}$-ring. Then $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ is a locally $\mathbb{K}$-ringed space.*

Definition 2.1.19. An *affine* $\mathbb{K}$*-scheme* is a locally ringed space $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic, in $\mathsf{LRSp}/\mathbb{K}$, to $\mathrm{Spec}(A)$ for some $\mathbb{K}$-ring A.

Definition 2.1.20. The category $\mathsf{AffSch}/\mathbb{K}$ is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the set of affine $\mathbb{K}$-schemes.

◇ ◇ ◇

From here on this is reading material

We now provide a more explicit description of the structure sheaf $\mathcal{O}_X$ os the affine scheme $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ in terms of its *Godement sheaf* from [Ye4, Def 3.6.11]. Recall that for a presheaf of $\mathbb{K}$-modules $\mathcal{M}$ on a space X, its Godement sheaf $\mathrm{GSh}(\mathcal{M})$ is defined by

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x$$

for an open set $U \subseteq X$. Then $\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$ is the subsheaf of *geometric sections.*

Specializing to our case it says the following:

Proposition 2.1.21. *Let $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$. Then $\mathcal{O}_X$ is the subsheaf of the Godement sheaf $\mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})$ consisting of the geometric sections. Specifically, let $U \subseteq X$ be an open set, and let*

$$f \in \Gamma\big(U, \mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

The section f belongs to $\Gamma(U, \mathcal{O}_X)$ iff for every point $x = \mathfrak{p} \in U$ there is an open set V such that $x \in V \subseteq U$, and elements $a \in A$ and $s \in \mathrm{S}(V) \subseteq A$, such that

$$f|_V = a \cdot s^{-1} \in \Gamma(V, \mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})) = \prod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Here are some example of affine schemes.

Example 2.1.22. Consider the ring $A = \mathbb{Z}$. The affine scheme $X := \mathrm{Spec}(\mathbb{Z})$ has these points:

- For every (positive) prime number p there is a maximal ideal $x = \mathfrak{m} := (p)$. These are closed points of X, since $\mathrm{Zer}(p) = \{x\}$. The local ring is

$$\mathcal{O}_{X,x} = \mathbb{Z}_{\mathfrak{m}} = \{a \cdot s^{-1} \mid s \notin \mathfrak{m}\} \subseteq \mathbb{Q}.$$

  The residue field is $\mathbb{F}_p$.
- The zero ideal $y = \mathfrak{p} := (0)$ is also prime. It is the *generic point* of X; namely its topological closure is X. To see this, take a nonempty open set $U \subseteq X$. We will prove that $y \in U$. By Prop 2.1.11 there is a nonempty $\mathrm{NZer}(s) \subseteq U$, so $s \neq 0$. Then $s \notin \mathfrak{p}$, so $y = \mathfrak{p} \in \mathrm{NZer}(s) \subseteq U$. The local ring and the residue field at $\mathfrak{p}$ are $\mathbb{Q}$.

Exercise 2.1.23. Analyze the affine scheme $\operatorname{Spec}(A)$ for the ring $A := \mathbb{K}[t]$, the polynomial ring in one variable over a field $\mathbb{K}$. State what is special when $\mathbb{K}$ is algebraically closed. (Hint: it is very similar to the example above.)

Example 2.1.24. Suppose $\mathbb{K}$ is a nonzero ring. For every $n \geq 0$ there is the affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

called the *n-dimensional affine space over* $\mathbb{K}$.

In Rem 1.2.8 I promised something, and it is fulfilled in the next example.

Example 2.1.25. We take a DVR A inside its fraction field K; to be concrete, we can take $K := \mathbb{Q}$ and $A := \mathbb{Z}_{\mathfrak{m}}$ where $\mathfrak{m} = (3)$. The affine scheme $(Y, \mathcal{O}_Y) = \operatorname{Spec}(K)$ has a single point, the zero ideal of K, which we call y_0. The affine scheme $(X, \mathcal{O}_X) = \operatorname{Spec}(A)$ has two points: the maximal ideal $x_1 = \mathfrak{m}$ and the zero ideal x_0. The open sets of Y are $Y = \{y_0\}$ and $\varnothing$. The open sets of X are X, $\{x_0\}$ and $\varnothing$.

We want to define a map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X),$$

and we deliberately do it the wrong way, so it will fail to be local.

Define $f : Y \to X$ by $f(y_0) := x_1$. To define ψ we need a ring homomorphism $\psi_U : \Gamma(U, \mathcal{O}_X) \to \Gamma(f^{-1}(U), \mathcal{O}_Y)$ for every $U \subseteq X$ open. The only nontrivial case is for $U = X$, and we let it be the inclusion $A \to K$. Now note that $\mathcal{O}_{X,x_1} = A$ and $\mathcal{O}_{Y,y_0} = K$, and the ring homomorphism $\psi_{y_0} : \mathcal{O}_{X,x_1} = A \to \mathcal{O}_{Y,y_0} = K$ is not local.

$\boxed{\text{End of Lecture 2}}$

Lecture 3, 17 March 2021

2.2. **More Properties of Affine Schemes.** In this subsection we shall encounter the first difficult theorem on affine schemes (Thm 2.2.14)

There will many lemmas, some not very interesting to prove, so I will let you read them at home (or prove as exercises).

I'd like to start by retracting the notation (2.1.4), because it seems to be confusing. We shall only use its variant $\mathrm{NZer}(s)$ from (2.1.9). We will sometimes write $\mathrm{NZer}_X(s)$ for the principal open set in $X := \mathrm{Spec}(A)$ defined by an element $s \in A$, and $\mathrm{Zer}_X(\mathfrak{a})$ for the closed set in X defined by an ideal $\mathfrak{a}$.

