Course Notes:

Algebraic Geometry – Schemes 2

Ben Gurion University
Spring Semester 2020-21

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Contents

Lecture 1, 3 March 2021

This course is a continuation of the course "Algebraic Geometry – Schemes 1" from the previous semester. The notes of that course are [Ye4].

1. Locally Ringed Spaces

1.1. **Recalling some Material.** We work over a base ring $\mathbb{K}$, which is some nonzero commutative ring. In particular examples $\mathbb{K}$ will be specified.

Recall that a ringed space over $\mathbb{K}$, or a $\mathbb{K}$-ringed space, is a pair $(X, \mathcal{O}_X)$, where X is a topological space X, and $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-rings on X.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be $\mathbb{K}$-ringed spaces. A *map of* $\mathbb{K}$*-ringed spaces*

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of topological spaces

$$f : Y \to X$$

together with a homomorphism of sheaves of $\mathbb{K}_X$-rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y).$$

We saw a few examples of ringed spaces.

Example 1.1.1. A not very interesting ringed space is $(X, \mathcal{O}_X)$, where X is some topological spaces, and $\mathcal{O}_X := \mathbb{K}_X$, the constant sheaf with values in $\mathbb{K}$.

A map $f : Y \to X$ in Top gives rise to a map of $\mathbb{K}$-ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

where $\psi : \mathbb{K}_X \to f_*(\mathbb{K}_Y)$ is gotten by sheafification from the obvious

The way to produce ψ is this. Recall that $\Gamma(U, \mathbb{K}_X)$ is the ring of continuous functions $a : U \to \mathbb{K}$, for the discrete topology on $\mathbb{K}$. Let $V := f^{-1}(U) \subseteq Y$. Then

$$\psi(a) \in \Gamma(U, f_*(\mathbb{K}_Y)) = \Gamma(V, \mathbb{K}_X)$$

is is the continuous function $\psi(a) := a \circ f|_V : V \to \mathbb{K}$.

We also had a list of more interesting ringed spaces, repeated in the next example.

Example 1.1.2. A category of *geometric spaces* is a category Sp belonging to the list below. For each type we indicate the structure sheaf $\mathcal{O}_X$ and the base ring $\mathbb{K}$.

(1) The category Top of topological spaces and continuous maps between them. Here the base ring is $\mathbb{K} = \mathbb{R}$, the field of real numbers. The sheaf $\mathcal{O}_X$ is the sheaf of continuous $\mathbb{R}$-valued functions.
(2) The category Mfld of real differentiable manifolds and differentiable maps between them, where by differentiable we mean of class C^∞. Here $\mathbb{K} = \mathbb{R}$. The sheaf $\mathcal{O}_X$ is the sheaf of differentiable $\mathbb{R}$-valued functions.
(3) The category Var of quasi-projective algebraic varieties over an algebraically closed field $\mathbb{K}$. The sheaf $\mathcal{O}_X$ is the sheaf of algebraic, or regular, $\mathbb{K}$-valued functions.

In each of these cases, a map $f : Y \to X$ in Sp induced a map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

of ringed spaces, which is by *pullback*. I.e. for $U \subseteq X$ open and $V := f^{-1}(U) \subseteq Y$ the ring homomorphism

$$\Gamma(U, \psi) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, f_*(\mathcal{O}_Y)) = \Gamma(V, \mathcal{O}_Y)$$

sends $a \in \Gamma(U, \mathcal{O}_X)$ to

$$\Gamma(U, \psi)(a) := a \circ f|_V : V \to \mathbb{K}. \tag{1.1.3}$$

For this reason we shall often write $f^* := \psi$, so the map of ringed spaces is (f, f^*).

In this way we obtain a functor

$$\mathrm{RS} : \mathsf{Sp} \to \mathsf{RSp}/\mathbb{K}, \quad X \mapsto (X, \mathcal{O}_X), \quad f \mapsto (f, f^*). \tag{1.1.4}$$

This functor is clearly faithful.

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To a ringed space $(X, \mathcal{O}_X)$ we attach the category $\mathsf{Mod}(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules.

There are several important internal operations on $\mathcal{O}_X$-modules.

Given a collection $\{\mathcal{M}_i\}_{i \in I}$ of $\mathcal{O}_X$-modules, we can consider the direct sum $\bigoplus_{i \in I} \mathcal{M}_i$ and the product $\prod_{i \in I} \mathcal{M}_i$.

For $\mathcal{M}, \mathcal{N} \in \mathsf{Mod}(\mathcal{O}_X)$ we have the $\mathcal{O}_X$-modules $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. We know that

$$\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) = \mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_U)}(\mathcal{M}|_U, \mathcal{N}|_U)). \tag{1.1.5}$$

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A sequence of homomorphisms

$$\cdots \to \mathcal{M}^i \xrightarrow{\phi^i} \mathcal{M}^{i+1} \xrightarrow{\phi^{i+1}} \mathcal{M}^{i+1} \to \cdots$$

in $\mathsf{Mod}(\mathcal{O}_X)$ is called *exact* if for every point $x \in X$ the sequence of homomorphisms

$$\cdots \to \mathcal{M}^i_x \xrightarrow{\phi^i_x} \mathcal{M}^{i+1}_x \xrightarrow{\phi^{i+1}_x} \mathcal{M}^{i+1}_x \to \cdots$$

in $\mathsf{Mod}(\mathcal{O}_{X,x})$ is exact.

To a homomorphism $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}(\mathcal{O}_X)$ we attach the kernel and the cokernel, and these make the sequence

$$0 \to \mathrm{Ker}(\phi) \to \mathcal{M} \xrightarrow{\phi} \mathcal{N} \to \mathrm{Coker}(\phi) \to 0$$

in $\mathsf{Mod}(\mathcal{O}_X)$ is exact.

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Suppose we are given a map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

There there are functors

$$f_* : \mathsf{Mod}(\mathcal{O}_Y) \to \mathsf{Mod}(\mathcal{O}_X)$$

and

$$f^* : \mathsf{Mod}(\mathcal{O}_X) \to \mathsf{Mod}(\mathcal{O}_Y).$$

Recall that $f_*(\mathcal{N})$ is defined by

$$\Gamma(U, f_*(\mathcal{N})) := \Gamma(f^{-1}(U), \mathcal{N}),$$

and

$$f^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{M}),$$

where $f^{-1}(\mathcal{M})$ is the sheaf associated to the presheaf

$$V \mapsto \lim_{\to} \Gamma(U, \mathcal{M})$$

where $U \subseteq X$ is open and $f(V) \subseteq U$.

These functors satisfy the adjunction formula

$$\mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_X)}(\mathcal{M}, f_*(\mathcal{N})) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Mod}(\mathcal{O}_Y)}(f^*(\mathcal{M}), \mathcal{N}). \tag{1.1.6}$$

This isomorphism is functorial in $\mathcal{M}$ and $\mathcal{N}$.

1.2. **Locally Ringed Spaces and Their Maps.** Now to new material.

Let's recall that a local ring is a ring A that has exactly one maximal ideal, say $\mathfrak{m}$. This is often denoted by $(A, \mathfrak{m})$. If $(B, \mathfrak{n})$ is another local ring, then a local ring homomorphism $\psi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is a ring homomorphism $\psi : A \to B$ s.t. $\psi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Definition 1.2.1. A ringed space $(X, \mathcal{O}_X) \in \mathsf{RSp}/\mathbb{K}$ is called a *locally ringed space* if for every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring.

Definition 1.2.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space over $\mathbb{K}$. Given a point $x \in X$, the maximal ideal of $\mathcal{O}_{X,x}$ is denoted by $\mathfrak{m}_x$, and the residue field is

$$\boldsymbol{k}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

is

Definition 1.2.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces over $\mathbb{K}$. A *map of locally ringed spaces over* $\mathbb{K}$ is a map of $\mathbb{K}$-ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

such that for every point $y \in Y$ the $\mathbb{K}$-ring homomorphism

$$\psi_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$$

between the stalks is a local homomorphism.

The category of locally ringed spaces over $\mathbb{K}$, with maps as above, is denoted by $\mathsf{LRSp}/\mathbb{K}$.

Definition 1.2.4. If $(X, \mathcal{O}_X)$ is a locally ringed space and $U \subseteq X$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is also a locally ringed space. It is called an *open subspace* of $(X, \mathcal{O}_X)$, and the inclusion map

$$(U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X)$$

is called an *open embedding*.

Proposition 1.2.5. *Consider one of our three favorite categories of geometric spaces* Sp *from Example 1.1.2, with corresponding base field* $\mathbb{K}$. *For* $X \in \mathsf{Sp}$ *the sheaf of functions is* $\mathcal{O}_X$.

(1) *The* $\mathbb{K}$*-ringed space* $(X, \mathcal{O}_X)$ *is a locally ringed space.*
(2) *Let* $f : Y \to X$ *be a map in* Sp*, and let*

$$(f, f^*) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

be the corresponding map of ringed spaces. Then (f, f^*) *is a map of locally ringed spaces over* $\mathbb{K}$.

(3) *We obtain a functor*

$$\mathrm{LRSp} : \mathsf{Sp} \to \mathsf{LRSp}/\mathbb{K}, \quad X \mapsto (X, \mathcal{O}_X), \quad f \mapsto (f, f^*).$$

Exercise 1.2.6. Prove Proposition 1.2.5.

The next theorem is supposed to tell us how powerful the concept of locally ringed spaces is.

Theorem 1.2.7. *Consider one of the three categories of geometric spaces* Sp *from Example* 1.1.2*, with corresponding base field* $\mathbb{K}$*.*
Then the functor

$$\mathrm{LRSp} : \mathsf{Sp} \to \mathsf{LRSp}/\mathbb{K}$$

is fully faithful.

The proof will be given next week.

Remark 1.2.8. Schemes are locally ringed spaces. When we learn about affine schemes I will give an example of a map of ringed spaces $(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ between affine schemes that is not a map of locally ringed spaces.

Exercise 1.2.9. Try to find an example of real differentiable manifolds X and Y, and a map of ringed spaces $(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ that is not a map of locally ringed spaces. (Note that $f : Y \to X$ will not be a map of manifolds.) Try looking this up in [Jo] or mathexchange...

End of Lecture 1

Lecture 2, 10 March 2021

We will start with the proof of Theorem 1.2.7. Then we will start learning about affine schemes (a change of the program).

Observe that Theorem 1.2.7 means that the geometry Sp, namely the information added to the underlying topological space X, is completely encoded in the abstract notion of "locally ringed spaces and their maps".

Proof of Theorem 1.2.7. The faithfulness of the functor LRSp is easy to see: the forgetful functor $\mathsf{Sp} \to \mathsf{Set}$ is faithful ($f = g$ iff they are equal as functions between sets), and it factors through $\mathsf{LRSp}/\mathbb{K}$.

The challenge is to prove fullness. Let

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

be a morphism in $\mathsf{LRSp}/\mathbb{K}$ between $(X, \mathcal{O}_X) = \mathrm{LRSp}(X)$ and $(Y, \mathcal{O}_Y) = \mathrm{LRSp}(Y)$ for two spaces $X, Y \in \mathsf{Sp}$. We must prove that the continuous map $f : Y \to X$ is a map in Sp, and that $\psi = f^*$.

Step 1. In this step we prove that $\psi = f^*$. Consider an open set $U \subseteq X$ and a function $a \in \Gamma(U, \mathcal{O}_X)$. We must prove that

$$\psi(a) = f^*(a) \in \Gamma(U, f_*(\mathcal{O}_Y)).$$

In other words, letting $V := f^{-1}(U) \subseteq Y$, we must prove that the functions

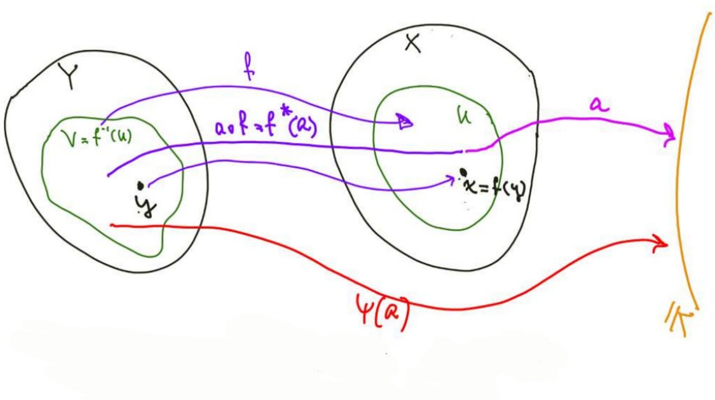
$$\psi(a),\ f^*(a) : V \to \mathbb{K}$$

are equal. Since $f^*(a) = a \circ f|_V$, this amounts to showing that for every point $y \in V$ there is equality

$$\psi(a)(y) = (a \circ f)(y) = a(x),$$

where $x := f(y) \in U$. See picture (1.2.10).

(1.2.10)



Let $\lambda := a(x) \in \mathbb{K}$. Because both ψ and f^* are $\mathbb{K}$-ring homomorphisms, it suffices to prove that the function $b := a - \lambda : U \to \mathbb{K}$ satisfies $\psi(b)(y) = f^*(b)(y)$. Now $f^*(b)(y) = b(x) = a(x) - \lambda = 0$. It remains to prove that $\psi(b)(y) = 0$.

Consider the commutative diagram

$$
\begin{tikzcd}
\Gamma(U, \mathcal{O}_X) \arrow[r, "\Gamma(U,\psi)"] \arrow[d, "\mathrm{rest}_{x/U}"] \arrow[dd, bend right, "\mathrm{ev}_x"'] & \Gamma(V, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{y/V}"'] \arrow[dd, bend left, "\mathrm{ev}_y"] \\
\mathcal{O}_{X,x} \arrow[r, "\psi_y"] \arrow[d, "\mathrm{ev}_x"] & \mathcal{O}_{Y,y} \arrow[d, "\mathrm{ev}_y"'] \\
\mathbb{K} \arrow[r, "\mathrm{id}"'] & \mathbb{K}
\end{tikzcd}
\tag{1.2.11}
$$

of $\mathbb{K}$-rings. The function b is in the upper left corner. The germ $\tilde{b} := \mathrm{rest}_{x/U}(b) \in \mathcal{O}_{X,x}$ belongs to the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, because $\mathrm{ev}_x(\tilde{b}) = b(x) = 0$. We know that ψ_y is a *local homomorphism*, and therefore $\psi_i(\tilde{b}) \in \mathfrak{m}_y$. It follows that

$$\psi(b)(y) = \mathrm{ev}_y(\psi(b)) = \mathrm{ev}_y(\psi_y(\tilde{b})) = 0 \in \mathbb{K}.$$

Step 2. Now we shall prove that $f : Y \to X$ is a map in Sp. For $\mathsf{SP} = \mathsf{Top}$ it is automatic. For the other two geometries this is a local question on Y. Take a point $y \in Y$, and let $x := f(y) \in X$. Choose an open neighborhood U of x in X that embeds into $\mathbf{A}^n(\mathbb{K})$, as an open subspace for $\mathsf{SP} = \mathsf{Mfld}$, and as a closed intersect open subvariety for $\mathsf{SP} = \mathsf{Var}$. Take $V := f^{-1}(U) \subseteq Y$, which is an open neighborhood of y. It suffices to prove that the map $f|_V : V \to U$ is a map in Sp.

Let

$$t_1, \ldots, t_n \in \Gamma\big(\mathbf{A}^n(\mathbb{K}), \mathcal{O}_{\mathbf{A}^n(\mathbb{K})}\big)$$

be the coordinate functions. The map $f|_V : V \to U$ is in Sp iff the functions

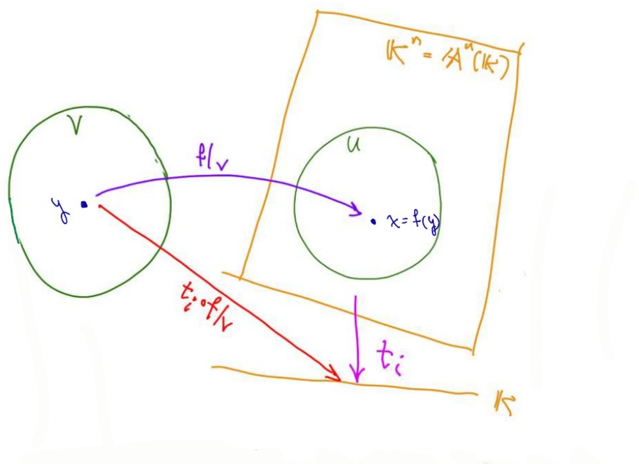
$$t_i \circ f|_V = f|_V^*(t_i) : V \to \mathbb{K}$$

are in Sp for all i; i.e. if

$$f|_V^*(t_i) \in \Gamma(V, \mathcal{O}_Y)$$

for all i. See diagram (1.2.12).

(1.2.12)



But by step 1 we know that

$$f|_V^*(t_i) = \psi(t_i) \in \Gamma(V, \mathcal{O}_Y). \qquad \square$$

2. Affine Schemes

2.1. Definitions and Basic Properties.

Affine schemes were introduced by A. Grothendieck in the 1950's, following attempts by geometers such as Serre, Chevalley, Weil, Zariski and others.

Definition 2.1.1 (Affine Schemes as Sets)**.** Let A be a $\mathbb{K}$-ring. The *prime spectrum* of A is the set $\operatorname{Spec}(A)$ of prime ideals of A.