Recall that for an ideal $\mathfrak{a} \subseteq A$, its *radical* of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A \mid a^i \in \mathfrak{a} \text{ for some } i > 0\} \subseteq A.$$

Lemma 2.2.1. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Zer}(\mathfrak{a})} \mathfrak{p}$$

and

$$\mathrm{Zer}(\mathfrak{a}) = \mathrm{Zer}(\sqrt{\mathfrak{a}}).$$

Exercise 2.2.2. Prove the lemma. (Hint: do not use the Nullstellensatz.)

Lemma 2.2.3. *For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ the following are equivalent:*

(i) $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.
(ii) $\mathrm{Zer}(\mathfrak{a}) = \mathrm{Zer}(\mathfrak{b})$.

Proof. Clear from Lemma 2.2.1. □

Definition 2.2.4. A topological space X is called *quasi-compact* if every open covering of X has a finite subcovering.

Remark 2.2.5. The term "compact" is usually reserved for spaces that are Hausdorff and quasi-compact. Schemes are almost never Hausdorff. There is an analogous notion of separation for scheme, that we will (hopefully) study later.

Proposition 2.2.6. *Let A be a ring. The topological space $X := \mathrm{Spec}(A)$ is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering, indexed by an infinite set I. For each i there is an ideal $\mathfrak{a}_i$ such that $U_i = X - \mathrm{Zer}_X(\mathfrak{a}_i)$; this is by the definition of the Zariski topology. Write $\mathfrak{a} := \sum_i \mathfrak{a}_i$. Then there is equality

$$\mathrm{Zer}_X(A) = \varnothing = \bigcap_{i \in I} \mathrm{Zer}_X(\mathfrak{a}_i) = \mathrm{Zer}_X(\mathfrak{a})$$

of subsets of X. By Lemma 2.2.3 we know that $\sqrt{A} = \sqrt{\mathfrak{a}}$. Since $1 \in A$, we see that $1 \in \sqrt{\mathfrak{a}}$, and hence $1 \in \mathfrak{a}$. This says that we can express 1 as a finite sum: $1 = \sum_{i \in I'} a_i$ with $I' \subseteq I$ a finite subset and $a_i \in \mathfrak{a}_i$. We see that $A = \sum_{i \in I'} \mathfrak{a}_i$, and therefore

$$\varnothing = \mathrm{Zer}_X(A) = \bigcap_{i \in I'} \mathrm{Zer}_X(\mathfrak{a}_i)$$

and $X = \bigcup_{i \in I'} U_i$. □

Lemma 2.2.7. *Let $s_1, \ldots, s_m \in A$. TFAE:*

(i) $\mathrm{Spec}(A) = \bigcup_i \mathrm{NZer}(s_i)$.

(ii) *There exists* $a_1, \ldots, a_m \in A$ *s.t.* $1_A = \sum_i a_i \cdot s_i$.

Proof.

(i) $\Rightarrow$ (ii): As in the proof of the proposition, $A = \sum_i \mathfrak{a}_i$, where $\mathfrak{a}_i := (s_i)$. So the element 1_A is a linear combination $1_A = \sum_i a_i \cdot s_i$ for some $a_i \in A$.

(ii) $\Rightarrow$ (i): Here $1_A \in \sum_i \mathfrak{a}_i$, so $A = \sum_i \mathfrak{a}_i$, and, as in the proof of the proposition, $\operatorname{Spec}(A) = \bigcup_i \operatorname{NZer}(s_i)$. □

Lemma 2.2.8. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $s \in A$ *and* $U := \operatorname{NZer}_X(s)$. *Then* s *is invertible in the ring* $\Gamma(U, \mathcal{O}_X)$.

Proof. Clearly $s \in \mathrm{S}(U) \subseteq A$. By definition of the presheaf $\mathcal{O}_X^{\mathrm{pre}}$, the element s is invertible in the ring $A_{\mathrm{S}(U)} = \Gamma(U, \mathcal{O}_X^{\mathrm{pre}})$. Due to the homomorphism of sheaves of rings $\mathcal{O}_X^{\mathrm{pre}} \to \mathcal{O}_X$ we see that s is invertible in the ring $\Gamma(U, \mathcal{O}_X)$. □

Lemma 2.2.9. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. *Suppose* $U \subseteq X$ *is an open set,* $f \in \Gamma(U, \mathcal{O}_X)$, *and* $x = \mathfrak{p} \in U$. *Then there are elements* $a, s \in A$ *such that, letting* $W := \operatorname{NZer}_X(s)$, *we have* $x \in W \subseteq U$ *and*

$$f|_W = a \cdot s^{-1} \in \Gamma(W, \mathcal{O}_X).$$

Proof. The proof is interesting, so I'll do it in class.

By Proposition 2.1.21 the point $x = \mathfrak{p} \in U$ has an open neighborhood $V \subseteq U$, and elements $b, t \in A$, such that $t \in \mathrm{S}(V)$ and $f|_V = b \cdot t^{-1} \in \Gamma(V, \mathcal{O}_X)$.

According to Proposition 2.1.11 we can replace V with a smaller open neighborhood $\operatorname{NZer}_X(r)$ of x for some $r \in A$, i.e. $x \in \operatorname{NZer}_X(r) \subseteq V$.

Define the elements $a := b \cdot r \in A$ and $s := t \cdot r \in A$, and the open set $W := \operatorname{NZer}_X(s)$.

Since $\operatorname{NZer}_X(r) \subseteq V \subseteq \operatorname{NZer}_X(t)$, it follows that

$$\operatorname{NZer}_X(r) = \operatorname{NZer}_X(t \cdot r) = \operatorname{NZer}_X(s) = W.$$

By Lem 2.2.8 we know that s is invertible in $\Gamma(W, \mathcal{O}_X)$. Hence t and r are invertible in $\Gamma(W, \mathcal{O}_X)$. And

$$a \cdot s^{-1} = (b \cdot r) \cdot (t \cdot r)^{-1} = b \cdot t^{-1} = f|_W \in \Gamma(W, \mathcal{O}_X).$$ □

Lemma 2.2.10. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. *Suppose* $V \subseteq X$ *is a quasi-compact open set, and* $f \in \Gamma(V, \mathcal{O}_X)$. *Then there are finitely many elements* $a_1, \ldots, a_m, s_1, \ldots, s_m \in A$ *such that, writing* $W_i := \operatorname{NZer}_X(s_i)$, *we have* $V = \bigcup_{i=1}^m W_i$, *and*

$$f|_{W_i} = a_i \cdot s_i^{-1} \in \Gamma(W_i, \mathcal{O}_X)$$

for every i.