Example 2.1.2. Before going any further, here is an attempt to explain the name "spectrum". Let $\mathbb{K}$ be an algebraically closed field, and let $a \in \operatorname{Mat}_{n \times n}(\mathbb{K})$ be a matrix of size $n \geq 1$. Define $A := \mathbb{K}[a] \subseteq \operatorname{Mat}_{n \times n}(\mathbb{K})$, so A is a commutative subring of the NC ring $\operatorname{Mat}_{n \times n}(\mathbb{K})$. What is the spectrum of A? It is the set of eigenvalues of a. Indeed, if $p(t) \in \mathbb{K}[t]$ is the minimal polynomial of a, then $A \cong \mathbb{K}[t]/(p(t))$ as $\mathbb{K}$-rings. If $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ are the eigenvalues, then $p(t) = \prod_i (t - \lambda_i)$, and the prime ideals of A are the maximal ideals $(a - \lambda_i) \subseteq A$.

For an ideal $\mathfrak{a} \subseteq A$ we define its *zero locus*

$$\operatorname{Zer}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A) \tag{2.1.3}$$

and *nonzero locus*

$$\operatorname{NZer}(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \not\subseteq \mathfrak{p}\} \subseteq \operatorname{Spec}(A). \tag{2.1.4}$$

Of course

$$\operatorname{Spec}(A) = \operatorname{Zer}(\mathfrak{a}) \sqcup \operatorname{NZer}(\mathfrak{a}),$$

a disjoint union.

Proposition 2.1.5 (The Zariski Topology)**.** *Let A be a ring. The collection of subsets $\{\operatorname{NZer}(\mathfrak{a})\}$, indexed by the ideals $\mathfrak{a} \subseteq A$, is a topology on the set* $\operatorname{Spec}(A)$. *It is called the* Zariski topology.

Like many of the results here, you have seen them before in the course on classical algebraic geometry, so the proofs are left as exercises.

Exercise 2.1.6. Prove the proposition above.

Here is an explanation of the terminology. To a prime ideal $\mathfrak{p}$ we associate the local ring $A_{\mathfrak{p}}$ and the residue field $\boldsymbol{k}(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. An element $a \in A$ has a residue class $a(\mathfrak{p}) \in \boldsymbol{k}(\mathfrak{p})$, coming from the canonical ring homomorphism $A \to \boldsymbol{k}(\mathfrak{p})$. In this way the elements of the ring A are "functions", but the value $a(\mathfrak{p})$ of $a \in A$ at each point $\mathfrak{p}$ is in a field depending on $\mathfrak{p}$...

Anyhow:

$$\operatorname{Zer}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid a(\mathfrak{p}) = 0 \ \text{ for all } \ a \in \mathfrak{a}\}. \tag{2.1.7}$$

This formula interprets the set $\operatorname{Zer}(\mathfrak{a})$ as the "set of zeroes of all functions $a \in \mathfrak{a}$".

Exercise 2.1.8. Prove the formula above.

For an element $s \in A$ we define

$$\operatorname{NZer}(s) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s \notin \mathfrak{p}\}. \tag{2.1.9}$$

This is an open set: it it the complement of the closed set $\operatorname{Zer}(\mathfrak{a})$, where $\mathfrak{a} := (s)$, the principal ideal generated by s. We call such an open set a *principal open set.* Analogously to (2.1.7) we have

$$\operatorname{NZer}(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid s(\mathfrak{p}) \neq 0\}. \tag{2.1.10}$$

Proposition 2.1.11. *The principal open sets are a basis of the topology of* $\mathrm{Spec}(A)$. *Namely every open set set* U *is a union* $U = \bigcup_i \mathrm{NZer}(s_i)$ *for a suitable collection* $\{s_i\}$ *of elements of* A.

Exercise 2.1.12. Prove the proposition above. (Hint: it is easy.)

[**comment**: (210405 AY) new definition next:]

Definition 2.1.13. Let A be a ring, and write $X := \mathrm{Spec}(A)$ for this topological space, with the Zariski topology. For an open set $U \subseteq X$ we let $\mathrm{S}(U) \subseteq A$ be the multiplicatively closed set

$$\mathrm{S}(U) := \{s \in A \mid s(\mathfrak{p}) \neq 0 \ \text{ for all } \ \mathfrak{p} \in U\}.$$

Let $A_{\mathrm{S}(U)}$ be the localization of A w.r.t. $\mathrm{S}(U)$.

Lemma 2.1.14. *The assignment* $U \mapsto \mathrm{S}(U)$ *is a presheaf of* $\mathbb{K}$*-rings on* $X = \mathrm{Spec}(A)$, *which we denote by* $\mathcal{O}_X^{\mathrm{pre}}$.

Proof. This is easy: if $V \subseteq U$ is a smaller open set, then $\mathrm{S}(U) \subseteq \mathrm{S}(V)$, so by the universal property of localization there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(V)}$. □

Definition 2.1.15. Let A be a ring. The *structure sheaf* of $X := \mathrm{Spec}(A)$ is the sheaf of rings $\mathcal{O}_X := \mathrm{Sh}(\mathcal{O}_X^{\mathrm{pre}})$.

The *affine scheme* $\mathrm{Spec}(A)$ is the $\mathbb{K}$-ringed space $(X, \mathcal{O}_X)$.

Proposition 2.1.16. *Let* A *be a ring and write* $X := \mathrm{Spec}(A)$. *For every point* $x = \mathfrak{p} \in X$ *the stalk of* $\mathcal{O}_X$ *at* x *is* $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, *the local ring at* $\mathfrak{p}$. *More precisely, there is a unique* A*-ring isomorphism* $\mathcal{O}_{X,x} \cong A_{\mathfrak{p}}$.

The proof was not done in the lecture. Please read it.

Proof. Let $\mathrm{S}(x) := A - \mathfrak{p}$, the complement of $\mathfrak{p}$. By definition we have $A_{\mathfrak{p}} = A_{\mathrm{S}(x)}$. The universal property of localization says that there is at most one A-ring isomorphism $A_{\mathfrak{p}} \xrightarrow{\simeq} \mathcal{O}_{X,x}$. We will produce it.

Let us denote by $\boldsymbol{U}_x$ the set of principal open neighborhoods of x. So

$$\boldsymbol{U}_x = \{\mathrm{NZer}(s)\}_{s \in \mathrm{S}(x)}.$$

By Proposition 2.1.11, $\boldsymbol{U}_x$ is a basis of open neighborhoods of x. Because the stalks of the presheaf $\mathcal{O}_X^{\mathrm{pre}}$ and its associated sheaf $\mathcal{O}_X$ are the same, we have

$$\mathcal{O}_{X,x} = \lim_{U \to} \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = \lim_{U \to} A_{\mathrm{S}(U)} \tag{2.1.17}$$

where U runs over the poset $\boldsymbol{U}_x$.

For each $U = \mathrm{NZer}(s) \in \boldsymbol{U}_x$ we have $\mathrm{S}(U) \subseteq \mathrm{S}(x)$, so there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to A_{\mathrm{S}(x)} = A_{\mathfrak{p}}$. Going to the direct limit in U we get an A-ring homomorphism

$$\psi : \mathcal{O}_{X,x} \to A_{\mathfrak{p}}.$$

In the other direction, every element $s \in \mathrm{S}(x)$ belongs to $\mathrm{S}(U)$ for $U = \mathrm{NZer}(s) \in \boldsymbol{U}_x$, and hence s is invertible in $A_{\mathrm{S}(U)}$. This means that every $s \in \mathrm{S}(x)$ is invertible in the A-ring $\mathcal{O}_{X,x} = \lim_{\to} A_{\mathrm{S}(U)}$. Hence there is a unique A-ring homomorphism

$$\phi : A_{\mathfrak{p}} = A_{\mathrm{S}(x)} \to \mathcal{O}_{X,x}.$$

The homomorphism ϕ is surjective, because every $f \in \mathcal{O}_{X,x}$ is the image of some fraction $a/s \in A_{\mathrm{S}(U)}$, see (2.1.17). But $s \in \mathrm{S}(x)$, so $a/s \in A_{\mathfrak{p}}$ and $f = \phi(a/s)$.

Finally consider the A-ring homomorphism $\psi \circ \phi$ from $A_{\mathfrak{p}}$ to itself. By uniqueness there is equality $\psi \circ \phi = \mathrm{id}_{A_{\mathfrak{p}}}$. This shows that ϕ is injective. In conclusion, ϕ is an isomorphism of A-rings. □

Corollary 2.1.18. *Let A be a $\mathbb{K}$-ring. Then $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ is a locally $\mathbb{K}$-ringed space.*

Definition 2.1.19. An *affine* $\mathbb{K}$*-scheme* is a locally ringed space $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ which is isomorphic, in $\mathsf{LRSp}/\mathbb{K}$, to $\mathrm{Spec}(A)$ for some $\mathbb{K}$-ring A.

Definition 2.1.20. The category $\mathsf{AffSch}/\mathbb{K}$ is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the set of affine $\mathbb{K}$-schemes.

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From here on this is reading material

We now provide a more explicit description of the structure sheaf $\mathcal{O}_X$ of the affine scheme $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ in terms of its *Godement sheaf* from [Ye4, Def 3.6.11]. Recall that for a presheaf of $\mathbb{K}$-modules $\mathcal{M}$ on a space X, its Godement sheaf $\mathrm{GSh}(\mathcal{M})$ is defined by

$$\Gamma(U, \mathrm{GSh}(\mathcal{M})) := \prod_{x \in U} \mathcal{M}_x$$

for an open set $U \subseteq X$. Then $\mathrm{Sh}(\mathcal{M}) \subseteq \mathrm{GSh}(\mathcal{M})$ is the subsheaf of *geometric sections.*

Specializing to our case it says the following:

Proposition 2.1.21. *Let $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$. Then $\mathcal{O}_X$ is the subsheaf of the Godement sheaf $\mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})$ consisting of the geometric sections. Specifically, let $U \subseteq X$ be an open set, and let*

$$f \in \Gamma\big(U, \mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

The section f belongs to $\Gamma(U, \mathcal{O}_X)$ iff for every point $x = \mathfrak{p} \in U$ there is an open set V such that $x \in V \subseteq U$, and elements $a \in A$ and $s \in \mathrm{S}(V) \subseteq A$, such that

$$f|_V = a \cdot s^{-1} \in \Gamma(V, \mathrm{GSh}(\mathcal{O}_X^{\mathrm{pre}})) = \prod_{\mathfrak{q} \in V} A_{\mathfrak{q}}.$$

Here are some example of affine schemes.

Example 2.1.22. Consider the ring $A = \mathbb{Z}$. The affine scheme $X := \mathrm{Spec}(\mathbb{Z})$ has these points:

- For every (positive) prime number p there is a maximal ideal $x = \mathfrak{m} := (p)$. These are closed points of X, since $\mathrm{Zer}(p) = \{x\}$. The local ring is
$$\mathcal{O}_{X,x} = \mathbb{Z}_{\mathfrak{m}} = \{a \cdot s^{-1} \mid s \notin \mathfrak{m}\} \subseteq \mathbb{Q}.$$
The residue field is $\mathbb{F}_p$.
- The zero ideal $y = \mathfrak{p} := (0)$ is also prime. It is the *generic point* of X; namely its topological closure is X. To see this, take a nonempty open set $U \subseteq X$. We will prove that $y \in U$. By Prop 2.1.11 there is a nonempty $\mathrm{NZer}(s) \subseteq U$, so $s \neq 0$. Then $s \notin \mathfrak{p}$, so $y = \mathfrak{p} \in \mathrm{NZer}(s) \subseteq U$. The local ring and the residue field at $\mathfrak{p}$ are $\mathbb{Q}$.

Exercise 2.1.23. Analyze the affine scheme $\operatorname{Spec}(A)$ for the ring $A := \mathbb{K}[t]$, the polynomial ring in one variable over a field $\mathbb{K}$. State what is special when $\mathbb{K}$ is algebraically closed. (Hint: it is very similar to the example above.)

Example 2.1.24. Suppose $\mathbb{K}$ is a nonzero ring. For every $n \geq 0$ there is the affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}(\mathbb{K}[t_1, \ldots, t_n])$$

called the *n-dimensional affine space over* $\mathbb{K}$.

In Rem 1.2.8 I promised something, and it is fulfilled in the next example.

Example 2.1.25. We take a DVR A inside its fraction field K; to be concrete, we can take $K := \mathbb{Q}$ and $A := \mathbb{Z}_{\mathfrak{m}}$ where $\mathfrak{m} = (3)$. The affine scheme $(Y, \mathcal{O}_Y) = \operatorname{Spec}(K)$ has a single point, the zero ideal of K, which we call y_0. The affine scheme $(X, \mathcal{O}_X) = \operatorname{Spec}(A)$ has two points: the maximal ideal $x_1 = \mathfrak{m}$ and the zero ideal x_0. The open sets of Y are $Y = \{y_0\}$ and $\varnothing$. The open sets of X are X, $\{x_0\}$ and $\varnothing$.

We want to define a map of ringed spaces

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X),$$

and we deliberately do it the wrong way, so it will fail to be local.

Define $f : Y \to X$ by $f(y_0) := x_1$. To define ψ we need a ring homomorphism $\psi_U : \Gamma(U, \mathcal{O}_X) \to \Gamma(f^{-1}(U), \mathcal{O}_Y)$ for every $U \subseteq X$ open. The only nontrivial case is for $U = X$, and we let it be the inclusion $A \to K$. Now note that $\mathcal{O}_{X,x_1} = A$ and $\mathcal{O}_{Y,y_0} = K$, and the ring homomorphism $\psi_{y_0} : \mathcal{O}_{X,x_1} = A \to \mathcal{O}_{Y,y_0} = K$ is not local.

$\boxed{\text{End of Lecture 2}}$

Lecture 3, 17 March 2021

2.2. **More Properties of Affine Schemes.** In this subsection we shall encounter the first difficult theorem on affine schemes (Thm 2.2.14)

There will many lemmas, some not very interesting to prove, so I will let you read them at home (or prove as exercises).

I'd like to start by retracting the notation (2.1.4), because it seems to be confusing. We shall only use its variant $\mathrm{NZer}(s)$ from (2.1.9). We will sometimes write $\mathrm{NZer}_X(s)$ for the principal open set in $X := \mathrm{Spec}(A)$ defined by an element $s \in A$, and $\mathrm{Zer}_X(\mathfrak{a})$ for the closed set in X defined by an ideal $\mathfrak{a}$.

Recall that for an ideal $\mathfrak{a} \subseteq A$, its *radical* of $\mathfrak{a}$ is the ideal

$$\sqrt{\mathfrak{a}} := \{a \in A \mid a^i \in \mathfrak{a} \text{ for some } i > 0\} \subseteq A.$$

Lemma 2.2.1. *Let $\mathfrak{a} \subseteq A$ be an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Zer}(\mathfrak{a})} \mathfrak{p}$$

and

$$\mathrm{Zer}(\mathfrak{a}) = \mathrm{Zer}(\sqrt{\mathfrak{a}}).$$

Exercise 2.2.2. Prove the lemma. (Hint: do not use the Nullstellensatz.)

Lemma 2.2.3. *For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ the following are equivalent:*

(i) $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

(ii) $\mathrm{Zer}(\mathfrak{a}) = \mathrm{Zer}(\mathfrak{b})$.

Proof. Clear from Lemma 2.2.1. □

Definition 2.2.4. A topological space X is called *quasi-compact* if every open covering of X has a finite subcovering.

Remark 2.2.5. The term "compact" is usually reserved for spaces that are Hausdorff and quasi-compact. Schemes are almost never Hausdorff. There is an analogous notion of separation for scheme, that we will (hopefully) study later.

Proposition 2.2.6. *Let A be a ring. The topological space $X := \mathrm{Spec}(A)$ is quasi-compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering, indexed by an infinite set I. For each i there is an ideal $\mathfrak{a}_i$ such that $U_i = X - \mathrm{Zer}_X(\mathfrak{a}_i)$; this is by the definition of the Zariski topology. Write $\mathfrak{a} := \sum_i \mathfrak{a}_i$. Then there is equality

$$\mathrm{Zer}_X(A) = \varnothing = \bigcap_{i \in I} \mathrm{Zer}_X(\mathfrak{a}_i) = \mathrm{Zer}_X(\mathfrak{a})$$

of subsets of X. By Lemma 2.2.3 we know that $\sqrt{A} = \sqrt{\mathfrak{a}}$. Since $1 \in A$, we see that $1 \in \sqrt{\mathfrak{a}}$, and hence $1 \in \mathfrak{a}$. This says that we can express 1 as a finite sum: $1 = \sum_{i \in I'} a_i$ with $I' \subseteq I$ a finite subset and $a_i \in \mathfrak{a}_i$. We see that $A = \sum_{i \in I'} \mathfrak{a}_i$, and therefore

$$\varnothing = \mathrm{Zer}_X(A) = \bigcap_{i \in I'} \mathrm{Zer}_X(\mathfrak{a}_i)$$

and $X = \bigcup_{i \in I'} U_i$. □

Lemma 2.2.7. *Let $s_1, \ldots, s_m \in A$. TFAE:*

(i) $\mathrm{Spec}(A) = \bigcup_i \mathrm{NZer}(s_i)$.

(ii) *There exists* $a_1, \dots, a_m \in A$ *s.t.* $1_A = \sum_i a_i \cdot s_i$.

Proof.

(i) $\Rightarrow$ (ii): As in the proof of the proposition, $A = \sum_i \mathfrak{a}_i$, where $\mathfrak{a}_i := (s_i)$. So the element 1_A is a linear combination $1_A = \sum_i a_i \cdot s_i$ for some $a_i \in A$.