Proof. By Lemma 2.2.9, for every point $x = \mathfrak{p} \in V$ there are elements $a_x, s_x \in A$ such that, writing $W_x := \operatorname{NZer}_X(s_x)$, we have $x \in W_x \subseteq V$ and

$$f|_{W_x} = a_x \cdot s_x^{-1} \in \Gamma(W_x, \mathcal{O}_X).$$

We have an open covering $V = \bigcup_{x \in V} W_x$. Because of quasi-compactness we can pass to a finite subcovering, that is to a finite subset $\{x_1, \ldots, x_m\}$ of V. Finally, by letting $a_i := a_{x_i}$, $s_i := s_{x_i}$ and $W_i := W_{x_i}$ we are done. □

A ring homomorphism $\psi : A \to B$ induces a map of sets

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A) \tag{2.2.11}$$

with formula

$$\operatorname{Spec}(\psi)(\mathfrak{q}) := \psi^{-1}(\mathfrak{q}).$$

If it is not clear to you why the ideal $\mathfrak{p} := \psi^{-1}(\mathfrak{q}) \subseteq A$ is prime, then prove it!

Lemma 2.2.12. *Let $\psi : A \to B$ be a ring homomorphism.*

(1) *The map $\operatorname{Spec}(\psi)$ is continuous.*
(2) *Suppose $B = A_s$ for some $s \in A$ and ψ is the localization homomorphism. Then the image of $\operatorname{Spec}(\psi)$ is the open set $\operatorname{NZer}(s) \subseteq \operatorname{Spec}(A)$, and*

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(A_s) \to \operatorname{NZer}(s)$$

is a homeomorphism.

Exercise 2.2.13. Prove the last lemma. (Hint: for (i) show that $f := \operatorname{Spec}(\psi)$ satisfies $f^{-1}(\operatorname{NZer}(s)) = \operatorname{NZer}(\psi(s))$.)

Here is the main theorem of this subsection. The proof is by a magical juggling of denominators.

Theorem 2.2.14. *Let A be a ring and $s \in A$ an element. Consider the affine scheme $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ and the principal open set $U := \operatorname{NZer}_X(s)$. There is a unique A-ring isomorphism $A_s \cong \Gamma(U, \mathcal{O}_X)$.*

Proof.

Step 1. Since the element s is invertible in the ring $\Gamma(U, \mathcal{O}_X)$, there is a unique A-ring homomorphism

$$\phi : A_s \to \Gamma(U, \mathcal{O}_X). \tag{2.2.15}$$

We need to prove that ϕ is bijective.

Step 2. In this step we we will prove that ϕ is injective. Since $\mathcal{O}_X$ is a subsheaf of the Godement sheaf $\operatorname{GSh}(\mathcal{O}_X)$, there is an embedding of A-rings

$$\Gamma(U, \mathcal{O}_X) \rightarrowtail \Gamma\big(U, \operatorname{GSh}(\mathcal{O}_X)\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

It thus suffices to prove that the composed homomorphism

$$\phi' : A_s \to \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \tag{2.2.16}$$

is injective.

Let's write $B := A_s$. By Lemma 2.2.12 we know that $\operatorname{Spec}(B) = U$ as topological subspaces of X. For every $\mathfrak{p} \in U$ the element s is invertible in $A_{\mathfrak{p}}$, and hence the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is bijective. So we can rewrite (2.2.16) as $\phi' : B \to \prod_{\mathfrak{p} \in U} B_{\mathfrak{p}}$, and we need to prove it is injective.

Suppose $b \in B$ is such that $\phi'(b) = 0$. We need to prove that $b = 0$.

The vanishing of $\phi'(b)$ means that $\phi'_{\mathfrak{p}}(b) = 0$ in $B_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. Next, the vanishing of b in $B_{\mathfrak{p}}$ means that there is some element $t_{\mathfrak{p}} \in B$ such that $t_{\mathfrak{p}}(\mathfrak{p}) \neq 0$, i.e. $\mathfrak{p} \in \operatorname{NZer}(t_{\mathfrak{p}})$, yet $t_{\mathfrak{p}} \cdot b = 0$ in B.

Now $U = \operatorname{Spec}(B) = \bigcup_{\mathfrak{p} \in U} \operatorname{NZer}(t_{\mathfrak{p}})$. By quasi-compactness of U we can pass to a finite subcovering, indexed by $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Let $t_i := t_{\mathfrak{p}_i} \in B$. Then $t_i \cdot b = 0$ in B, and $\operatorname{Spec}(B) = \bigcup_{i=1}^m \operatorname{NZer}(t_i)$. According to Lemma 2.2.7 there are elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Then $b = 1_B \cdot b = \sum_i c_i \cdot t_i \cdot b = 0$.

End of live lecture

Please read end of proof at home.

Step 3. We now prove that the homomorphism ϕ from (2.2.15) is surjective. Recall that $B = A_s$ and $U = \operatorname{Spec}(B)$. Take an element $f \in \Gamma(U, \mathcal{O}_X)$. We know that $U = \operatorname{Spec}(B)$ is a quasi-compact topological space. By Lem 2.2.10 there are finitely many elements $a_i, s_i \in A$, $1 \leq i \leq m$, such that, letting $U_i := \operatorname{NZer}_U(s_i)$, we have $U = \bigcup_{i=1}^m U_i$, and $f|_{U_i} = a_i/s_i \in \Gamma(U_i, \mathcal{O}_X)$ for every i.