(ii) $\Rightarrow$ (i): Here $1_A \in \sum_i \mathfrak{a}_i$, so $A = \sum_i \mathfrak{a}_i$, and, as in the proof of the proposition, $\operatorname{Spec}(A) = \bigcup_i \operatorname{NZer}(s_i)$. □

Lemma 2.2.8. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$*,* $s \in A$ *and* $U := \operatorname{NZer}_X(s)$*. Then* s *is invertible in the ring* $\Gamma(U, \mathcal{O}_X)$*.*

Proof. Clearly $s \in \mathrm{S}(U) \subseteq A$. By definition of the presheaf $\mathcal{O}_X^{\mathrm{pre}}$, the element s is invertible in the ring $A_{\mathrm{S}(U)} = \Gamma(U, \mathcal{O}_X^{\mathrm{pre}})$. Due to the homomorphism of sheaves of rings $\mathcal{O}_X^{\mathrm{pre}} \to \mathcal{O}_X$ we see that s is invertible in the ring $\Gamma(U, \mathcal{O}_X)$. □

Lemma 2.2.9. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$*. Suppose* $U \subseteq X$ *is an open set,* $f \in \Gamma(U, \mathcal{O}_X)$*, and* $x = \mathfrak{p} \in U$*. Then there are elements* $a, s \in A$ *such that, letting* $W := \operatorname{NZer}_X(s)$*, we have* $x \in W \subseteq U$ *and*

$$f|_W = a \cdot s^{-1} \in \Gamma(W, \mathcal{O}_X).$$

Proof. The proof is interesting, so I'll do it in class.

By Proposition 2.1.21 the point $x = \mathfrak{p} \in U$ has an open neighborhood $V \subseteq U$, and elements $b, t \in A$, such that $t \in \mathrm{S}(V)$ and $f|_V = b \cdot t^{-1} \in \Gamma(V, \mathcal{O}_X)$.

According to Proposition 2.1.11 we can replace V with a smaller open neighborhood $\operatorname{NZer}_X(r)$ of x for some $r \in A$, i.e. $x \in \operatorname{NZer}_X(r) \subseteq V$.

Define the elements $a := b \cdot r \in A$ and $s := t \cdot r \in A$, and the open set $W := \operatorname{NZer}_X(s)$.

Since $\operatorname{NZer}_X(r) \subseteq V \subseteq \operatorname{NZer}_X(t)$, it follows that

$$\operatorname{NZer}_X(r) = \operatorname{NZer}_X(t \cdot r) = \operatorname{NZer}_X(s) = W.$$

By Lem 2.2.8 we know that s is invertible in $\Gamma(W, \mathcal{O}_X)$. Hence t and r are invertible in $\Gamma(W, \mathcal{O}_X)$. And

$$a \cdot s^{-1} = (b \cdot r) \cdot (t \cdot r)^{-1} = b \cdot t^{-1} = f|_W \in \Gamma(W, \mathcal{O}_X).$$ □

Lemma 2.2.10. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$*. Suppose* $V \subseteq X$ *is a quasi-compact open set, and* $f \in \Gamma(V, \mathcal{O}_X)$*. Then there are finitely many elements* $a_1, \dots, a_m, s_1, \dots, s_m \in A$ *such that, writing* $W_i := \operatorname{NZer}_X(s_i)$*, we have* $V = \bigcup_{i=1}^m W_i$*, and*

$$f|_{W_i} = a_i \cdot s_i^{-1} \in \Gamma(W_i, \mathcal{O}_X)$$

for every i*.*

Proof. By Lemma 2.2.9, for every point $x = \mathfrak{p} \in V$ there are elements $a_x, s_x \in A$ such that, writing $W_x := \operatorname{NZer}_X(s_x)$, we have $x \in W_x \subseteq V$ and

$$f|_{W_x} = a_x \cdot s_x^{-1} \in \Gamma(W_x, \mathcal{O}_X).$$

We have an open covering $V = \bigcup_{x \in V} W_x$. Because of quasi-compactness we can pass to a finite subcovering, that is to a finite subset $\{x_1, \dots, x_m\}$ of V. Finally, by letting $a_i := a_{x_i}$, $s_i := s_{x_i}$ and $W_i := W_{x_i}$ we are done. □

A ring homomorphism $\psi : A \to B$ induces a map of sets

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A) \tag{2.2.11}$$

with formula

$$\operatorname{Spec}(\psi)(\mathfrak{q}) := \psi^{-1}(\mathfrak{q}).$$

If it is not clear to you why the ideal $\mathfrak{p} := \psi^{-1}(\mathfrak{q}) \subseteq A$ is prime, then prove it!

Lemma 2.2.12. *Let $\psi : A \to B$ be a ring homomorphism.*

(1) *The map $\operatorname{Spec}(\psi)$ is continuous.*
(2) *Suppose $B = A_s$ for some $s \in A$ and ψ is the localization homomorphism. Then the image of $\operatorname{Spec}(\psi)$ is the open set $\operatorname{NZer}(s) \subseteq \operatorname{Spec}(A)$, and*

$$\operatorname{Spec}(\psi) : \operatorname{Spec}(A_s) \to \operatorname{NZer}(s)$$

is a homeomorphism.

Exercise 2.2.13. Prove the last lemma. (Hint: for (i) show that $f := \operatorname{Spec}(\psi)$ satisfies $f^{-1}(\operatorname{NZer}(s)) = \operatorname{NZer}(\psi(s))$.)

Here is the main theorem of this subsection. The proof is by a magical juggling of denominators.

Theorem 2.2.14. *Let A be a ring and $s \in A$ an element. Consider the affine scheme $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ and the principal open set $U := \operatorname{NZer}_X(s)$. There is a unique A-ring isomorphism $A_s \cong \Gamma(U, \mathcal{O}_X)$.*

Proof.

Step 1. Since the element s is invertible in the ring $\Gamma(U, \mathcal{O}_X)$, there is a unique A-ring homomorphism

$$\phi : A_s \to \Gamma(U, \mathcal{O}_X). \tag{2.2.15}$$

We need to prove that ϕ is bijective.

Step 2. In this step we we will prove that ϕ is injective. Since $\mathcal{O}_X$ is a subsheaf of the Godement sheaf $\operatorname{GSh}(\mathcal{O}_X)$, there is an embedding of A-rings

$$\Gamma(U, \mathcal{O}_X) \rightarrowtail \Gamma\big(U, \operatorname{GSh}(\mathcal{O}_X)\big) = \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

It thus suffices to prove that the composed homomorphism

$$\phi' : A_s \to \prod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \tag{2.2.16}$$

is injective.

Let's write $B := A_s$. By Lemma 2.2.12 we know that $\operatorname{Spec}(B) = U$ as topological subspaces of X. For every $\mathfrak{p} \in U$ the element s is invertible in $A_{\mathfrak{p}}$, and hence the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is bijective. So we can rewrite (2.2.16) as $\phi' : B \to \prod_{\mathfrak{p} \in U} B_{\mathfrak{p}}$, and we need to prove it is injective.

Suppose $b \in B$ is such that $\phi'(b) = 0$. We need to prove that $b = 0$.

The vanishing of $\phi'(b)$ means that $\phi'_{\mathfrak{p}}(b) = 0$ in $B_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$. Next, the vanishing of b in $B_{\mathfrak{p}}$ means that there is some element $t_{\mathfrak{p}} \in B$ such that $t_{\mathfrak{p}}(\mathfrak{p}) \neq 0$, i.e. $\mathfrak{p} \in \operatorname{NZer}(t_{\mathfrak{p}})$, yet $t_{\mathfrak{p}} \cdot b = 0$ in B.

Now $U = \operatorname{Spec}(B) = \bigcup_{\mathfrak{p} \in U} \operatorname{NZer}(t_{\mathfrak{p}})$. By quasi-compactness of U we can pass to a finite subcovering, indexed by $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Let $t_i := t_{\mathfrak{p}_i} \in B$. Then $t_i \cdot b = 0$ in B, and $\operatorname{Spec}(B) = \bigcup_{i=1}^m \operatorname{NZer}(t_i)$. According to Lemma 2.2.7 there are elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Then $b = 1_B \cdot b = \sum_i c_i \cdot t_i \cdot b = 0$.

End of live lecture

Please read end of proof at home.

Step 3. We now prove that the homomorphism ϕ from (2.2.15) is surjective. Recall that $B = A_s$ and $U = \operatorname{Spec}(B)$. Take an element $f \in \Gamma(U, \mathcal{O}_X)$. We know that $U = \operatorname{Spec}(B)$ is a quasi-compact topological space. By Lem 2.2.10 there are finitely many elements $a_i, s_i \in A$, $1 \leq i \leq m$, such that, letting $U_i := \operatorname{NZer}_U(s_i)$, we have $U = \bigcup_{i=1}^m U_i$, and $f|_{U_i} = a_i/s_i \in \Gamma(U_i, \mathcal{O}_X)$ for every i.

Step 2, when applied to the element $s_i \cdot s_j \cdot s \in A$ instead of to s, shows that for every i, j, letting $U_{i,j} := \operatorname{NZer}_U(s_i \cdot s_i) = U_i \cap U_j$, the A-ring homomorphism

$$\phi_{i,j} : A_{s_i \cdot s_j \cdot s} = B_{s_i \cdot s_j} \to \Gamma\big(U_{i,j}, \mathcal{O}_X\big)$$

is injective. We know that

$$f|_{U_{i,j}} = a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1} \in \Gamma\big(U_{i,j}, \mathcal{O}_X\big).$$

Therefore $a_i \cdot s_i^{-1} = a_j \cdot s_j^{-1}$ in $B_{s_i \cdot s_j}$. The kernel of the localization homomorphism $B \to B_{s_i \cdot s_j}$ is known: there is a positive integer $l_{i,j}$ such that

$$(s_i \cdot s_j)^{l_{i,j}} \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0$$

in B. Taking $l := \max(\{l_{i,j}\})$ we obtain

$$(s_i \cdot s_j)^l \cdot (a_i \cdot s_j - a_j \cdot s_i) = 0 \tag{2.2.17}$$

in B.

Define $b_i := a_i \cdot s_i^l$ and $t_i := s_i^{l+1}$. Then $\operatorname{NZer}_U(t_i) = \operatorname{NZer}_U(s_i) = U_i$ and

$$f|_{U_i} = a_i \cdot s_i^{-1} = b_i \cdot t_i^{-1} \in \Gamma(U_i, \mathcal{O}_X). \tag{2.2.18}$$

Also, from (2.2.17) we have

$$t_i \cdot b_j = s_i^{l+1} \cdot a_j \cdot s_j^l = (s_i \cdot s_j)^l \cdot s_i \cdot a_j = (s_i \cdot s_j)^l \cdot s_j \cdot a_i = t_j \cdot b_i \tag{2.2.19}$$

in B.

Since

$$\operatorname{Spec}(B) = U = \bigcup_{i=1}^m U_i, \tag{2.2.20}$$

and $U_i = \operatorname{NZer}(t_i)$, by Lem 2.2.7 we can find elements $c_1, \ldots, c_m \in B$ such that $1_B = \sum_i c_i \cdot t_i$. Let

$$b := \sum_i c_i \cdot b_i \in B.$$

For every i we have – using (2.2.19) –

$$\begin{aligned} \phi(b)|_{U_i} &= \sum_j c_j \cdot b_j|_{U_i} = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_i \cdot b_j\Big)|_{U_i} = t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j \cdot b_i\Big)|_{U_i} \\ &= t_i^{-1} \cdot \Big(\sum_j c_j \cdot t_j\Big) \cdot b_i|_{U_i} = t_i^{-1} \cdot b_i|_{U_i} = f|_{U_i} \end{aligned}$$

in $\Gamma(U_i, \mathcal{O}_X)$. But by (2.2.20) and the sheaf axioms this implies that $\phi(b) = f$ in $\Gamma(U, \mathcal{O}_X)$. □

Corollary 2.2.21. *For a ring A, with $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, the canonical ring homomorphism $A \to \Gamma(X, \mathcal{O}_X)$ is bijective.*

Proof. Take $s = 1$ in Thm 2.2.14. □

$\boxed{\text{More material for the vacation}}$

Here are three more exercises, all interesting in my opinion. The first two are about the notion of quasi-compactness.

It is easy to see that if X is a quasi-compact topological space (e.g. X is the underlying topological space of an affine scheme) and $Z \subseteq X$ is a closed subset, then Z (with the induced topology) is also quasi-compact. (You probably saw this in a basic topology class.) But what about an open subset $U \subseteq X$?

Exercise 2.2.22. Let A be a noetherian ring and let $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$. Let $U \subseteq X$ be an open subset. Prove that U is a quasi-compact topological space. (This is of medium difficulty. I will give hints by email to those who write to me.)

Exercise 2.2.23. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[t_1, t_2, \ldots]$ be the polynomial ring in countably many variables. Let $\mathfrak{m} \subseteq A$ be the ideal generated by the variables. Note that $\mathfrak{m}$ is a maximal ideal. Define $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$, $Z := \mathrm{Zer}_X(\mathfrak{m})$, and $U := X - Z$. Since $\mathfrak{m}$ is maximal we have that $Z = \{\mathfrak{m}\}$. Thus $U = X - \{\mathfrak{m}\}$.

The ring A is not noetherian. Indeed, the ideal $\mathfrak{m}$ is not finitely generated. Prove this. (This is of medium difficulty.)

The topological space X is of course quasi-compact. The goal is to prove that the open set U is not quasi-compact.

For every i let $V_i := \mathrm{NZer}_X(t_i)$, which is an open set in X. Show that $U = \bigcup_i V_i$. (This is easy.)

Now prove that the open covering $\{V_i\}_{i \geq 1}$ of U has no finite subcovering. (This is hard, but I think I know how to prove it. I will give hints to those who ask in private emails.)

The last exercise will give us an example of a scheme which is not affine. Of course we did not learn about such schemes yet; but once we do, we will go back to this exercise and make the observation that $(U, \mathcal{O}_X|_U)$ is a scheme that's not affine.

Exercise 2.2.24. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[t_1, t_2]$ be the polynomial ring in two variables. The affine scheme $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ is called the 2*-dimensional affine space over* $\mathbb{K}$, with notation $\mathbf{A}^2_{\mathbb{K}}$.

Let $\mathfrak{m}$ be the maximal ideal in A generated by the variables, let $Z := \mathrm{Zer}_X(\mathfrak{m})$ and let $U := X - Z$. Thus (like in the previous exercise) $U = X - \{\mathfrak{m}\}$, the complement of the origin.

We know (by Cor 2.2.21) that $\Gamma(X, \mathcal{O}_X) = A$. The exercise is to prove that $\Gamma(U, \mathcal{O}_X) = A$, namely that the restriction ring homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is bijective.

Here are a few hints. First, let $V_i := \mathrm{NZer}_X(t_i)$, so $U = V_1 \cup V_2$. Also $V_1 \cap V_2 = \mathrm{NZer}_X(t_1 \cdot t_2)$. Using this open covering and the sheaf condition, there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_X) \to \Gamma(V_1, \mathcal{O}_X) \oplus \Gamma(V_2, \mathcal{O}_X) \to \Gamma(V_1 \cap V_2, \mathcal{O}_X). \tag{2.2.25}$$

By Thm 2.2.14 we know that $\Gamma(V_1, \mathcal{O}_X) = A_{t_1}$, $\Gamma(V_2, \mathcal{O}_X) = A_{t_2}$ and $\Gamma(V_1 \cap V_2, \mathcal{O}_X) = A_{t_1 \cdot t_2}$. This last ring is the ring of Laurent polynomials, and the first two rings are subrings of it.

Show that the exact sequence (2.2.25) can be interpreted as saying that

$$\Gamma(U, \mathcal{O}_X) = A_{t_1} \cap A_{t_2} \subseteq A_{t_1 \cdot t_2}.$$

Finally use the fact that A is a UFD. (Find a reference for this.)

End of Lecture 3

Lecture 4, 7 April 2021

First:

Solution of Exercise 2.2.23. (Different from the solution Yotam sent earlier.) Recall that $\mathbb{K}$ is a field, $A := \mathbb{K}[t_0, t_1, \ldots]$ is the polynomial ring in countably many variables, $\mathfrak{m} \subseteq A$ is the ideal generated by the variables, $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $Z := \operatorname{Zer}_X(\mathfrak{m})$, and $U := X - Z$. Since $\mathfrak{m}$ is maximal we have $Z = \{\mathfrak{m}\}$. For every i let $V_i := \operatorname{NZer}_X(t_i)$, which is an open set in X. It is easy to see that $U = \bigcup_{i \in \mathbb{N}} V_i$. The exercise is to prove that there is no finite subcovering.

Assume that there is a finite subcovering. Then $U = \bigcup_{i \leq n} V_i$ for some $n \in \mathbb{N}$. This implies that

$$\operatorname{Zer}_X(\mathfrak{m}) = \{\mathfrak{m}\} = \bigcap_{i \leq n} \operatorname{Zer}_X(t_i) = \operatorname{Zer}_X(\mathfrak{p}),$$

where $\mathfrak{p} := (t_0, \ldots, t_n)$. The ideals $\mathfrak{m}$ and $\mathfrak{p}$ are prime, and hence radical. (An ideal $\mathfrak{a}$ is called radical if $\mathfrak{a} = \sqrt{\mathfrak{a}}$; and $\mathfrak{a}$ is radical iff the only nilpotent element in ring $A/\mathfrak{a}$ is 0.) By Lem 2.2.3 we get $\mathfrak{m} = \mathfrak{p}$. This says that $t_{n+1} \in \mathfrak{p}$, which is impossible; there are two ways to see that:

(1) One way to see it is by a direct linear algebra calculation.
(2) Suppose $t_{n+1} = \sum_{i \leq n} a_i \cdot t_i$ for some $a_i \in A$. Looking at the $\mathbb{K}$-ring homomorphism $\phi : A \to \mathbb{K}$ sending $\phi(t_i) := 0$ for $i \neq n+1$ and $\phi(t_{n+1}) := 1$, we get $0 = 1$ in $\mathbb{K}$.