Step 2, when applied to the element $s_i \cdot s_j \cdot s \in A$ instead of to s, shows that for every i, j, letting $U_{i,j} := \operatorname{NZer}_U(s_i \cdot s_i) = U_i \cap U_j$, the A-ring homomorphism

$$\phi_{i,j} : A_{s_i \cdot s_j \cdot s} = B_{s_i \cdot s_j} \to \Gamma\big(U_{i,j}, \mathcal{O}_X\big)$$

is injective. We know that

$$f|_{U_{i,j}} = a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1} \in \Gamma\big(U_{i,j}, \mathcal{O}_X\big).$$

Therefore $a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1}$ in $B_{s_i \cdot s_j}$. The kernel of the localization homomorphism $B \to B_{s_i \cdot s_j}$ is known: there is a positive integer $l_{i,j}$ such that

$$(s_i \cdot s_j)^{l_{i,j}} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0$$

in B. Taking $l := \max(\{l_{i,j}\})$ we obtain

$$(s_i \cdot s_j)^l \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0 \tag{2.2.17}$$

in B.

Define $b_i := a_i \cdot s_i^l$ and $t_i := s_i^{l+1}$. Then $\operatorname{NZer}_U(t_i) = \operatorname{NZer}_U(s_i) = U_i$ and

$$f|_{U_i} = a_i \cdot s_i^{-1} = b_i \cdot t_i^{-1} \in \Gamma(U_i, \mathcal{O}_X). \tag{2.2.18}$$

Also, from (2.2.17) we have

$$t_i \cdot b_j = s_i^{l+1} \cdot a_j \cdot s_j^l = (s_i \cdot s_j)^l \cdot s_i \cdot a_j = (s_i \cdot s_j)^l \cdot s_j \cdot a_i = t_j \cdot b_i \tag{2.2.19}$$

in B.

Since

$$\operatorname{Spec}(B) = U = \bigcup_{i=1}^m U_i, \tag{2.2.20}$$

and $U_i = \operatorname{NZer}(t_i)$, by Lem 2.2.7 we can find elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Let

$$b := \sum_i c_i \cdot b_i \in B.$$

For every i we have – using (2.2.19) –

$$\begin{aligned} \phi(b)|_{U_i} &= \sum_j c_j \cdot b_j|_{U_i} = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_i \cdot b_j\Big)|_{U_i} = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j \cdot b_i\Big)|_{U_i} \\ &= t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j\Big) \cdot b_i|_{U_i} = t_i^{-1} \cdot b_i|_{U_i} = f|_{U_i} \end{aligned}$$

in $\Gamma(U_i, \mathcal{O}_X)$. But by (2.2.20) and the sheaf axioms this implies that $\phi(b) = f$ in $\Gamma(U, \mathcal{O}_X)$. □

Corollary 2.2.21. *For a ring A, with $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, the canonical ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ is bijective.*

Proof. Take $s = 1$ in Thm 2.2.14. □

More material for the vacation

Here are three more exercises, all interesting in my opinion. The first two are about the notion of quasi-compactness.

It is easy to see that if X is a quasi-compact topological space (e.g. X is the underlying toplogical space of an affine scheme) and $Z \subseteq X$ is a closed subset, then Z (with the induced topology) is also quasi-compact. (You probably saw this in a basic topology class.) But what about an open subset $U \subseteq X$?

Exercise 2.2.22. Let A be a noetherian ring and let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. Let $U \subseteq X$ be an open subset. Prove that U is a quasi-compact topological space. (This is of medium difficulty. I will give hints by email to those who write to me.)

Exercise 2.2.23. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[t_1, t_2, \ldots]$ be the polynomial ring in countably many variables. Let $\mathfrak{m} \subseteq A$ be the ideal generated by the variables. Note that $\mathfrak{m}$ is a maximal ideal. Define $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $Z := \operatorname{Zer}_X(\mathfrak{m})$, and $U := X - Z$. Since $\mathfrak{m}$ is maximal we have that $Z = \{\mathfrak{m}\}$. Thus $U = X - \{\mathfrak{m}\}$.

The ring A is not noetherian. Indeed, the ideal $\mathfrak{m}$ is not finitely generated. Prove this. (This is of medium difficulty.)

The topological space X is of course quasi-compact. The goal is to prove that the open set U is not quasi-compact.

For every i let $V_i := \operatorname{NZer}_X(t_i)$, which is an open set in X. Show that $U = \bigcup_i V_i$. (This is easy.)

Now prove that the open covering $\{V_i\}_{i \geq 1}$ of U has no finite subcovering. (This is hard, but I think I know how to prove it. I will give hints to those who ask in private emails.)

The last exercise will give us an example of a scheme which is not affine. Of course we did not learn about such schemes yet; but once we do, we will go back to this exercise and make the observation that $(U, \mathcal{O}_X|_U)$ is a scheme that's not affine.

Exercise 2.2.24. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[t_1, t_2]$ be the polynomial ring in two variables. The affine scheme $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ is called the *2-dimensional affine space over* $\mathbb{K}$, with notation $\mathbf{A}^2_{\mathbb{K}}$.

Let $\mathfrak{m}$ be the maximal ideal in A generated by the variables, let $Z := \operatorname{Zer}_X(\mathfrak{m})$ and let $U := X - Z$. Thus (like in the previous exercise) $U = X - \{\mathfrak{m}\}$, the complement of the origin.

We know (by Cor 2.2.21) that $\Gamma(X, \mathcal{O}_X) = A$. The exercise is to prove that $\Gamma(U, \mathcal{O}_X) = A$, namely that the restriction ring homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is bijective.

Here are a few hints. First, let $V_i := \operatorname{NZer}_X(t_i)$, so $U = V_1 \cup V_2$. Also $V_1 \cap V_2 = \operatorname{NZer}_X(t_1 \cdot t_2)$. Using this open covering and the sheaf condition, there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_X) \to \Gamma(V_1, \mathcal{O}_X) \oplus \Gamma(V_2, \mathcal{O}_X) \to \Gamma(V_1 \cap V_2, \mathcal{O}_X). \tag{2.2.25}$$

By Thm 2.2.14 we know that $\Gamma(V_1, \mathcal{O}_X) = A_{t_1}$, $\Gamma(V_2, \mathcal{O}_X) = A_{t_2}$ and $\Gamma(V_1 \cap V_2, \mathcal{O}_X) = A_{t_1 \cdot t_2}$. This last ring is the ring of Laurent polynomials, and the first two rings are subrings of it.

Show that the exact sequence (2.2.25) can be interpreted as saying that

$$\Gamma(U, \mathcal{O}_X) = A_{t_1} \cap A_{t_2} \subseteq A_{t_1 \cdot t_2}.$$

Finally use the fact that A is a UFD. (Find a reference for this.)

$\boxed{\text{End of Lecture 3}}$

Lecture 4, 7 April 2021

First:

Solution of Exercise 2.2.23. (Different from the solution Yotam sent earlier.) Recall that $\mathbb{K}$ is a field, $A := \mathbb{K}[t_0, t_1, \ldots]$ is the polynomial ring in countably many variables, $\mathfrak{m} \subsetneq A$ is the ideal generated by the variables, $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $Z := \operatorname{Zer}_X(\mathfrak{m})$, and $U := X - Z$. Since $\mathfrak{m}$ is maximal we have $Z = \{\mathfrak{m}\}$. For every i let $V_i := \operatorname{NZer}_X(t_i)$, which is an open set in X. It is easy to see that $U = \bigcup_{i \in \mathbb{N}} V_i$. The exercise is to prove that there is no finite subcovering.

Assume that there is a finite subcovering. Then $U = \bigcup_{i \leq n} V_i$ for some $n \in \mathbb{N}$. This implies that

$$\operatorname{Zer}_X(\mathfrak{m}) = \{\mathfrak{m}\} = \bigcap_{i \leq n} \operatorname{Zer}_X(t_i) = \operatorname{Zer}_X(\mathfrak{p}),$$

where $\mathfrak{p} := (t_0, \ldots, t_n)$. The ideals $\mathfrak{m}$ and $\mathfrak{p}$ are prime, and hence radical. (An ideal $\mathfrak{a}$ is called radical if $\mathfrak{a} = \sqrt{\mathfrak{a}}$; and $\mathfrak{a}$ is radical iff the only nilpotent element in ring $A/\mathfrak{a}$ is 0.) By Lem 2.2.3 we get $\mathfrak{m} = \mathfrak{p}$. This says that $t_{n+1} \in \mathfrak{p}$, which is impossible; there are two ways to see that:

(1) One way to see it is by a direct linear algebra calculation.
(2) Suppose $t_{n+1} = \sum_{i \leq n} a_i \cdot t_i$ for some $a_i \in A$. Looking at the $\mathbb{K}$-ring homomorphism $\phi : A \to \mathbb{K}$ sending $\phi(t_i) := 0$ for $i \neq n+1$ and $\phi(t_{n+1}) := 1$, we get $0 = 1$ in $\mathbb{K}$.

2.3. **Maps of Affine Schemes.** Recall that an affine $\mathbb{K}$-scheme is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$, which is isomorphic, in the category $\mathsf{LRSp}/\mathbb{K}$, to $\operatorname{Spec}(A)$ for some ring A.

The category of affine $\mathbb{K}$-schemes is denoted by $\mathsf{AffSch}/\mathbb{K}$. It is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the affine schemes. Also Recall that $\mathsf{Rng}/\mathbb{K}$ is the category of (commutative) $\mathbb{K}$-rings.

Proposition 2.3.1. *The assignment that sends a locally ringed space $(X, \mathcal{O}_X)$ to the ring $\Gamma(X, \mathcal{O}_X)$, and a map of locally ringed spaces $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to the ring homomorphism*

$$\Gamma(X, \phi) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X),$$

is a functor

$$\Gamma : (\mathsf{LRSp}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Exercise 2.3.2. Prove proposition 2.3.1. (Easy.)

Remark 2.3.3. We are interested in a functor

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}$$

which will be adjoint to Γ. In formula (2.2.11) and Lemma 2.2.12(1) we saw how to produce a functor Spec with values in topological spaces.

Lifting (or upgrading) this functor to $\mathsf{AffSch}/\mathbb{K}$, which is the same as lifting it to $\mathsf{LRSp}/\mathbb{K}$, can be done directly. This is the next optional exercise. We will do it indirectly using Thm 2.3.12.

Exercise 2.3.4. (Optional) Given a ring homomorphism $\psi : A \to B$, let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $(Y, \mathcal{O}_Y) := \operatorname{Spec}(B)$ and $f := \operatorname{Spec}(\psi) : Y \to X$, try to construct a homomorphism of sheaves of rings $\tilde{\psi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ on X such that $\Gamma(X, \tilde{\psi}) = \psi$. See [Har, Prop II.2.3].

Definition 2.3.5. Let $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$ and let $s \in \Gamma(Y, \mathcal{O}_Y)$. For a point $y \in Y$ we denote by $s(y)$ the image of the element s in the residue field $\boldsymbol{k}(y)$, via the canonical homomorphisms

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\mathrm{rest}_{y/Y}} \mathcal{O}_{Y,y} \twoheadrightarrow \boldsymbol{k}(y).$$

And we write

$$\mathrm{NZer}_Y(s) := \{y \in Y \mid s(y) \neq 0\}.$$

Lemma 2.3.6. *Let $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$ and $s \in \Gamma(Y, \mathcal{O}_Y)$. Then:*

(1) *The set $\mathrm{NZer}_Y(s)$ is open in Y.*
(2) *The element s is invertible in the ring $\Gamma(\mathrm{NZer}_Y(s), \mathcal{O}_Y)$.*

The proof was not done in class; please read it.