2.3. **Maps of Affine Schemes.** Recall that an affine $\mathbb{K}$-scheme is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$, which is isomorphic, in the category $\mathsf{LRSp}/\mathbb{K}$, to $\operatorname{Spec}(A)$ for some ring A.

The category of affine $\mathbb{K}$-schemes is denoted by $\mathsf{AffSch}/\mathbb{K}$. It is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the affine schemes. Also Recall that $\mathsf{Rng}/\mathbb{K}$ is the category of (commutative) $\mathbb{K}$-rings.

Proposition 2.3.1. *The assignment that sends a locally ringed space $(X, \mathcal{O}_X)$ to the ring $\Gamma(X, \mathcal{O}_X)$, and a map of locally ringed spaces $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to the ring homomorphism*

$$\Gamma(X, \phi) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X),$$

is a functor

$$\Gamma : (\mathsf{LRSp}/\mathbb{K})^{\mathrm{op}} \to \mathsf{Rng}/\mathbb{K}.$$

Exercise 2.3.2. Prove proposition 2.3.1. (Easy.)

Remark 2.3.3. We are interested in a functor

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}$$

which will be adjoint to Γ. In formula (2.2.11) and Lemma 2.2.12(1) we saw how to produce a functor Spec with values in topological spaces.

Lifting (or upgrading) this functor to $\mathsf{AffSch}/\mathbb{K}$, which is the same as lifting it to $\mathsf{LRSp}/\mathbb{K}$, can be done directly. This is the next optional exercise. We will do it indirectly using Thm 2.3.12.

Exercise 2.3.4. (Optional) Given a ring homomorphism $\psi : A \to B$, let $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$, $(Y, \mathcal{O}_Y) := \operatorname{Spec}(B)$ and $f := \operatorname{Spec}(\psi) : Y \to X$, try to construct a homomorphism of sheaves of rings $\tilde{\psi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ on X such that $\Gamma(X, \tilde{\psi}) = \psi$. See [Har, Prop II.2.3].

Definition 2.3.5. Let $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$ and let $s \in \Gamma(Y, \mathcal{O}_Y)$. For a point $y \in Y$ we denote by $s(y)$ the image of the element s in the residue field $\boldsymbol{k}(y)$, via the canonical homomorphisms

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\mathrm{rest}_{y/Y}} \mathcal{O}_{Y,y} \twoheadrightarrow \boldsymbol{k}(y).$$

And we write

$$\mathrm{NZer}_Y(s) := \{y \in Y \mid s(y) \neq 0\}.$$

Lemma 2.3.6. *Let $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$ and $s \in \Gamma(Y, \mathcal{O}_Y)$. Then:*

(1) *The set $\mathrm{NZer}_Y(s)$ is open in Y.*
(2) *The element s is invertible in the ring $\Gamma(\mathrm{NZer}_Y(s), \mathcal{O}_Y)$.*

The proof was not done in class; please read it.

Proof.
(1) Write $V := \mathrm{NZer}_Y(s)$. Take a point $y \in V$. Since $s(y) \neq 0$, it is an invertible element of the field $\boldsymbol{k}(y)$. Because the stalk $\mathcal{O}_{Y,y}$ is a local ring, it follows that s is an invertible element of $\mathcal{O}_{Y,y}$. Let $t \in \mathcal{O}_{Y,y}$ be the inverse of s, so $s \cdot t = 1$ in $\mathcal{O}_{Y,y}$. There is an open neighborhood W of y such that $t \in \Gamma(W, \mathcal{O}_Y)$. There is a smaller open neighborhood W' of y s.t. $s \cdot t = 1$ in $\Gamma(W', \mathcal{O}_Y)$. But then $s(y') \neq 0$ for all $y' \in W'$, so $W' \subseteq V$.

(2) Let $V := \mathrm{NZer}_Y(s)$ as before. As we saw above, every point $y \in V$ has an open neighborhood $W_y \subseteq V$ and an element $t_y \in \Gamma(W_y, \mathcal{O}_Y)$ such that $s \cdot t_y = 1$. This means that $t_y = s^{-1}$ in $\Gamma(W_y, \mathcal{O}_Y)$, a fact that makes t_y unique. We conclude that $t_y = t_{y'}$ in $\Gamma(V_y \cap V_{y'}, \mathcal{O}_Y)$. By the sheaf property we get $t \in \Gamma(V, \mathcal{O}_Y)$, and it satisfies $s \cdot t = 1$. □

Recall that for a ring A we denote by $A^{\times}$ its multiplicative group, i.e. the set of invertible elements of A, equipped with the operation of multiplication.

Exercise 2.3.7. Let $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$.

(1) Show that the assignment $U \mapsto \Gamma(U, \mathcal{O}_X)^{\times}$ is a sheaf of groups on X. It is denoted by $\mathcal{O}_X^{\times}$ and also by $\mathrm{GL}_1(\mathcal{O}_X)$.
(2) Show that for the stalks at for every point $x \in X$ there is a canonical group isomorphism $(\mathcal{O}_X^{\times})_x \cong (\mathcal{O}_{X,x})^{\times}$.

Lemma 2.3.8. *Let $(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a map in $\mathsf{LRSp}/\mathbb{K}$, $A := \Gamma(X, \mathcal{O}_X)$, $B := \Gamma(Y, \mathcal{O}_Y)$ and $\phi := \Gamma(X, \tilde{\phi}) : A \to B$.*

(1) *Take $s \in A$ and let $U := \mathrm{NZer}_X(s) \subseteq X$ and $V := f^{-1}(U) \subseteq Y$. Then $V = \mathrm{NZer}_Y(\phi(s))$.*
(2) *For an open set $U \subseteq X$ let*

$$\mathrm{S}(U) := \{s \in A \mid s(x) \neq 0 \ \text{ for all } \ x \in U\}.$$

Then the image of $\mathrm{S}(U)$ in $\Gamma(U, \mathcal{O}_X)$ consists of invertible elements. It follows there there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to \Gamma(U, \mathcal{O}_X)$.
(3) *Let $V := f^{-1}(U) \subseteq Y$. Then the image of $\phi(\mathrm{S}(U))$ in $\Gamma(V, \mathcal{O}_Y)$ consists of invertible elements. It follows there there is a unique A-ring homomorphism $A_{\mathrm{S}(U)} \to \Gamma(V, \mathcal{O}_V)$.*

Note that item (3) is a special case of item (2) with $(f, \tilde{\phi}) = \mathrm{id}$. Also, when $(X, \mathcal{O}_X)$ is an affine scheme, the set $\mathrm{S}(U)$ is the same as in Definition 2.1.13.

Exercise 2.3.9. Prove Lemma 2.3.8.

Lemma 2.3.10. *Let* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$ *for some ring* A*. Then*

$$\Gamma(X, \mathcal{O}_X^{\times}) = A^{\times} = \mathrm{S}(X)$$

as subsets of A*. Therefore* $A_{\mathrm{S}(X)} = A$ *as rings; to be precise, the canonical ring homomorphism* $A \to A_{\mathrm{S}(X)}$ *is bijective.*

Exercise 2.3.11. Prove Lemma 2.3.10.

Theorem 2.3.12. *Let* $A \in \mathsf{Rng}/\mathbb{K}$ *and let* $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$*. Write* $B := \Gamma(Y, \mathcal{O}_Y)$ *and* $(X, \mathcal{O}_X) := \operatorname{Spec}(A)$*. Given a* $\mathbb{K}$*-ring homomorphism* $\phi : A \to B$*, there is a unique map*

$$(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ *such that*

$$\Gamma(X, \tilde{\phi}) = \phi : A \to B.$$

A weaker statement can be found in [Har, Exer II.2.4].

The proof was not done in class; please read it. There might be errors – please verify!

Proof.

Step 1. We prove uniqueness of f in Top, which is the same as uniqueness in Set.

Take a point $y \in Y$, and let $x = \mathfrak{p} := f(y) \in X$. Because $(f, \tilde{\phi})$ is a map in $\mathsf{Rng}/\mathbb{K}$, the ring homomorphism $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism. We are given that $\Gamma(X, \tilde{\phi}) = \phi$, so we have a commutative diagram of rings

(2.3.13)
$$\begin{tikzcd}
A \arrow[r, "\phi = \Gamma(X,\tilde{\phi})"] \arrow[d, "\mathrm{rest}_{x/U}"'] & B \arrow[d, "\mathrm{rest}_{y/U}"] \\
\mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] \arrow[d, two heads] & \mathcal{O}_{Y,y} \arrow[d, two heads] \\
\boldsymbol{k}(x) \arrow[r, tail] & \boldsymbol{k}(y)
\end{tikzcd}$$

The bottom square commutes because $\tilde{\phi}_y$ is a local homomorphism. The homomorphism $\boldsymbol{k}(x) \to \boldsymbol{k}(y)$ is injective. Comparing the two paths in this diagram we see that

$$\operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) = \operatorname{Ker}\big(A \to \boldsymbol{k}(\mathfrak{p})\big) = \mathfrak{p}.$$

The homomorphism $A \xrightarrow{\phi} B \to \boldsymbol{k}(y)$ depends only on y and ϕ, and it determines $f(y) = \mathfrak{p} = x$.

Step 2. Now we prove that the homomorphism of sheaves of rings on X

$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

is unique. Since the principal open sets $U = \operatorname{NZer}_X(s)(s) \subseteq X$, for $s \in A$, are a basis for the topology, it is enough to prove the uniqueness of the ring homomorphism

$$\Gamma(U, \tilde{\phi}) : \Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_Y),$$

for $U := \operatorname{NZer}_X(s)$ and $V := f^{-1}(U) \subseteq Y$.

By Theorem 2.2.14 we know that $\Gamma(U, \mathcal{O}_X) = A_s$. Let's examine this commutative diagram of rings

(2.3.14)
$$\begin{tikzcd} A = \Gamma(X, \mathcal{O}_X) \arrow[r, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d, "\mathrm{rest}_{U/X}"'] & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{V/Y}"] \\ A_s = \Gamma(U, \mathcal{O}_X) \arrow[r, "\Gamma(U, \tilde{\phi})"] & \Gamma(V, \mathcal{O}_Y) \end{tikzcd}$$

The path going right and then down depends only on ϕ, f and U. (In step 1 we already determined f.) Because the left vertical arrow is a localization, it follows that the ring homomorphism $\Gamma(U, \tilde{\phi})$ is unique.

Step 3. Here we start with the existence. We define the function $f : Y \to X$ by the formula from step 1, namely a point $y \in Y$ is sent to the point

(2.3.15)
$$x = \mathfrak{p} := \operatorname{Ker}\big(A \xrightarrow{\phi} B \to \boldsymbol{k}(y)\big) \in \operatorname{Spec}(A) = X.$$

We need to prove that f is continuous. It suffices to show that for every principal open set $U = \mathrm{NZer}_X(s) \subseteq X$, its preimage $f^{-1}(U)$ is open. But by Lemma 2.3.8(1) we have $f^{-1}(U) = \mathrm{NZer}_X(\phi(s))$, and this is open in Y.

Step 4. Now we construct the homomorphism of sheaves of rings

(2.3.16)
$$\tilde{\phi} : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the universal property of sheafification, it suffices to construct a homomorphism of presheaves of rings

(2.3.17)
$$\tilde{\phi}^{\mathrm{pre}} : \mathcal{O}_X^{\mathrm{pre}} \to f_*(\mathcal{O}_Y),$$

and then to take $\tilde{\phi} := \mathrm{Sh}(\tilde{\phi}^{\mathrm{pre}})$.

For every $U \subseteq X$ open we have by definition $\Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = A_{\mathrm{S}(U)}$, where $\mathrm{S}(U) \subseteq A$ is the multiplicatively closed set from Definition 2.1.13 and Lemma 2.3.8(2). By Lemma 2.3.8(3) there is a unique A-ring homomorphism

$$\tilde{\phi}_U^{\mathrm{pre}} : \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) = A_{\mathrm{S}(U)} \to \Gamma(f^{-1}(U), \mathcal{O}_Y) = \Gamma(U, f_*(\mathcal{O}_Y)).$$

As U varies this become a homomorphism of presheaves of rings (2.3.17).

Finally, Lemma 2.3.10 says that $A_{\mathrm{S}(X)} = A$. It follows that

$$\Gamma(X, \tilde{\phi}) = \Gamma(X, \tilde{\phi}^{\mathrm{pre}}) = \phi$$

as homomorphisms $A \to B$, as required.

Step 5. It remains to prove that $(f, \tilde{\phi})$ is local, i.e. $\tilde{\phi}_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a local homomorphism for every $y \in Y$ and $x = \mathfrak{p} := f(y)$.

By the definition of f, see formula (2.3.15), the solid diagram of rings below is commutative:

(2.3.18)
$$\begin{tikzcd} A = \Gamma(X, \mathcal{O}_X) \arrow[r, "\phi"] \arrow[d, two heads, "\mathrm{can}"'] & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{y/Y}"] \\ A/\mathfrak{p} \arrow[d, tail, "\mathrm{can}"'] \arrow[dr, tail] & \mathcal{O}_{Y,y} \arrow[d, two heads, "\mathrm{can}"] \\ \boldsymbol{k}(\mathfrak{p}) \arrow[r, dashed] & \boldsymbol{k}(y) \end{tikzcd}$$

Because the slanted arrow is an injection, it extends to the field of fractions $\boldsymbol{k}(\mathfrak{p})$, i.e. the dashed arrow exists.

On the other hand, our construction in step 4 is such that for every open neighborhood U of x, and for every open set V in Y such that $y \in V \subseteq f^{-1}(U)$, there is a commutative diagram

(2.3.19)
$$\begin{tikzcd}
A = \Gamma(X, \mathcal{O}_X) \arrow[r, "\phi = \Gamma(X, \tilde{\phi})"] \arrow[d, "\mathrm{can}"'] & B = \Gamma(Y, \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{f^{-1}(U)/Y}"] \\
A_{\mathrm{S}(U)} = \Gamma(U, \mathcal{O}_X^{\mathrm{pre}}) \arrow[r, "\tilde{\phi}_U^{\mathrm{pre}}"] \arrow[dr] & \Gamma(f^{-1}(U), \mathcal{O}_Y) \arrow[d, "\mathrm{rest}_{V/f^{-1}(U)}"] \\
 & \Gamma(V, \mathcal{O}_Y)
\end{tikzcd}$$

Passing to direct limits in U and V we get this commutative diagram:

(2.3.20)
$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\
A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] & \mathcal{O}_{Y,y}
\end{tikzcd}$$

In the next diagram the top square is the commutative diagram (2.3.20) and the boundary is the boundary of the commutative diagram (2.3.18):

(2.3.21)
$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\
A_{\mathfrak{p}} = \mathcal{O}_{X,x} \arrow[r, "\tilde{\phi}_y"] \arrow[d, two heads, "\mathrm{can}"'] & \mathcal{O}_{Y,y} \arrow[d, two heads, "\mathrm{can}"] \\
\boldsymbol{k}(\mathfrak{p}) \arrow[r] & \boldsymbol{k}(y)
\end{tikzcd}$$

The surjections are the canonical ones from a local ring to its residue field. Because $A_{\mathfrak{p}}$ is a localization of A, there is at most one A-ring homomorphism $A_{\mathfrak{p}} \to \boldsymbol{k}(y)$, and this implies that the bottom square in (2.3.21) is also commutative. Hence $\tilde{\phi}_y$ is a local homomorphism. □

The rest of the material was done in class

Corollary 2.3.22. *The assignment that sends a ring A to the affine scheme* $\operatorname{Spec}(A)$*, and a ring homomorphism ϕ to the map of affine schemes $(f, \tilde{\phi})$ from Theorem 2.3.12, is a functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}.$$

Proof. Say $\phi : A \to B$ and $\psi : B \to C$ are homomorphism in $\mathsf{Rng}/\mathbb{K}$, with corresponding maps of affine schemes

$$\operatorname{Spec}(\phi) = (f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

and

$$\operatorname{Spec}(\psi) = (g, \tilde{\psi}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y).$$

We get these maps of schemes

$$(f, \tilde{\phi}) \circ (g, \tilde{\psi}),\ \operatorname{Spec}(\psi \circ \phi) \ : \ (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X),$$

and we need to prove they are equal.

The condition in Thm 2.3.12 is that

$$\Gamma(Y, \tilde{\psi}) \circ \Gamma(X, \tilde{\phi}) = \psi \circ \phi = \Gamma(X, \psi \circ \phi).$$

The uniqueness clause in Thm 2.3.12 says that

$$\operatorname{Spec}(\phi) \circ \operatorname{Spec}(\psi) = \operatorname{Spec}(\psi \circ \phi).$$

Likewise $\operatorname{Spec}(\mathrm{id}_A) = \mathrm{id}_{\operatorname{Spec}(A)}$. □

Corollary 2.3.23. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{LRS}/\mathbb{K}$$

is right adjoint to the functor

$$\Gamma : \mathsf{LRS}/\mathbb{K} \to (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}}.$$

Proof. Thm 2.3.12 produces a bijection

$$\operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(A, \Gamma(Y, \mathcal{O}_Y)\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big((Y, \mathcal{O}_Y), \operatorname{Spec}(A)\big) \tag{2.3.24}$$

for $A \in \mathsf{Rng}/\mathbb{K}$ and $(Y, \mathcal{O}_Y) \in \mathsf{LRSp}/\mathbb{K}$. We need to prove that this is bifunctorial, i.e. it is functorial in A and $(Y, \mathcal{O}_Y)$. This is an exercise. □

Exercise 2.3.25. Prove that the bijection (2.3.24) is functorial in A and in $(Y, \mathcal{O}_Y)$.