Proof.
(1) Write $V := \mathrm{NZer}_Y(s)$. Take a point $y \in V$. Since $s(y) \neq 0$, it is an invertible element of the field $\boldsymbol{k}(y)$. Because the stalk $\mathcal{O}_{Y,y}$ is a local ring, it follows that s is an invertible element of $\mathcal{O}_{Y,y}$. Let $t \in \mathcal{O}_{Y,y}$ be the inverse of s, so $s \cdot t = 1$ in $\mathcal{O}_{Y,y}$. There is an open neighborhood W of y such that $t \in \Gamma(W, \mathcal{O}_Y)$. There is a smaller open neighborhood W' of y s.t. $s \cdot t = 1$ in $\Gamma(W', \mathcal{O}_Y)$. But then $s(y') \neq 0$ for all $y' \in W'$, so $W' \subseteq V$.

(2) Let $V := \mathrm{NZer}_Y(s)$ as before. As we saw above, every point $y \in V$ has an open neighborhood $W_y \subseteq V$ and an element $t_y \in \Gamma(W_y, \mathcal{O}_Y)$ such that $s \cdot t_y = 1$. This means that $t_y = s^{-1}$ in $\Gamma(W_y, \mathcal{O}_Y)$, a fact that makes t_y unique. We conclude that $t_y = t_{y'}$ in $\Gamma(V_y \cap V_{y'}, \mathcal{O}_Y)$. By the sheaf property we get $t \in \Gamma(V, \mathcal{O}_Y)$, and it satisfies $s \cdot t = 1$. □

Recall that for a ring A we denote by $A^{\times}$ its multiplicative group, i.e. the set of invertible elements of A, equipped with the operation of multiplication.

Exercise 2.3.7. Let $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$.

(1) Show that the assignment $U \mapsto \Gamma(U, \mathcal{O}_X)^{\times}$ is a sheaf of groups on X. It is denoted by $\mathcal{O}_X^{\times}$ and also by $\mathrm{GL}_1(\mathcal{O}_X)$.
(2) Show that for the stalks at for every point $x \in X$ there is a canonical group isomorphism $(\mathcal{O}_X^{\times})_x \cong (\mathcal{O}_{X,x})^{\times}$.

Lemma 2.3.8. *Let $(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map in $\mathsf{LRSp}/\mathbb{K}$, $A := \Gamma(X, \mathcal{O}_X)$, $B := \Gamma(Y, \mathcal{O}_Y)$ and $\phi := \Gamma(X, \tilde{\phi}) : A \to B$.*

(1) *Take $s \in A$ and let $U := \mathrm{NZer}_X(s) \subseteq X$ and $V := f^{-1}(U) \subseteq Y$. Then $V = \mathrm{NZer}_Y(\phi(s))$.*
(2) *For an open set $U \subseteq X$ let*
$$\mathrm{S}(U) := \{s \in A \mid s(x) \neq 0 \ \text{ for all } \ x \in U\}.$$
Then the image of $\mathrm{S}(U)$ in $\Gamma(U, \mathcal{O}_X)$ consists of invertible elements. It follows there there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to \Gamma(U, \mathcal{O}_X)$.
(3) *Let $V := f^{-1}(U) \subseteq Y$. Then the image of $\phi(\mathrm{S}(U))$ in $\Gamma(V, \mathcal{O}_Y)$ consists of invertible elements. It follows there there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to \Gamma(V, \mathcal{O}_V)$.*

Note that item (3) is a special case of item (2) with $(f, \tilde{\phi}) = \mathrm{id}$. Also, when $(X, \mathcal{O}_X)$ is an affine scheme, the set $\mathrm{S}(U)$ is the same as in Definition 2.1.13.

Exercise 2.3.9. Prove Lemma 2.3.8.

Lemma 2.3.10. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ *for some ring* A. *Then*

$$\Gamma(X, \mathcal{O}_X^{\times}) = A^{\times} = \mathrm{S}(X)$$

as subsets of A. *Therefore* $A_{\mathrm{S}(X)} = A$ *as rings; to be precise, the canonical ring homomorphism* $A \to A_{\mathrm{S}(X)}$ *is bijective.*

Exercise 2.3.11. Prove Lemma 2.3.10.

Theorem 2.3.12. *Let* $A \in \mathsf{Rng}/\mathbb{K}$ *and let* $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$. *Write* $B := \Gamma(Y, \mathcal{O}_Y)$ *and* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$. *Given a* $\mathbb{K}$*-ring homomorphism* $\phi : A \to B$, *there is a unique map*

$$(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ *such that*

$$\Gamma(X, \tilde{\phi}) = \phi : A \to B.$$

A weaker statement can be found in [Har, Exer II.2.4].

The proof was not done in class; please read it. There might be errors – please verify!

Proof.

Step 1. We prove uniqueness of f in Top, which is the same as uniqueness in Set.

Take a point $y \in Y$, and let $x = \mathfrak{p} := f(y) \in X$. Because $(f, \tilde{\phi})$ is a map in $\mathsf{Rng}/\mathbb{K}$, the ring homomorphism $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism. We are given that $\Gamma(X, \tilde{\phi}) = \phi$, so we have a commutative diagram of rings

(2.3.13)
$$\begin{tikzcd}
A \arrow[r, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d, "\mathrm{rest}_{x/U}"'] & B \arrow[d, "\mathrm{rest}_{y/U}"] \\
\mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] \arrow[d, two heads] & \mathcal{O}_{Y,y} \arrow[d, two heads] \\
\boldsymbol{k}(x) \arrow[r, tail] & \boldsymbol{k}(y)
\end{tikzcd}$$

The bottom square commutes because $\tilde{\phi}_y$ is a local homomorphism. The homomorphism $\boldsymbol{k}(x) \to \boldsymbol{k}(y)$ is injective. Comparing the two paths in this diagram we see that

$$\operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) = \operatorname{Ker}\big(A \to \boldsymbol{k}(\mathfrak{p})\big) = \mathfrak{p}.$$

The homomorphism $A \xrightarrow{\phi} B \to \boldsymbol{k}(y)$ depends only on y and ϕ, and it determines $f(y) = \mathfrak{p} = x$.

Step 2. Now we prove that the homomorphism of sheaves of rings on X

$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

is unique. Since the principal open sets $U = \operatorname{NZer}_X(s)(s) \subseteq X$, for $s \in A$, are a basis for the topology, it is enough to prove the uniqueness of the ring homomorphism

$$\Gamma(U, \tilde{\phi}) : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y),$$

for $U := \operatorname{NZer}_X(s)$ and $V := f^{-1}(U) \subseteq Y$.

By Theorem 2.2.14 we know that $\Gamma(U, \mathcal{O}_X) = A_s$. Let's examine this commutative diagram of rings

(2.3.14)
$$\begin{tikzcd} A = \Gamma(X, \mathcal{O}_X) \arrow[r, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d, "\mathrm{rest}_{U/X}"'] & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{V/Y}"] \\ A_s = \Gamma(U, \mathcal{O}_X) \arrow[r, "\Gamma(U, \tilde{\phi})"] & \Gamma(V, \mathcal{O}_Y) \end{tikzcd}$$

The path going right and then down depends only on ϕ, f and U. (In step 1 we already determined f.) Because the left vertical arrow is a localization, it follows that the ring homomorphism $\Gamma(U, \tilde{\phi})$ is unique.

Step 3. Here we start with the existence. We define the function $f : Y \to X$ by the formula from step 1, namely a point $y \in Y$ is sent to the point

(2.3.15)
$$x = \mathfrak{p} := \operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) \in \operatorname{Spec}(A) = X.$$

We need to prove that f is continuous. It suffices to show that for every principal open set $U = \operatorname{NZer}_X(s) \subseteq X$, its preimage $f^{-1}(U)$ is open. But by Lemma 2.3.8(1) we have $f^{-1}(U) = \operatorname{NZer}_X(\phi(s))$, and this is open in Y.

Step 4. Now we construct the homomorphism of sheaves of rings

(2.3.16)
$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the universal property of sheafication, it suffices to construct a homomorphism of presheaves of rings

(2.3.17)
$$\tilde{\phi}^{\mathrm{pre}} : \mathcal{O}_X^{\mathrm{pre}} \to f_*(\mathcal{O}_Y),$$

and then to take $\tilde{\phi} := \operatorname{Sh}(\tilde{\phi}^{\mathrm{pre}})$.

For every $U \subseteq X$ open we have by definition $\Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = A_{\mathrm{S}(U)}$, where $\mathrm{S}(U) \subseteq A$ is the multiplicatively closed set from Definition 2.1.13 and Lemma 2.3.8(2). By Lemma 2.3.8(3) there is a unique A-ring homomorphism

$$\tilde{\phi}_U^{\mathrm{pre}} : \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = A_{\mathrm{S}(U)} \to \Gamma(f^{-1}(U), \mathcal{O}_Y) = \Gamma(U, f_*(\mathcal{O}_Y).$$

As U varies this become a homomorphism of presheaves of rings (2.3.17).

Finally, Lemma 2.3.10 says that $A_{\mathrm{S}(X)} = A$. It follows that

$$\Gamma(X, \tilde{\phi}) = \Gamma(X, \tilde{\phi}^{\mathrm{pre}}) = \phi$$

as homomorphisms $A \to B$, as required.

Step 5. It remains to prove that $(f, \tilde{\phi})$ is local, i.e. $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism for every $y \in Y$ and $x = \mathfrak{p} := f(y)$.

By the definition of f, see formula (2.3.15), the solid diagram of rings below is commutative:

(2.3.18)
$$\begin{tikzcd} A = \Gamma(X, \mathcal{O}_X) \arrow[r, "\phi"] \arrow[d, two heads, "\mathrm{can}"'] & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{y/Y}"] \\ A/\mathfrak{p} \arrow[d, tail, "\mathrm{can}"'] \arrow[dr, tail] & \mathcal{O}_{Y,y} \arrow[d, two heads, "\mathrm{can}"] \\ \boldsymbol{k}(\mathfrak{p}) \arrow[r, dashed] & \boldsymbol{k}(y) \end{tikzcd}$$

Because the slanted arrow is an injection, it extends to the field of fractions $\boldsymbol{k}(\mathfrak{p})$, i.e. the dashed arrow exists.

On the other hand, our construction in step 4 is such that for every open neighborhood U of x, and for every open set V in Y such that $y \in V \subseteq f^{-1}(U)$, there is a commutative diagram

$$\begin{tikzcd} A = \Gamma(X, \mathcal{O}_X) \arrow[rr, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d, "\mathrm{can}"'] & & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{f^{-1}(U)/Y}"] \\ A_{\mathrm{S}(U)} = \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) \arrow[rr, "\tilde{\phi}_U^{\mathrm{pre}}"] \arrow[drr] & & \Gamma(f^{-1}(U), \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{V/f^{-1}(U)}"] \\ & & \Gamma(V, \mathcal{O}_Y) \end{tikzcd} \tag{2.3.19}$$

Passing to direct limits in U and V we get this commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] & \mathcal{O}_{Y,y} \end{tikzcd} \tag{2.3.20}$$

In the next diagram the top square is the commutative diagram (2.3.20) and the boundary is the boundary of the commutative diagram (2.3.18):

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] \arrow[d, two heads, "\mathrm{can}"'] & \mathcal{O}_{Y,y} \arrow[d, two heads, "\mathrm{can}"] \\ \boldsymbol{k}(\mathfrak{p}) \arrow[r] & \boldsymbol{k}(y) \end{tikzcd} \tag{2.3.21}$$