Corollary 2.3.26. *The functor*

$$\operatorname{Spec} : (\mathsf{Rng}/\mathbb{K})^{\mathrm{op}} \to \mathsf{AffSch}/\mathbb{K}$$

is an equivalence of categories, with quasi-inverse Γ.

Proof. By definition, $\mathsf{AffSch}/\mathbb{K}$ is the essential image in $\mathsf{LRSp}/\mathbb{K}$ of the functor Spec.

We need to prove that Spec is fully faithful. But this is an immediate consequence of Theorem 2.3.12 – for rings A and B there is a bijection

$$\operatorname{Spec} : \operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}(A, B) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{LRSp}/\mathbb{K}}\big(\operatorname{Spec}(B), \operatorname{Spec}(A)\big)\,,\ \phi \mapsto \operatorname{Spec}(\phi).$$ □

◇ ◇ ◇

Next week we will some examples of the "functor of points".

End of Lecture 4

Lecture 5, 21 April 2021

2.4. **The Functor of Points of an Affine Scheme.** Recall that $\mathbb{K}$ is our nonzero base ring.

Let $X = \operatorname{Spec}(A)$ be an affine scheme over $\mathbb{K}$. We are going to see how X gives rise to a functor from rings to sets. Later on this will be extended to an arbitrary scheme X.

Proposition 2.4.1. *Fix an affine $\mathbb{K}$-scheme $X = \operatorname{Spec}(A)$.*

For a ring $B \in \mathsf{Rng}/\mathbb{K}$ we define the set

$$X(B) := \operatorname{Hom}_{\mathsf{AffSch}/\mathbb{K}}\big(\operatorname{Spec}(B), X\big).$$

For a homomorphism $\phi : B \to C$ in $\mathsf{Rng}/\mathbb{K}$ we define the function

$$X(\phi) : X(B) \to X(A)$$

by the formula

$$X(\phi)(g) := g \circ \operatorname{Spec}(\phi)$$

for $g : \operatorname{Spec}(B) \to X$.

Then

$$X : \mathsf{Rng}/\mathbb{K} \to \mathsf{Set}$$

is a functor, called the functor of points of the affine scheme X*.*

Here is the diagram describing $X(\phi)(g)$.

$$\begin{tikzcd} \operatorname{Spec}(C) \arrow[r, "\operatorname{Spec}(\phi)"] \arrow[dr, "X(\phi)(g)"'] & \operatorname{Spec}(B) \arrow[d, "g"] \\ & X \end{tikzcd}$$

Exercise 2.4.2. Prove Prop 2.4.1.

Proposition 2.4.3. *In the situation of Prop 2.4.1 there is an isomorphism*

$$X(B) \cong \operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}(A, B)$$

of functors $\mathsf{Rng}/\mathbb{K} \to \mathsf{Set}$.

Exercise 2.4.4. Prove Prop 2.4.3.

Here are two examples.

Example 2.4.5. Let $n \in \mathbb{N}$. The *n-dimensional affine space over* $\mathbb{K}$ is the affine scheme

$$\mathbf{A}^n_{\mathbb{K}} := \operatorname{Spec}\big(\mathbb{K}[t_1, \ldots, t_n]\big),$$

where $\mathbb{K}[t_1, \ldots, t_n]$ is the polynomial ring in n variables.

Take an arbitrary $\mathbb{K}$-ring B. Then, using Prop 2.4.3 and the universal property of the polynomial ring, we have an isomorphism of sets

$$X(B) \cong \operatorname{Hom}_{\mathsf{Rng}/\mathbb{K}}\big(\mathbb{K}[t_1, \ldots, t_n], B\big) \cong B^n,$$

and it is functorial in B.

We see that the functor of points of $\mathbf{A}^n_{\mathbb{K}}$ is isomorphic to the functor $B \mapsto B^n$.

Example 2.4.6. First let us recall that for a nonzero ring B we have the group $\mathrm{GL}_n(B)$ of invertible $n \times n$ matrices with entries in B. These are that matrices $\boldsymbol{b}$ for which $\det(\boldsymbol{b})$ is an invertible element of B.

Indeed, the entries on the inverse matrix of $\boldsymbol{b}$ can be expressed using the Cramer Rule.

In particular, $\mathrm{GL}_1(B) = B^{\times}$.

Let $n \geq 1$, and for $1 \leq i, j \leq n$ let $t_{i,j}$ be a variable. These n^2 variables are viewed as a matrix $\boldsymbol{t} = [t_{i,j}]$ of size $n \times n$. Consider the polynomial ring $\mathbb{K}[\boldsymbol{t}]$ in these variables. Then the matrix $\boldsymbol{t} = [t_{i,j}]$ is an element of the noncommutative ring $\mathrm{Mat}_n(\mathbb{K}[\boldsymbol{t}])$.

The matrix $\boldsymbol{t}$ is called a *generic matrix*; but a better name for is it probably the *universal matrix*. The reason is this: Given a $\mathbb{K}$-ring B and a matrix $\boldsymbol{b} = [b_{i,j}] \in \mathrm{Mat}_n(B)$, there is a unique $\mathbb{K}$-ring homomorphism $\phi : \mathbb{K}[\boldsymbol{t}] \to B$ such that $\phi(t_{i,j}) = b_{i,j}$. Then ϕ extends to a homomorphism of NC rings

$$\phi_{\mathrm{mat}} : \mathrm{Mat}_n(\mathbb{K}[\boldsymbol{t}]) \to \mathrm{Mat}_n(B),$$

and $\phi_{\mathrm{mat}}(\boldsymbol{t}) = \boldsymbol{b}$.

The determinant of $\boldsymbol{t}$ is an element $\det(\boldsymbol{t}) \in \mathbb{K}[\boldsymbol{t}]$. Define the localized ring

$$A := \mathbb{K}[\boldsymbol{t}]_{\det(\boldsymbol{t})}.$$

The matrix $\boldsymbol{t}$, now seen as an $n \times n$ matrix with entries in A, is invertible.

Now consider the affine $\mathbb{K}$-scheme

$$\mathrm{GL}_{n,\mathbb{K}} := \mathrm{Spec}(A).$$

What is the functor of points of $\mathrm{GL}_{n,\mathbb{K}}$?

Take an arbitrary $\mathbb{K}$-ring B. Using Exa 2.4.5, for n^2 instead of n, we see that a point $\boldsymbol{b} \in \mathrm{GL}_{n,\mathbb{K}}(B)$ is an $n \times n$ matrix with entries in B, such that

$$\det(\boldsymbol{b}) = \det(\phi_{\mathrm{mat}}(\boldsymbol{t})) = \phi(\det(\boldsymbol{t})) \in B^{\times}.$$

Thus as sets we have

$$\mathrm{GL}_{n,\mathbb{K}}(B) \cong \mathrm{GL}_n(B) \tag{2.4.7}$$

and this isomorphism is functorial in B.

Exercise 2.4.8. Show that for a $\mathbb{K}$-ring homomorphism $\phi : B \to C$, the function

$$\mathrm{GL}_{n,\mathbb{K}}(\phi) : \mathrm{GL}_{n,\mathbb{K}}(B) \to \mathrm{GL}_{n,\mathbb{K}}(C)$$

is a group homomorphism.

Thus we actually have a functor

$$\mathrm{GL}_{n,\mathbb{K}} : \mathsf{Rng}/\mathbb{K} \to \mathsf{Grp}\,. \tag{2.4.9}$$

Remark 2.4.10. After we learn about fibered products of schemes, we will show that there are maps of schemes

$$\mathrm{mult} : \mathrm{GL}_{n,\mathbb{K}} \times \mathrm{GL}_{n,\mathbb{K}} \to \mathrm{GL}_{n,\mathbb{K}},$$

$$\mathrm{inv} : \mathrm{GL}_{n,\mathbb{K}} \to \mathrm{GL}_{n,\mathbb{K}}$$

and

$$\mathrm{unit} : \mathrm{Spec}(\mathbb{K}) \to \mathrm{GL}_{n,\mathbb{K}}$$

which satisfy that axioms of multiplication, inversion and unit in a group.

This fact will explain Exer 2.4.8.

An object like $\mathrm{GL}_{n,\mathbb{K}}$ is called an *affine group scheme.*

3. Schemes

3.1. **Definitions.** Recall that for a sheaf $\mathcal{M}$ on a space X, and an open set $U \subseteq X$, the restriction of $\mathcal{M}$ to U is the sheaf $\mathcal{M}|_U$ on U such that $\Gamma(V, \mathcal{M}|_U) = \Gamma(V, \mathcal{M})$ for every open set $V \subseteq U$. The stalks of $\mathcal{M}$ and $\mathcal{M}|_U$ at points $x \in U$ are the same.

If $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ and $U \subseteq X$ is an open set, then $(U, \mathcal{O}_X|_U)$ is also a locally ringed space, and the inclusion

$$g : (U, \mathcal{O}_X|_U) \to (X, \mathcal{O}_X) \tag{3.1.1}$$

is a map in $\mathsf{LRSp}/\mathbb{K}$.

Also recall that an affine scheme is an object $(X, \mathcal{O}_X) \in \mathsf{LRSp}/\mathbb{K}$ that is isomorphic to $\mathrm{Spec}(A)$ for some $A \in \mathsf{Rng}/\mathbb{K}$.

Out of laziness, I will often leave the base ring $\mathbb{K}$ implicit here.

Definition 3.1.2. Let $(X, \mathcal{O}_X)$ be a LR space and let $U \subseteq X$ be an open subset. The LR space $(U, \mathcal{O}_X|_U)$ is called an *open LR subspace* of $(X, \mathcal{O}_X)$.

Example 3.1.3. Let A be a ring and $s \in A$. Define $(X, \mathcal{O}_X) := \mathrm{Spec}(A)$ and the principal open set $U := \mathrm{NZer}_X(s) \subseteq X$. We also have the affine scheme $\mathrm{Spec}(A_s)$, where A_s is the localized ring. Let $\lambda : A \to A_s$ be the canonical ring homomorphism.

Using Thm 2.3.12 we see that there is a unique isomorphism h in LRSp making the next diagram commutative.

$$\begin{tikzcd}
\mathrm{Spec}(A_s) \arrow[r, dashed, "h", "\simeq"'] \arrow[d, "\mathrm{Spec}(\lambda)"'] & (U, \mathcal{O}_X|_U) \arrow[d, "g"] \\
\mathrm{Spec}(A) \arrow[r, "="] & (X, \mathcal{O}_X)
\end{tikzcd}$$

Here g is the inclusion (3.1.1).

Definition 3.1.4. A $\mathbb{K}$-*scheme* is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$ satisfying this condition: There is an open covering $X = \bigcup_{i \in I} U_i$, where for each $i \in I$ the open subspace $(U_i, \mathcal{O}_X|_{U_i})$ of $(X, \mathcal{O}_X)$ is an affine $\mathbb{K}$-scheme.

Explanation: This means that for every index i there is an isomorphism $(U_i, \mathcal{O}_X|_{U_i}) \cong \mathrm{Spec}(A_i)$ in $\mathsf{LRSp}/\mathbb{K}$, for some $\mathbb{K}$-ring A_i.

Definition 3.1.5. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be $\mathbb{K}$-schemes. A map of $\mathbb{K}$-schemes

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is, by definition, a maps of LR $\mathbb{K}$-spaces. Thus the category $\mathsf{Sch}/\mathbb{K}$ of $\mathbb{K}$-schemes is the full subcategory of $\mathsf{LRSp}/\mathbb{K}$ on the $\mathbb{K}$-schemes.

End of Lecture 5

Lecture 6, 28 April 2021

We are continuing with our study of basic properties of schemes.

Recall that a $\mathbb{K}$-*scheme* is a locally ringed $\mathbb{K}$-space $(X, \mathcal{O}_X)$ satisfying this condition:

- There is an open covering $X = \bigcup_{i \in I} U_i$, where for each $i \in I$ the open subspace $(U_i, \mathcal{O}_X|_{U_i})$ of $(X, \mathcal{O}_X)$ is isomorphic, in $\mathsf{LRSp}/\mathbb{K}$, to $\mathrm{Spec}(A_i)$ for some $\mathbb{K}$-ring A_i.

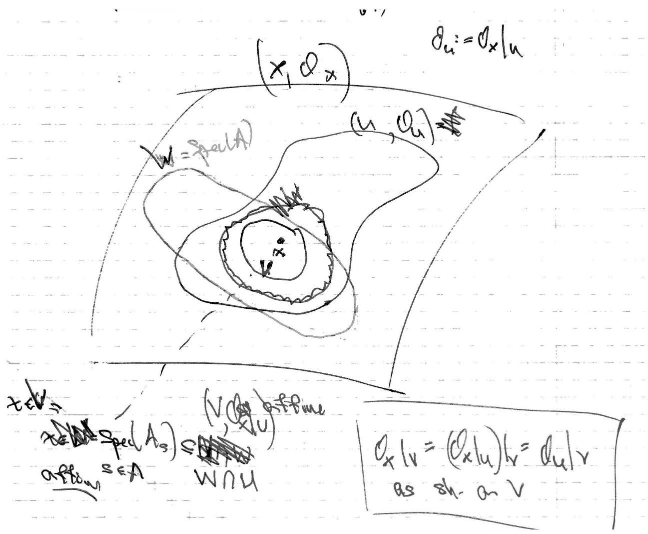
A map of schemes is, by definition, a map of LR $\mathbb{K}$-spaces. Thus $\mathsf{Sch}/\mathbb{K}$ is a full subcategory of $\mathsf{LRSp}/\mathbb{K}$.

Proposition 3.1.6. *Let $(X, \mathcal{O}_X)$ be a $\mathbb{K}$-scheme and let $U \subseteq X$ be an open set. Then the open subspace $(U, \mathcal{O}_X|_U)$ is a $\mathbb{K}$-scheme.*

[**comment**: (210429 AY) The proof, Definition 3.1.8 and Exa 3.1.10, were not done in class. Please read!]

Proof. Let's write $\mathcal{O}_U := \mathcal{O}_X|_U$. So we need to prove that $(U, \mathcal{O}_U)$ is a $\mathbb{K}$-scheme. This means that for every point $x \in U$ we need to find an open neighborhood V of x in U such that the open subspace $(V, \mathcal{O}_U|_V)$ is an affine $\mathbb{K}$-scheme. See picture (3.1).

(3.1.7)



We know (by Definition 3.1.4) that there is an open neighborhood W of x in X such that the open subspace $(W, \mathcal{O}_X|_W)$ is an affine $\mathbb{K}$-scheme; say $(W, \mathcal{O}_X|_W) = \mathrm{Spec}(A)$. Because $U \cap W$ is open in W, we can find an element $s \in A$ such that $x \in \mathrm{NZer}_X(s) \subseteq U \cap W$. Write $V := \mathrm{NZer}_X(s)$. Now $(V, \mathcal{O}_U|_V) = (V, \mathcal{O}_X|_V)$, and by Thm Thm 2.3.12 (see also Exa 3.1.3) we know that $(V, \mathcal{O}_X|_V) \cong \mathrm{Spec}(A_s)$ as LR spaces. □

This proposition makes the next definition sensible.

Definition 3.1.8. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) An *open subscheme* of $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}_U)$ such that $U \subseteq X$ is an open subset, and $\mathcal{O}_U = \mathcal{O}_X|_U$.

(2) An *affine open subscheme* of $(X, \mathcal{O}_X)$ is an open subscheme $(U, \mathcal{O}_U)$ which is an affine scheme, i.e. $(U, \mathcal{O}_U) \cong \mathrm{Spec}(A)$ for some ring A.

(3) An *affine open set* of $(X, \mathcal{O}_X)$ is an open subset $U \subseteq X$ such that the open subscheme $(U, \mathcal{O}_X|_U)$ is an affine scheme.

In item (2) above the ring A is $\Gamma(U, \mathcal{O}_U)$, by Corollary 2.2.21.

[**comment**: (210429 AY) The next def was not in the lecture]

Definition 3.1.9. Let

$$(f, \phi) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$$

be a map of LR spaces. If (f, ϕ) factors through an isomorphism

$$(f', \phi') : (U, \mathcal{O}_U) \xrightarrow{\simeq} (U', \mathcal{O}_{U'}),$$

where $(U', \mathcal{O}_{U'})$ is an open subspace of $(X, \mathcal{O}_X)$, then (f, ϕ) is called an *open embedding of LR spaces*.

Here is an example of a scheme which is not affine.

[**comment**: (210504 AY) Changes next Exa]

Example 3.1.10. The open subscheme $(U, \mathcal{O}_U) := (U, \mathcal{O}_X|_U)$ of $(X, \mathcal{O}_X) := \mathbf{A}^2_{\mathbb{K}}$ from Exer 2.2.24 is not affine.

This is because the inclusion map

$$(g, \psi) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$$

is not an isomorphism, yet the ring homomorphism

$$\Gamma(X, \psi) : \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$$

is an isomorphism.

Now use Cor 2.3.26.

Or you can use Thm 2.3.12 directly, as follows. If the LR space $(U, \mathcal{O}_U)$ were an affine scheme, then the theorem would give a map of LR spaces $(h, \theta) : (X, \mathcal{O}_X) \to (U, \mathcal{O}_U)$ s.t. $\Gamma(U, \theta)$ is the inverse of the ring isom $\Gamma(X, \psi)$. By uniqueness, there would be equality

$$(h, \theta) \circ (g, \psi) = \mathrm{Id}_{(U, \mathcal{O}_U)} .$$

This is impossible because $g : U \to X$ is not bijective.

The feature above is true for the complement of the origin in $\mathbf{A}^n_{\mathbb{K}}$, for every $n \geq 2$ and field $\mathbb{K}$.

3.2. **Closed Subschemes.**

The material in this subsection was improvised during the lecture.

[**comment**: The typed version below is more concise than the live improvised lecture, and corrected. Still it is too much material for one lecture!!

Do not try to solve all exercises. It will be too much work.

Try to solve Exer 3.2.11, but if you get stuck then ask me or Mattia.

Please read all the material below carefully and verify the proofs that I did provide. Also report all errors you find!

Notify me by email no later than Tuesday if you want me to talk about some of the material next Wednesday in Lect 7. Otherwise I will proceed with new material...]

Here is a general concept of *closed subspaces*, for LR $\mathbb{K}$-spaces, i.e. in $\mathsf{LRSp}/\mathbb{K}$.

We start with a more general setup.

Definition 3.2.1. Let $(X, \mathcal{O}_X)$ be a $\mathbb{K}$-ringed space.

(1) An $\mathcal{O}_X$*-ring* is a sheaf $\mathcal{A}$ of $\mathbb{K}$-rings on X, equipped with a homomorphism $\phi_{\mathcal{A}} : \mathcal{O}_X \to \mathcal{A}$ of sheaves of $\mathbb{K}$-rings, called the structure homomorphism.
(2) Suppose $\mathcal{A}$ and $\mathcal{B}$ are of $\mathcal{O}_X$-rings. A *homomorphism of* $\mathcal{O}_X$*-rings* is a homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ of sheaves of $\mathbb{K}$-rings, such that $\psi \circ \phi_{\mathcal{A}} = \phi_{\mathcal{B}}$.
(3) The category of $\mathcal{O}_X$-rings is denoted by $\mathsf{Rng}/\mathcal{O}_X$.

For the sake of having less cluttered notation, we mostly leave the base ring $\mathbb{K}$ implicit in the future; thus LRSp will mean $\mathsf{LRSp}/\mathbb{K}$, and "LR space" will mean "locally ringed $\mathbb{K}$-space". Likewise RSp will mean $\mathsf{RSp}/\mathbb{K}$, and "ringed space" will mean "ringed $\mathbb{K}$-space".

Example 3.2.2. When $\mathcal{O}_X = \mathbb{K}_X$, then $\mathsf{Rng}/\mathcal{O}_X$ is the category of sheaves of $\mathbb{K}$-rings on X as defined before.

When X is a single point then $\mathsf{Rng}/\mathcal{O}_X$ is the same as Rng/A, where $A := \Gamma(X, \mathcal{O}_X)$.

Definition 3.2.3. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) A *quotient ring of* $\mathcal{O}_X$ is an $\mathcal{O}_X$-ring $\mathcal{A}$, such that the structure homomorphism $\phi_{\mathcal{A}} : \mathcal{O}_X \to \mathcal{A}$ is surjective.
(2) Suppose $\mathcal{A}$ and $\mathcal{B}$ are quotients of $\mathcal{O}_X$. A *homomorphism of quotients* is a homomorphism of $\mathcal{O}_X$-rings.

Thus the quotients of $\mathcal{O}_X$ form a full subcategory of $\mathsf{Rng}/\mathcal{O}_X$.

Definition 3.2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. An *ideal sheaf in* $\mathcal{O}_X$ is a subsheaf $\mathcal{I} \subseteq \mathcal{O}_X$, which is an $\mathcal{O}_X$-submodule.

Proposition 3.2.5. *Let* $(X, \mathcal{O}_X)$ *be a ringed space.*

(1) *Given a quotient ring* $\mathcal{A}$ *of* $\mathcal{O}_X$*, the sheaf* $\mathcal{I} := \operatorname{Ker}(\phi_{\mathcal{A}})$ *is an ideal sheaf in* $\mathcal{O}_X$*.*
(2) *Given an ideal sheaf* $\mathcal{I} \subseteq \mathcal{O}_X$*, the quotient* $\mathcal{O}_X$*-module* $\mathcal{A} := \mathcal{O}_X/\mathcal{I}$*, with its canonical* $\mathcal{O}_X$*-module homomorphism* $\phi_{\mathcal{A}} : \mathcal{O}_X \to \mathcal{A}$*, is a quotient ring of* $\mathcal{O}_X$*.*
(3) *Suppose* $\psi : \mathcal{A} \to \mathcal{B}$ *is a homomorphism of quotient rings of* $\mathcal{O}_X$*, and let* $\mathcal{I} := \operatorname{Ker}(\phi_{\mathcal{A}})$ *and* $\mathcal{J} := \operatorname{Ker}(\phi_{\mathcal{B}})$*. Then* $\mathcal{I} \subseteq \mathcal{J}$*. Moreover,* $\mathcal{I} = \mathcal{J}$ *iff* ψ *is an isomorphism.*

Exercise 3.2.6. Prove Prop 3.2.5. Also write the commutative diagram in $\mathsf{Mod}(\mathcal{O}_X)$ with exact rows that corresponds to item (3)..

[**comment**: (210504 AY) new warning next]

Warning: The quotient sheaf $\mathcal{A} := \mathcal{O}_X/\mathcal{I}$ is the sheaf of rings associated to the presheaf of rings $U \mapsto \Gamma(U, \mathcal{O}_X) \,/\, \Gamma(U, \mathcal{I})$.

Definition 3.2.7. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The *support of* $\mathcal{M}$ is the subset

$$\operatorname{Supp}_X(\mathcal{M}) := \{x \in X \mid \mathcal{M}_x \neq 0\} \subseteq X.$$

Note that $\mathcal{M}_x$ is an $\mathcal{O}_{X,x}$-module.

Definition 3.2.8. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{M}$ an $\mathcal{O}_X$-module.

(1) Let $U \subseteq X$ be an open set and $m_1, \dots, m_r \in \Gamma(U, \mathcal{M})$. We say that the sequence $(m_1, \dots, m_r)$ *generates* $\mathcal{M}|_U$ *as an* $\mathcal{O}_X|_U$*-module* if the homomorphism of $\mathcal{O}_X$-modules

$$\mu : (\mathcal{O}_X|_U)^{\oplus r} \to \mathcal{M}|_U, \quad \mu(a_1, \dots, a_r) := (a_1 \cdot m_1, \dots, a_r \cdot m_r),$$

is surjective.

(2) Let $U \subseteq X$ be an open subset. If there exists a finite sequence of elements that generates $\mathcal{M}|_U$, then we call $\mathcal{M}|_U$ a *finitely generated* $\mathcal{O}_X|_U$*-module.*

(3) We say that $\mathcal{M}$ is a *locally finitely generated* $\mathcal{O}_X$*-module* if there is an open covering $X = \bigcup_{i \in I} U_i$, such that $\mathcal{M}|_{U_i}$ is a finitely generated $\mathcal{O}_X|_{U_i}$-module for every $i \in I$.

Example 3.2.9. Suppose $\mathcal{A}$ is a quotient sheaf of rings of $\mathcal{O}_X$. Then $\mathcal{A}$ is generated as an $\mathcal{O}_X$-module by the global section $1 \in \Gamma(X, \mathcal{A})$. Hence $\mathcal{A}$ is a finitely generated $\mathcal{O}_X$-module.

The support is very often a *closed* subset of X, when dealing with *locally ringed spaces.*

Theorem 3.2.10. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathcal{M}$ be a locally finitely generated $\mathcal{O}_X$-module. Then* $\operatorname{Supp}_X(\mathcal{M})$ *is a closed subset of X.*

Exercise 3.2.11. (Hard) Prove Thm 3.2.10.

Hints:

(1) A subset $Y \subseteq X$ is closed iff for every point $x \in X$ there is an open neighborhood $U \subseteq X$ of x such that $Y \cap U$ is closed in U.

(2) Take $U \subseteq X$ open and $m \in \Gamma(U, \mathcal{M})$. For every point $x \in U$ let $m_x := \operatorname{rest}_{x/U}(m) \in \mathcal{M}_x$. Define

$$\operatorname{Supp}_U(m) := \{x \in U \mid m_x \neq 0\} \subseteq U.$$

Show that $\operatorname{Supp}_U(m)$ is closed in U.

(3) Take $U \subseteq X$ open such that $\mathcal{M}|_U$ a finitely generated $\mathcal{O}_X|_U$-module, with sequence of generators $(m_1, \dots, m_r)$. Then

$$\operatorname{Supp}_X(\mathcal{M}) \cap U = \bigcup_{i=1,\dots,r} \operatorname{Supp}_U(m).$$

The next exercise should tell us to avoid a common confusion between supports and zero loci.

[**comment**: (210504 AY) new text next]

Let $\mathcal{M}$ be a sheaf on a topological space X. By "local section m of $\mathcal{M}$" we mean an element $m \in \Gamma(U, \mathcal{M})$ for some open set $U \subseteq X$.

Proposition 3.2.12. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, let $s \in \Gamma(X, \mathcal{O}_X)$, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal sheaf generated by s, namely $\mathcal{I} = \operatorname{Im}(\mu)$, where $\mu : \mathcal{O}_X \to \mathcal{O}_X$ is $\mu(a) := a \cdot s$. Let $\mathcal{B} := \mathcal{O}_X / \mathcal{I}$, the quotient ring. Then*

$$\operatorname{Supp}_X(\mathcal{B}) = \operatorname{Zer}_X(s)$$

Exercise 3.2.13. Prove Prop 3.2.12.

Exercise 3.2.14. Find a LR space $(X, \mathcal{O}_X)$ and an $\mathcal{O}_X$-module $\mathcal{M}$ whose support in not closed in X. (It might be is easier to do this for schemes.)

[**comment**: (210504 AY) small change next def]

Definition 3.2.15. Let $(X, \mathcal{O}_X)$ be a locally ringed space. A *closed LR subspace of* $(X, \mathcal{O}_X)$ is a locally ringed space $(Y, \mathcal{O}_Y)$, such that these two conditions hold:

(i) Y is a closed subset of X, with the induced subspace topology.
(ii) $\mathcal{O}_Y$ is a quotient ring of $\mathcal{O}_X$.

Proposition 3.2.16. *Let $(X, \mathcal{O}_X)$ be a LR space, and let $(Y, \mathcal{O}_Y)$ be a closed subspace of $(X, \mathcal{O}_X)$. Denote the inclusion map of topological spaces by $f : Y \to X$, and the surjection of sheaves of rings by $\phi : \mathcal{O}_X \to \mathcal{O}_Y$. Then*

$$(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

is a map of LR spaces.

Exercise 3.2.17. Prove Prop 3.2.16.

Definition 3.2.18. Let

$$(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

be a map of LR spaces. If (f, ϕ) factors through an isomorphism

$$(f', \phi') : (Y, \mathcal{O}_Y) \xrightarrow{\simeq} (Y', \mathcal{O}_{Y'})$$

where $(Y', \mathcal{O}_{Y'})$ is a closed subspace of $(X, \mathcal{O}_X)$, then (f, ϕ) is called a *closed embedding of LR spaces.*

[**comment**: (210511 AY) small change today in thm]

Theorem 3.2.19. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$. Define the sheaf of rings $\mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}$ and the subset $Y := \operatorname{Supp}_X(\mathcal{O}_Y)$. Then $(Y, \mathcal{O}_Y)$ is a LR closed subspace of $(X, \mathcal{O}_X)$.*

The proof is in Lect 8.

Definition 3.2.20.

(1) A ring A is called *reduced* if the only nilpotent element in it is 0.
(2) A ringed space $(X, \mathcal{O}_X)$ is called *reduced* if for every open subset $U \subseteq X$ the ring $\Gamma(U, \mathcal{O}_X)$ is reduced.

Proposition 3.2.21. *Let $(X, \mathcal{O}_X)$ be a ringed space. TFAE:*

(i) *$(X, \mathcal{O}_X)$ be a reduced ringed space (Def 3.2.20).*
(ii) *For every point $x \in X$ the ring $\mathcal{O}_{X,x}$ is reduced.*

Exercise 3.2.22. Prove Prop 3.2.21.

[**comment**: (210504 AY) changes next exer]

Exercise 3.2.23. (Maybe hard) Let $(X, \mathcal{O}_X)$ be a locally ringed space, let $Y \subseteq X$ be a closed subset, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be the subpresheaf defined by

$$\Gamma(U, \mathcal{I}) := \{a \in \Gamma(U, \mathcal{O}_X) \mid a(y) = 0 \text{ for all } y \in Y \cap U\}.$$

Here $a(y)$ is the residue class of a in the residue field $\boldsymbol{k}(y)$.

(1) Show that $\mathcal{I}$ is an ideal sheaf in $\mathcal{O}_X$.
(2) Define the sheaf of rings $\mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}$. Show that $\operatorname{Supp}_X(\mathcal{O}_Y) = Y$.
(3) Conclude that $(Y, \mathcal{O}_Y)$ is a closed LR subspace of $(X, \mathcal{O}_X)$.
(4) Prove that $(Y, \mathcal{O}_Y)$ is a reduced LR space.
(5) Prove that every closed subspace $(Y', \mathcal{O}_{Y'})$ of $(X, \mathcal{O}_X)$ with $Y' = Y$ contains $(Y, \mathcal{O}_Y)$ as a closed subspace.

The scheme $(Y, \mathcal{O}_Y)$ is called the *reduced closed subscheme of* $(X, \mathcal{O}_X)$ *supported in* Y.

[**comment**: (210504 AY) The next two examples are repeated with more details in next lecture.]

Example 3.2.24. Here is a way to get many closed subspaces of a LR space $(X, \mathcal{O}_X)$ with the same underlying closed subset Y.

We start with some given closed subspace $(Y_0, \mathcal{O}_{Y_0})$ of $(X, \mathcal{O}_X)$ having $Y_0 = Y$; e.g. we can take the reduced closed subscheme from Exer 3.2.23. Define the ideal sheaf

$$\mathcal{I}_0 := \mathrm{Ker}(\phi_{\mathcal{O}_{Y_0}}) \subseteq \mathcal{O}_X$$

where $\phi_{\mathcal{O}_{Y_0}} : \mathcal{O}_X \to \mathcal{O}_{Y_0}$ is the given ring surjections.

For every $n \geq 0$ let $\mathcal{I}_n$ be the ideal $\mathcal{I}_n := (\mathcal{I}_0)^{n+1}$. By this we mean the sheafification of the presheaf sending U to the $n+1$ power of the ideal $\Gamma(U, \mathcal{I}_0)$ in the ring $\Gamma(U, \mathcal{O}_X)$.

We then define the closed subspace $(Y_n, \mathcal{O}_{Y_n})$ of $(X, \mathcal{O}_X)$ using Thm 3.2.19 with the ideal sheaf $\mathcal{I}_n$.

The closed subspace $(Y_n, \mathcal{O}_{Y_n})$ is called the *n-th infinitesimal thickening of* $(Y_0, \mathcal{O}_{Y_0})$. It will usually not be reduced.

Exercise 3.2.25. Work out the details of the previous example for the scheme $(X, \mathcal{O}_X) := \mathbf{A}^1_{\mathbb{K}} = \mathrm{Spec}(\mathbb{K}[t])$, where $\mathbb{K}$ is a field, $y \in X$ is the origin (i.e. the point satisfying $t(y) = 0$), and $(Y_0, \mathcal{O}_{Y_0})$ is the reduced closed subspace supported on $Y := \{y\} = \mathrm{Zer}_X(t)$.

Verify that $(Y_n, \mathcal{O}_{Y_n})$ are all affine schemes, and find the rings $\Gamma(Y_n, \mathcal{O}_{Y_n})$.

$\boxed{\text{End of Lecture 6}}$

Lecture 7, 5 May 2021

I have decided to continue speaking about closed LR subspaces this week, emphasizing some of the ideas from last week, introducing closed subschemes, and doing some examples.

[**comment**: (210504 AY) There are several new corrections in the notes of Lect 6. Make sure to read the notes of Lect 6 again before the class tomorrow. (There is no need to read the prenotes for Lect 7.)]

First: Did everybody read and verify the proof of Thm 2.3.12?

Second: There is an improvement to Exa 3.1.10.

Third: Did everybody solve Exer 3.2.11, namely to prove Thm 3.2.10? If not then please work on it with Mattia. It is important.

◇ ◇ ◇

3.3. **Closed Topological Subspaces.** Definition 3.2.15 was a bit ambiguous – how can we consider $\mathcal{O}_Y$ as a sheaf on X? In this subsection we explain this feature of closed embeddings, in the general context of topological spaces.

[**comment**: (210506 AY) The notes below are written not in the order of the live lecture, but in a better order.]

The next definition clarifies things that are (probably unintentionally) obscured in topology classes.

Definition 3.3.1. Let X be a topological space.

(1) A *topological subspace* of X is a topological space Y such that Y is a subset of X, and the topology of Y is the induced subspace topology from X.
(2) A map $f : Y \to X$ of topological spaces is an *embedding of topological spaces* if, letting $Y' := f(Y) \subseteq X$ be the image of f with the induced subspace topology, the map $f : Y \to Y'$ is a homomorphism.
(3) A map $f : Y \to X$ of topological spaces is called a *closed* (resp. *open*) *embedding of topological spaces* if it is an embedding as defined above, and $f(Y)$ is closed (resp. open) in X.