The surjections are the canonical ones from a local ring to its residue field. Because $A_{\mathfrak{p}}$ is a localization of A, there is at most one A-ring homomorphism $A_{\mathfrak{p}} \to \boldsymbol{k}(y)$, and this implies that the bottom square in (2.3.21) is also commutative. Hence $\tilde{\phi}_y$ is a local homomorphism. □

$\boxed{\text{The rest of the material was done in class}}$

Corollary 2.3.22. *The assignment that sends a ring A to the affine scheme* $\mathrm{Spec}(A)$*, and a ring homomorphism ϕ to the map of affine schemes $(f, \tilde{\phi})$ from Theorem 2.3.12, is a functor*

$$\mathrm{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}.$$

Proof. Say $\phi : A \to B$ and $\psi : B \to C$ are homomorphism in $\mathsf{Rng}/\mathbb{K}$, with corresponding maps of affine schemes

$$\mathrm{Spec}(\phi) = (f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

and

$$\mathrm{Spec}(\psi) = (g, \tilde{\psi}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y).$$

We get these maps of schemes

$$(f, \tilde{\phi}) \circ (g, \tilde{\psi}),\ \mathrm{Spec}(\psi \circ \phi) \ : \ (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X),$$

and we need to prove they are equal.

The condition in Thm 2.3.12 is that

$$\Gamma(Y, \tilde{\psi}) \circ \Gamma(X, \tilde{\phi}) = \psi \circ \phi = \Gamma(X, \psi \circ \phi).$$

The uniqueness clause in Thm 2.3.12 says that

$$\mathrm{Spec}(\phi) \circ \mathrm{Spec}(\psi) = \mathrm{Spec}(\psi \circ \phi).$$

Likewise $\mathrm{Spec}(\mathrm{id}_A) = \mathrm{id}_{\mathrm{Spec}(A)}$. □

Corollary 2.3.23. *The functor*

$$\mathrm{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{LRS}/\mathbb{K}$$

is right adjoint to the functor

$$\Gamma : \mathsf{LRS}/\mathbb{K} \to (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}}.$$

Proof. Thm 2.3.12 produces a bijection

$$\mathrm{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(A, \Gamma(Y, \mathcal{O}_Y)\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big((Y, \mathcal{O}_Y), \mathrm{Spec}(A)\big) \tag{2.3.24}$$

for $A \in \mathsf{Rng}/\mathbb{K}$ and $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$. We need to prove that this is bifunctorial, i.e. it is functorial in A and $(Y, \mathcal{O}_Y)$. This is an exercise. □

Exercise 2.3.25. Prove that the bijection (2.3.24) is functorial in A and in $(Y, \mathcal{O}_Y)$.

Corollary 2.3.26. *The functor*

$$\mathrm{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}$$

is an equivalence of categories, with quasi-inverse Γ.

Proof. By definition, $\mathsf{AffSch}/\mathbb{K}$ is the essential image in $\mathsf{LRSp}/\mathbb{K}$ of the functor Spec.

We need to prove that Spec is fully faithful. But this is an immediate consequence of Theorem 2.3.12 – for rings A and B there is a bijection

$$\mathrm{Spec} : \mathrm{Hom}_{\mathsf{Rng}/\mathbb{K}}(A, B) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big(\mathrm{Spec}(B), \mathrm{Spec}(A)\big),\ \phi \mapsto \mathrm{Spec}(\phi).$$ □

◇ ◇ ◇

Next week we will some examples of the "functor of points".

$\boxed{\text{End of Lecture 4}}$

References

[AlKl] A. Altman and S. Kleiman, "A Term of Commutative Algebra", free online at http://www.centerofmathematics.com/wwcomstore/index.php/commalg.html.

[Eis] D. Eisenbud, "Commutative Algebra", Springer, 1994.

[EiHa] D. Eisenbud and J. Harris, "The Geometry of Schemes", Springer, 2000.

[Har] R. Hartshorne, "Algebraic Geometry", Springer-Verlag, New-York, 1977.

[HiSt] P.J. Hilton and U. Stammbach, "A Course in Homological Algebra", Springer, 1971.

[Jo] D. Joyce, "Algebraic Geometry over C^∞-Rings", Memoirs AMS, 2019.

[Lee] John M. Lee, "Introduction to Smooth Manifolds", LNM **218**, Springer, 2013.

[KaSc] M. Kashiwara and P. Schapira, "Sheaves on manifolds", Springer-Verlag, 1990.

[Mac1] S. Maclane, "Homology", Springer, 1994 (reprint).

[Mac2] S. Maclane, "Categories for the Working Mathematician", Springer, 1978.

[Mats] H. Matsumura, "Commutative Ring Theory", Cambridge University Press, 1986.

[Rot] J. Rotman, "An Introduction to Homological Algebra", Academic Press, 1979.

[Row] L.R. Rowen, "Ring Theory" (Student Edition), Academic Press, 1991.

[We] C. Weibel, "An introduction to homological algebra", Cambridge Studies in Advanced Math. **38**, 1994.

[Ye1] A. Yekutieli, "Derived Categories", Cambridge University Press, 2019; prepublication version https://arxiv.org/abs/1610.09640v4

[Ye2] A. Yekutieli, "Commutative Algebra", Course Notes (2019-20), https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/comm-alg/crs-notes_200305.pdf.

[Ye3] A. Yekutieli, "Homological Algebra", Course Notes (2019-20), https://www.math.bgu.ac.il/~amyekut/teaching/2019-20/homol-alg/crs-notes_200626-d3.pdf.

[Ye4] A. Yekutieli, "Schemes 1", Course Notes (2020-21), https://www.math.bgu.ac.il/~amyekut/teaching/2020-21/schemes-1/notes-210113c.pdf.

[Ye5] A. Yekutieli, "Schemes 2", Course Notes (2018-19), https://www.math.bgu.ac.il/~amyekut//teaching/2018-19/schemes-2/notes-190618-d2.pdf.

Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel.
Email: amyekut@math.bgu.ac.il, *Web*: http://www.math.bgu.ac.il/~amyekut.