Example 3.3.2. Let X be a topological space. If $Y \subseteq X$ is a closed topological subspace, then the inclusion $f : Y \to X$ is a closed embedding.

Now to sheaves.

Definition 3.3.3. Let X be a topological space and $Y \subseteq X$ a closed subspace.

(1) We say that a sheaf $\mathcal{M} \in \mathsf{Ab}(X)$ is *supported in* Y if $\operatorname{Supp}_X(\mathcal{M}) \subseteq Y$.
(2) The full subcategory of $\mathsf{Ab}(X)$ on the sheaves supported in Y is denoted by $\mathsf{Ab}_Y(X)$.

Let $f : Y \to X$ be a map of topological spaces. Given $\mathcal{N} \in \mathsf{Ab}(Y)$, there is a homomorphism

$$\alpha_{\mathcal{N}} : f^{-1}(f_*(\mathcal{N})) \to \mathcal{N} \tag{3.3.4}$$

in $\mathsf{Ab}(Y)$, which corresponds by adjunction to $\mathrm{id}_{f_*(\mathcal{N})}$; see [Ye4, Prop 4.3.17], based on [Ye4, Thm 4.3.9]. (There is a typo in that proposition, $f_*(\mathcal{M})$ should be $f_*(\mathcal{N})$.) We call it the *adjunction homomorphism of* $\mathcal{N}$.

Likewise, given $\mathcal{M} \in \mathsf{Ab}(X)$, there is a homomorphism

$$\beta_{\mathcal{M}} : \mathcal{M} \to f_*(f^{-1}(\mathcal{M})) \tag{3.3.5}$$

$\mathsf{Ab}(X)$, which corresponds by adjunction to $\mathrm{id}_{f^{-1}(\mathcal{M})}$.

Finally, given $\mathcal{M} \in \mathsf{Ab}(X)$ and $y \in Y$, there is an isomorphism

$$\beta_{\mathcal{M},y} : \mathcal{M}_{f(y)} \xrightarrow{\simeq} (f^{-1}(\mathcal{M}))_y \tag{3.3.6}$$

in Ab, functorial in $\mathcal{M}$. This isomorphism arises from the direct limits occurring in the constructions of the stalks and of f^{-1}; see proof of [Ye4, Thm 4.3.9].

[**comment**: (210511 AY) changes from here to Cor 3.3.11 today]

Lemma 3.3.7. *Let X be a topological space and $Y \subseteq X$ a closed subspace, with closed embedding $f : Y \to X$. Let $\mathcal{N} \in \mathsf{Ab}(Y)$.*

(1) *The pushforward sheaf $f_*(\mathcal{N}) \in \mathsf{Ab}(X)$ satisfies $f_*(\mathcal{N})_x = 0$ for all $x \in X - Y$. Thus $f_*(\mathcal{N}) \in \mathsf{Ab}_Y(X)$.*

(2) *For $y \in Y$ the homomorphism*

$$\alpha_{\mathcal{N},y} : f_*(\mathcal{N})_y \to \mathcal{N}_y$$

in Ab is an isomorphism.

(3) *The functorial homomorphism*

$$\alpha_{\mathcal{N}} : f^{-1}(f_*(\mathcal{N})) \to \mathcal{N}$$

in $\mathsf{Ab}(Y)$ is an isomorphism.

Lemma 3.3.8. *Let X be a topological space and $Y \subseteq X$ a closed subspace, with closed embedding $f : Y \to X$, and let $\mathcal{M} \in \mathsf{Ab}_Y(X)$.*

(1) *For $y \in Y$ the homomorphism*

$$\beta_{\mathcal{M},y} : \mathcal{M}_y \to f_*(f^{-1}(\mathcal{M}))_y$$

in Ab is an isomor[phism.

(2) *The functorial homomorphism*

$$\beta_{\mathcal{M}} : \mathcal{M} \to f_*(f^{-1}(\mathcal{M}))$$

in $\mathsf{Ab}_Y(X)$ is an isomorphism.

Note that $f_*(f^{-1}(\mathcal{M}))$ belongs to $\mathsf{Ab}_Y(X)$ by Lem 3.3.7.

Exercise 3.3.9. Prove Lemmas 3.3.7 and 3.3.8.

Theorem 3.3.10. *Let X be a topological space and $Y \subseteq X$ a closed subspace, with closed embedding $f : Y \to X$.*

(1) *The functor*

$$f_* : \mathsf{Ab}(Y) \to \mathsf{Ab}_Y(X)$$

is an equivalence, with quasi-inverse f^{-1}.

(2) *Let $\mathcal{N} \in \mathsf{Ab}(Y)$, with pushforward $\mathcal{M} := f_*(\mathcal{N}) \in \mathsf{Ab}_Y(X)$. Then for every point $y \in Y$ the homomorphism*

$$\alpha_{\mathcal{N},y} : \mathcal{M}_y \to \mathcal{N}_y$$

in Ab is an isomorphism.

Proof. This is clear from Lemmas 3.3.7 and 3.3.8. □

Corollary 3.3.11. *In the situation of Thm 3.3.10, the functor*

$$f_* : \mathsf{Ab}(Y) \to \mathsf{Ab}(X)$$

is exact.

Proof. Immediate from Thm 3.3.10(2). □

Corollary 3.3.12. *In the situation of Thm 3.3.10, the functor*

$$f_* : \mathsf{Rng}(Y) \to \mathsf{Rng}_Y(X)$$

is an equivalence, with quasi-inverse f^{-1}.

Proof. The functors f_* and f^{-1} act on sheaves of rings, and the isomorphisms α and β in the lemmas are in $\mathsf{Rng}(Y)$ and $\mathsf{Rng}_Y(X)$, respectively. □

◇ ◇ ◇

Here is a remark on terminology.

Remark 3.3.13. It is important to compare our definitions of *LR subspace of a LR space* to those of Hartshorne in [Har, Section II.3]. Hartshorne uses the term "immersion" for what we call "embedding"; and he restricts attention to schemes only.

Our Definition of *open LR subspace* of a LR space (Def 3.1.2) coincides with that of [Har, Section II.3].

But our Definition of *closed LR subspace* of a LR space (Def 3.2.15) differs from that of [Har, Section II.3]. Hartshorne defines a closed subspace to be the isomorphism class of a closed embedding (immersion), see Def 3.2.18. The two definitions are equivalent due to Prop 3.2.16.

Here is another important remark.

Remark 3.3.14. Let $(X, \mathcal{O}_X)$ be a LR space. For every point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring, and hence it is nonzero. Therefore $\mathrm{Supp}_X(\mathcal{O}_X) = X$.

Exercise 3.3.15. Find an example of a ringed space $(X, \mathcal{O}_X)$ such that $\mathrm{Supp}_X(\mathcal{O}_X) \subsetneqq X$. (Of course this is not a LR space.)

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3.4. **Back to Closed LR Subspaces.** Here are variations of Defs 3.2.15 and 3.2.18, which are really more important for us. Cor 3.3.12 allows us to treat $\mathcal{O}_Y$ as a sheaf of rings on X.

Definition 3.4.1. Let $(X, \mathcal{O}_X)$ be a scheme. A *closed subscheme of* $(X, \mathcal{O}_X)$ is a scheme $(Y, \mathcal{O}_Y)$, such that these two conditions hold:

(i) Y is a closed topological subspace of X.
(ii) $\mathcal{O}_Y$ is a quotient sheaf of rings of $\mathcal{O}_X$.

Condition (ii) makes implict use of Cor 3.3.12.

According to Prop 3.2.16, we have a map of schemes

(3.4.2) $$(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X).$$

Here $f : Y \to X$ is the embedding of top spaces, and $\phi : \mathcal{O}_X \to \mathcal{O}_Y$ is the surjection of sheaves of rings.

Definition 3.4.3. A *closed embedding of schemes* is a closed embedding LR spaces, as in Def 3.2.18, between schemes.

Remark 3.4.4. We will prove later that if $(X, \mathcal{O}_X)$ is an affine scheme, and if $(Y, \mathcal{O}_Y)$ is a closed subscheme of it, then $(Y, \mathcal{O}_Y)$ is an affine scheme too.

This is a deep fact, and we shall prove it using *quasi-coherent sheaves.*

Next week I will give an example (maybe two) of an open embedding $g : U \to X$, which is not closed, and for which $g_* : \mathsf{Ab}(U) \to \mathsf{Ab}(X)$ is not exact.

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I still did not prove Thm 3.2.19.

[**comment**: (210506 AY) This is from Lect 6. I forgot about this – sorry! will do it next week, first thing.]

For now let's assume that Thm 3.2.19 holds, for the following example. It is Example 3.2.24 with some more details.

Example 3.4.5. We are given a LR space $(X, \mathcal{O}_X)$ and a closed subspace $(Y_0, \mathcal{O}_{Y_0})$. Perhaps both are schemes.

The closed embedding is

$$(f_0, \phi_0) : (Y_0, \mathcal{O}_{Y_0}) \to (X, \mathcal{O}_X).$$

Define the ideal sheaf

$$\mathcal{I}_0 := \mathrm{Ker}(\phi_0) \subseteq \mathcal{O}_X.$$

Thus

$$\mathcal{O}_{Y_0} = \mathcal{O}_X / \mathcal{I}_0.$$

Here is one possibilty to arrive at this situation. We take some closed subset $Y = Y_0 \subseteq X$, and define the ideal $\mathcal{I} = \mathcal{I}_0$ like is Exer 3.2.23. In this case $(Y_0, \mathcal{O}_{Y_0})$ is the reduced LR subspace supported on Y.

Anyhow, for every $n \geq 0$ let $\mathcal{I}_n$ be the ideal $\mathcal{I}_n := (\mathcal{I}_0)^{n+1}$. By this we mean the sheafification of the presheaf sending U to the $n + 1$ power of the ideal $\Gamma(U, \mathcal{I}_0)$ in the ring $\Gamma(U, \mathcal{O}_X)$.

We then define the closed subspace $(Y_n, \mathcal{O}_{Y_n})$ of $(X, \mathcal{O}_X)$ using Thm 3.2.19 with the ideal sheaf $\mathcal{I}_n$.

The closed subspace $(Y_n, \mathcal{O}_{Y_n})$ is called the *n-th infinitesimal thickening of* $(Y_0, \mathcal{O}_{Y_0})$. It will usually not be reduced. The underlying closed subset is $Y_n = Y_0$.

Since there is inclusion $\mathcal{I}_{n+1} \subseteq \mathcal{I}_n$ of of ideal sheaves, there is a corresponding homomorphism $\phi_n : \mathcal{O}_{Y_{n+1}} \to \mathcal{O}_{Y_n}$ of quotient rings of $\mathcal{O}_X$.

Writing $f_n : Y_n \to Y_{n+1}$ for the identity of this close subset, we obtain a map of LR spaces

$$(f_n, \phi_n) : (Y_n, \mathcal{O}_{Y_n}) \to (Y_{n+1}, \mathcal{O}_{Y_{n+1}}).$$

Putting all of these together we get a direct system

$$\big\{(Y_n, \mathcal{O}_{Y_n})\big\}_{n \in \mathbb{N}}$$

of closed LR subspaces of $(X, \mathcal{O}_X)$.

Does the direct limit exist?

I do not know in general. But for the nice case in Exer 3.2.25 I know the answer, and will work it out in detail next week.

[**comment**: (210506 AY) Next week I will first prove Thm 3.2.19, and then work out the solution of Exer 3.2.25 in detail (the direct limit is a formal scheme...)]

$\boxed{\text{End of Lecture 7}}$

Lecture 8, 12 May 2021

We shall start by recalling

- Definition 3.2.15 on LR subspaces.
- Definitions 3.2.7 and 3.3.3 on supports of sheaves.
- Theorem 3.3.10 and Corollary 3.3.12 on the pushforward functor f_* for a closed embedding $f : Y \to X$ of topological spaces.
- For a LR space $(Y, \mathcal{O}_Y)$ we have $\mathrm{Supp}_Y(\mathcal{O}_Y) = Y$.

I made some small changes in these results, so please read them again at home.

Now finally the proof of Thm 3.2.19, repeated here.

Theorem 3.4.6. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}_X$. Define the sheaf of rings $\mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}$ and the subspace $Y := \mathrm{Supp}_X(\mathcal{O}_Y)$. Then $(Y, \mathcal{O}_Y)$ is a LR closed subspace of $(X, \mathcal{O}_X)$.*

Proof. As an $\mathcal{O}_X$-module the sheaf $\mathcal{O}_Y$ is generated by single blobal section $1 \in \Gamma(X, \mathcal{O}_X)$. According to Theorem 3.2.10 the subset $Y := \mathrm{Supp}_X(\mathcal{O}_Y)$ is closed in X.

By Corollary 3.3.12 we may view $\mathcal{O}_Y$ as a sheaf of rings on Y.

It remains to prove that the ringed space $(Y, \mathcal{O}_Y)$ is LR. Let's denote the surjection of sheaves of rings on X by $\phi : \mathcal{O}_X \to \mathcal{O}_Y$.

Take a point $y \in Y$. The stalk $\mathcal{O}_{Y,y}$ is nonzero since $y \in \mathrm{Supp}_X(\mathcal{O}_Y)$. But we have a surjection of rings

$$\phi_y : \mathcal{O}_{X,y} \to \mathcal{O}_{Y,y}.$$

This implies (cf. next lemma) that $\mathcal{O}_{Y,y}$ is a local ring. □

Lemma 3.4.7. *Let $(A, \mathfrak{m})$ be a local ring, and let $\psi : A \to B$ be a surjective ring homomorphism. If $B \neq 0$, then B is a local ring, with maximal ideal $\mathfrak{n} := \psi(\mathfrak{m})$, $\psi : A \to B$ is a local homomorphism, and the induced homomorphism on residue fields $\bar{\psi} : A/\mathfrak{m} \to B/\mathfrak{n}$ is an isomorphism.*

Exercise 3.4.8. Prove this lemma.

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Example 3.4.9. Here we give a counterexample to Cor 3.3.11, when $g : U \to X$ is an open embedding, which is not closed, and for which $g_* : \mathsf{Ab}(U) \to \mathsf{Ab}(X)$ is not exact.

The setting is the one from Exercise 2.2.24, with a small change of notation. We take a field $\mathbb{K}$, and $A := \mathbb{K}[t_0, t_1]$, the polynomial ring in two variables. Then we take the affine scheme $(X, \mathcal{O}_X) := \mathrm{Spec}(A) = \mathbf{A}^2_{\mathbb{K}}$, the 2-dimensional affine space over $\mathbb{K}$.

Let $\mathfrak{m} = z \in X$ be the maximal ideal in A generated by the variables, let $Z = \{z\} = \mathrm{Zer}_X(\mathfrak{m})$, and let $U := X - Z$. We get an open subscheme $(U, \mathcal{O}_U)$ with $\mathcal{O}_U := \mathcal{O}_X|_U$. The open embedding is

$$(g, \psi) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X).$$

We will cook up an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0 \tag{3.4.10}$$

in $\mathsf{Mod}(\mathcal{O}_U)$ such that the sequence

$$0 \to g_*(\mathcal{M}') \to g_*(\mathcal{M}) \to g_*(\mathcal{M}'') \to 0 \tag{3.4.11}$$

in $\mathsf{Mod}(\mathcal{O}_X)$ is not exact.

We start with the *augmented Koszul complex*, in which the homomorphisms are matrix multiplications:

$$0 \to A \xrightarrow{[t_1, -t_0]} A^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} t_0 \\ t_1 \end{smallmatrix}\right]} A \to A/\mathfrak{m} \to 0. \tag{3.4.12}$$

This is an exact sequence in $\mathsf{Mod}(A)$. Of course $A/\mathfrak{m} \cong \mathbb{K}$.

We can form an exact sequence in $\mathsf{Mod}(\mathcal{O}_X)$ by sheafifying (3.4.12), and this is what we get:

$$0 \to \mathcal{O}_X \xrightarrow{[t_1, -t_0]} \mathcal{O}_X^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} t_0 \\ t_1 \end{smallmatrix}\right]} \mathcal{O}_X \to \boldsymbol{k}(z) \to 0. \tag{3.4.13}$$

Here $\boldsymbol{k}(z)$ is the sheaf supported on Z; in other words, we have the closed subscheme $(Z, \mathcal{O}_Z) \cong \operatorname{Spec}(\mathbb{K})$ of $(X, \mathcal{O}_X)$.

Next we restrict (3.4.13) to U, or in other words, apply g^{-1} to it. We get this exact sequence in $\mathsf{Mod}(\mathcal{O}_U)$:

$$0 \to \mathcal{O}_U \xrightarrow{[t_1, -t_0]} \mathcal{O}_U^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} t_0 \\ t_1 \end{smallmatrix}\right]} \mathcal{O}_U \to 0. \tag{3.4.14}$$

This the sequence (3.4.10) we want.

The proof in Exercise 2.2.24 that $\Gamma(U, \mathcal{O}_U) = A$, slightly modified, says that $g_*(\mathcal{O}_U) = \mathcal{O}_X$. Hence, applying g_* to (3.4.14) we obtain this sequence

$$0 \to \mathcal{O}_X \xrightarrow{[t_1, -t_0]} \mathcal{O}_X^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} t_0 \\ t_1 \end{smallmatrix}\right]} \mathcal{O}_X \to 0$$

in $\mathsf{Mod}(\mathcal{O}_X)$. This is not exact at the third nonzero term, i.e. at $\mathcal{O}_X = \mathcal{M}''$, because by (3.4.13) the cokernel there is nonzero.

Later when we talk about $\mathbf{P}^1_{\mathbb{K}}$ and the canonical projection

$$\pi : U = \mathbf{A}^2_{\mathbb{K}} - \{z\} \to \mathbf{P}^1_{\mathbb{K}}$$

we shall return to this example (please remind me!).

$\boxed{\text{End of Live Lecture 8}}$

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Please read the next long example on your own. If there are problems, send questions to the whole course list.

In Example 3.4.17 below we are going to solve Exercise 3.2.25 in full detail. But before that a lemma.

Lemma 3.4.15. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be LR $\mathbb{K}$-spaces, such that Y has a single point y. Given a point $x \in X$ and a local $\mathbb{K}$-ring homomorphism $\phi : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$, there is a unique map of LR $\mathbb{K}$-spaces*

$$(f, \tilde{\phi}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

such that $f(y) = x$ and $\tilde{\phi}_y = \phi$, as ring homomorphisms $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$.

Exercise 3.4.16. Prove this lemma.

Here is the example.

Example 3.4.17. $\mathbb{K}$ is a field, $A := \mathbb{K}[t]$ is the polynomial ring in one variable, and $(X, \mathcal{O}_X) := \mathbf{A}^1_{\mathbb{K}} = \operatorname{Spec}(A)$ is the affine line over $\mathbb{K}$. The point $y \in X$ is the origin, i.e. the point satisfying $t(y) = 0$, or if you wish $y = \mathfrak{m} = (t)$, the maximal ideal generated by t. And Y is the closed subset $Y := \{y\} = \operatorname{Zer}_X(t) \subseteq X$.

For every $n \in \mathbb{N}$ let

$$A_n := A/(t^{n+1}),$$

a quotient ring of A. And let

$$(Y_n, \mathcal{O}_{Y_n}) := \operatorname{Spec}(A_n).$$

Note that $Y_n = Y = \{y\}$ as sets and as topological spaces, because

$$\operatorname{Zer}_X(t^{n+1}) = \operatorname{Zer}_X(t) = \{y\}.$$

Because Y_n has a single point, we have

$$\Gamma(Y_n, \mathcal{O}_{Y_n}) = \mathcal{O}_{Y_n,y} = A_n.$$

When $n = 0$ we get a reduced scheme, but for $n \geq 1$ the scheme $\Gamma(Y_n, \mathcal{O}_{Y_n})$ is not reduced.

The surjective ring homomorphism

$$\phi_n : A \to A_n$$

gives rise to a closed embedding

$$(f_n, \tilde{\phi}_n) := \operatorname{Spec}(\phi_n) : (Y_n, \mathcal{O}_{Y_n}) \to (Y, \mathcal{O}_Y).$$

Let

$$\phi_{n+1,n} : A_{n+1} \to A_n$$

be the ring surjection. We get an inverse system of A-rings $\{A_n\}_{n\in\mathbb{N}}$.

The inverse limit in the category Rng is

$$\widehat{A} := \lim_{\leftarrow n} A_n = \mathbb{K}[[t]], \tag{3.4.18}$$

the ring of formal powers series. The ring $\mathbb{K}[[t]]$ can also be regarded as the completion of the ring $\mathbb{K}[t]$ with respect to the t-adic topology.

If you never saw the ring $\mathbb{K}[[t]]$ before, then read about adic completions in [AlKl, Chapter 22], [Eis, Chapter 7] or [Mats, pages 55 and 271].)

The ring $\widehat{A}$ is local, with maximal ideal $\widehat{\mathfrak{m}} = (t)$, and residue field $\widehat{A}/\widehat{\mathfrak{m}} = \mathbb{K}$. The homomorphism $A \to \widehat{A}$ sends $\mathfrak{m}$ to $\widehat{\mathfrak{m}}$, the induced homomorphism on residue fields $\widehat{A}/\widehat{\mathfrak{m}} \to A/\mathfrak{m}$ is bijective, and hence $A \to \widehat{A}$ factors through the local ring $A_{\mathfrak{m}} = \mathcal{O}_{X,y}$.

For every n there is a surjective local ring homomorphism

$$\widehat{\phi}_n : \widehat{A} \to A_n \cong \widehat{A}/\widehat{\mathfrak{m}}^{n+1}.$$

Look at the direct system of affine schemes

$$\big\{(Y_n, \mathcal{O}_{Y_n})\big\}_{n\in\mathbb{N}},$$

with transitions

$$\operatorname{Spec}(\phi_{n+1,n}) : (Y_n, \mathcal{O}_{Y_n}) \to (Y_{n+1}, \mathcal{O}_{Y_{n+1}}). \tag{3.4.19}$$

Does it have a direct limit in $\mathsf{LRSp}/\mathbb{K}$? If so, what is it? Is it a scheme?

The answer is this: define the locally ringed space $(Y, \widehat{\mathcal{O}}_Y)$, which again has only one point y, and

$$\Gamma(Y, \widehat{\mathcal{O}}_Y) = \widehat{\mathcal{O}}_{Y,y} = \widehat{A}.$$

According to Lemma 3.4.15, the ring homomorphisms

$$A \to \widehat{A} \to A_n$$

for $n \in \mathbb{N}$ give rise to maps

$$(Y_n, \mathcal{O}_{Y_n}) \to (Y, \widehat{\mathcal{O}}_Y) \to (Y, \mathcal{O}_Y)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Going to the limit in n we get a map

$$(g, \psi) : \lim_{n\to} (Y_n, \mathcal{O}_{Y_n}) \to (Y, \widehat{\mathcal{O}}_Y).$$

in $\mathsf{LRSp}/\mathbb{K}$.

Let us prove that (g, ψ) is an isomorphism. Take an arbitrary $(Z, \mathcal{O}_Z)$ in $\mathsf{LRSp}/\mathbb{K}$, with maps

$$(h_n, \theta_n) : (Y_n, \mathcal{O}_{Y_n}) \to (Z, \mathcal{O}_Z)$$

that respect the transitions (3.4.19).

Let $z := h_n(y) \in Z$; it is the same point for all n. For every n there is a local ring homomorphism

$$\theta_{n,y} : \mathcal{O}_{Z,z} \to \mathcal{O}_{Y_n,y} = A_n.$$

In the limit we get a unique local ring homomorphism

$$\lim_{\leftarrow n} \theta_{n,y} : \mathcal{O}_{Z,z} \to \widehat{\mathcal{O}}_{Y,y} = \widehat{A}. \tag{3.4.20}$$

By Lemma 3.4.15 there is a unique map

$$\widehat{\theta} : (Y, \widehat{\mathcal{O}}_Y) \to (Z, \mathcal{O}_Z)$$

in $\mathsf{LRSp}/\mathbb{K}$ that coincudes with (3.4.20) on stalks.

There is also a map

$$(Y, \widehat{\mathcal{O}}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$, but it is not a closed embedding, since $A \to \widehat{A}$ is not surjective.

The LR space $(Y, \widehat{\mathcal{O}}_Y)$ is not a scheme. If it had been a scheme, then, having only one point, it whould have to be an affine scheme. Yet $\widehat{A}$ is an integral domain (it is in fact a DVR), so $\operatorname{Spec}(\widehat{A})$ has two points: the maximal ideal $\widehat{\mathfrak{m}} = (t)$ and the prime ideal $\mathfrak{p} = (0)$.

The LR space $(Y, \widehat{\mathcal{O}}_Y)$ is a *formal scheme*, see [Har, Section II.9].

$\boxed{\text{End of Lecture 8}}$

Lecture 9, 19 May 2021

3.5. **Gluing Schemes.** In the first course we saw how to glue topological spaces (this was not really new for us), and how to glue sheaves on topological spaces (this was new).

The next theorem tells us how to glue ringed spaces, and the important special case will be that of gluing schemes.

Theorem 3.5.1 (Gluing Locally Ringed Spaces). *Let $\big\{(X_i, \mathcal{O}_{X_i})\big\}_{i \in I}$ be a collection of objects of* $\mathsf{LRSp}/\mathbb{K}$ *indexed by an ordered set I.*

We are given these data:

- *For every $i < j$ in I there are open subsets $U_{i,j} \subseteq X_i$ and $U_{j,i} \subseteq X_j$.*
- *For every $i < j$ in I there is an isomorphism*

$$(g_{i,j}, \psi_{i,j}) : (U_{i,j}, \mathcal{O}_{X_i}|_{U_{i,j}}) \xrightarrow{\simeq} (U_{j,i}, \mathcal{O}_{X_j}|_{U_{j,i}})$$

in $\mathsf{LRSp}/\mathbb{K}$.

The condition is this:

(C) *For every $i < j < k$ in I there is equality*

$$(g_{j,k}, \psi_{j,k}) \circ (g_{i,j}, \psi_{i,j}) = (g_{i,k}, \psi_{i,k})$$

of isomorphisms

$$\big(U_{i,j} \cap U_{i,k}, \mathcal{O}_{X_i}|_{U_{i,j} \cap U_{i,k}}\big) \xrightarrow{\simeq} \big(U_{k,i} \cap U_{k,j}, \mathcal{O}_{X_k}|_{U_{k,i} \cap U_{k,j}}\big).$$

Then there is an object $(X, \mathcal{O}_X)$ in $\mathsf{LRSp}/\mathbb{K}$*, with an open covering $X = \bigcup_{i \in I} U_i$, and with an isomorphism*

$$(g_i, \psi_i) : (X_i, \mathcal{O}_{X_i}) \xrightarrow{\simeq} (U_i, \mathcal{O}_X|_{U_i})$$

for every $i \in I$, satisfying

$$(g_i, \psi_i) = (g_j, \psi_j) \circ (g_{i,j}, \psi_{i,j})$$

for every $i < j$, as isomorphisms

$$\big(U_{i,j}, \mathcal{O}_{X_i}|_{U_{i,j}}\big) \xrightarrow{\simeq} \big(U_i \cap U_j, \mathcal{O}_X|_{U_i \cap U_j}\big).$$

Moreover, the object $(X, \mathcal{O}_X)$, with its collection of isomorphisms $\{(g_i, \psi_i)\}_{i \in I}$, is unique up to a unique isomorphism.

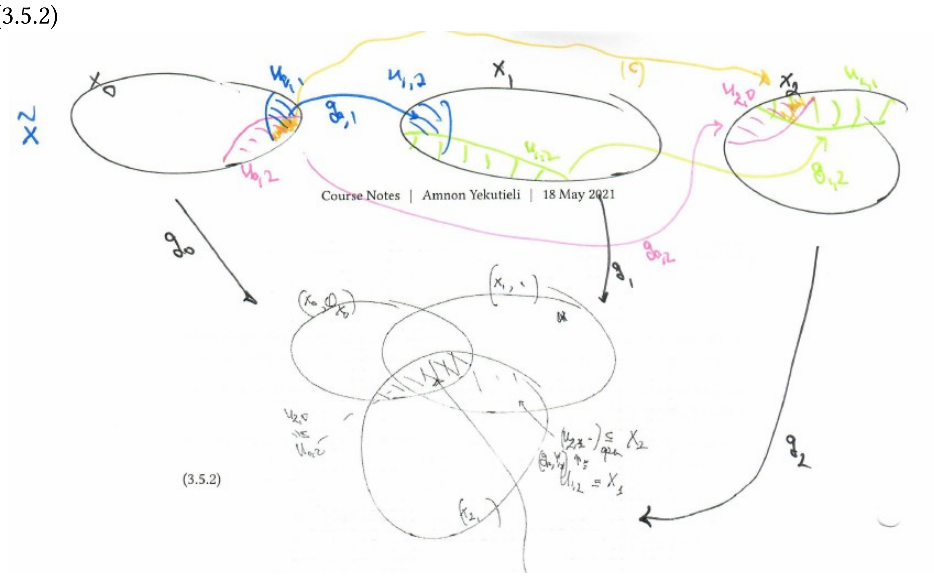
See picture (3.5.2) for an illustration, with $I = \{0, 1, 2\}$. The condition (C) is called the 2*-cocycle condition.*

Remark 3.5.3. The version of the theorem I gave is the *reduced* version, with an ordered indexing set. It is possible to take I to be just set (not ordered), but then extra conditions are required.

The "correct" way to state things is in *simplicial language.* You can look this up on the internet. Or you can take a look here:
`https://www.math.bgu.ac.il/~amyekut/lectures/higher-descent/notes.pdf`.

The theorem is [Har, Exer II.2.12], but it is stated incorectly (in my copy of the book).

(3.5.2)



Proof.

Step 1. Here we construct the set X.

Let

$$\tilde{X} := \coprod_{i \in I} X_i,$$

the disjoint union.

There is a relation $\sim$ on the set $\tilde{X}$, which is the closure under reflexivity, transitivity and symmetry of the generating relations $x_i \sim x_j$ for all $i < j$, $x_i \in U_{i,j} \subseteq X_i$ and $x_j := g_{i,j}(x_i) \in U_{j,i} \subseteq X_j$.

This is an *equivalence relation*, and we define the quotient set

$$X := \tilde{X}/{\sim} \tag{3.5.4}$$

with the projection map

$$p : \tilde{X} \twoheadrightarrow X. \tag{3.5.5}$$

For every i the projection

$$g_i := p|_{X_i} : X_i \to X \tag{3.5.6}$$

is injective.

Letting

$$U_i := g_i(X_i) \subseteq X \tag{3.5.7}$$

we get a covering

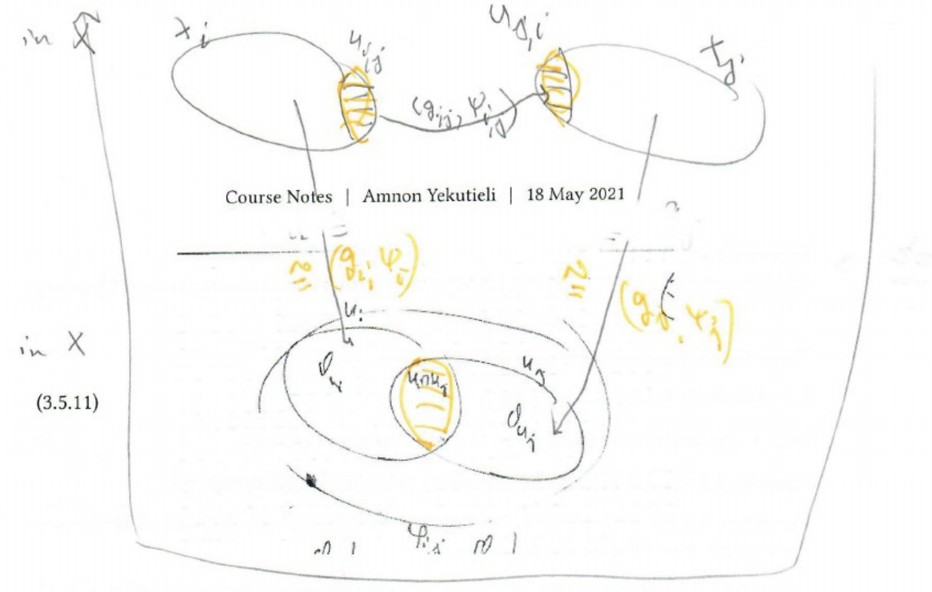
$$X = \bigcup_{i \in I} U_i. \tag{3.5.8}$$

and a bijection

$$g_i : X_i \xrightarrow{\simeq} U_i. \tag{3.5.9}$$

Verifying all these assertions is an exercise (∗).

(3.5.12)



Step 2. Here we construct the topological space X.

The disjoint union $\tilde{X}$ has a unique topology on it such that $X_i \subseteq \tilde{X}$ is an open subspace.

Consider the canonical projection $p : \tilde{X} \to X$, and put on X the quotient topology.

For every i the inclusion

$$g_i = p|_{X_i} : X_i \to X$$

is an open embedding of topological spaces. This is an exercise ($*$).

Step 3. Here we construct the sheaf of rings $\mathcal{O}_X$.

For every index i define $\mathcal{O}_{U_i}$ to be the sheaf

(3.5.10) $$\mathcal{O}_{U_i} := (g_i)_*(\mathcal{O}_{X_i})$$

on U_i. We get an isomorphism of LR spaces

(3.5.11) $$(g_i, \psi_i) : (X_i, \mathcal{O}_{X_i}) \xrightarrow{\simeq} (U_i, \mathcal{O}_{U_i}).$$

Consider the collection of sheaves $\{\mathcal{O}_{U_i}\}_{i \in I}$ on the open covering $\{U_i\}_{i \in I}$ of X.

For $i < j$ define the isomorphism

$$\phi_{i,j} : \mathcal{O}_{U_i}|_{U_i \cap U_j} \xrightarrow{\simeq} \mathcal{O}_{U_j}|_{U_i \cap U_j}$$

by the commutative diagram

$$\begin{tikzcd}
(g_i)_*(\mathcal{O}_{X_i}|_{U_{i,j}}) \arrow[r, "(g_j)_*(\psi_{i,j})"] \arrow[d, "\psi_i"', "\simeq"] & (g_j)_*(\mathcal{O}_{X_j}|_{U_{j,i}}) \arrow[d, "\simeq"', "\psi_j"] \\
\mathcal{O}_{U_i}|_{U_i \cap U_j} \arrow[r, "\phi_{i,j}", "\simeq"'] & \mathcal{O}_{U_j}|_{U_i \cap U_j}
\end{tikzcd}$$

See Figure (3.5.12).

The 2-cocycle condition (C) implies that the isomorphisms $\{\phi_{i,j}\}_{i<j}$ satisfy the 2-cocycle condition, see [Ye4, Theorem 3.7.11]. Therefore they can be glued to a sheaf of rings $\mathcal{O}_X$ on X, with an isomorphism

$$\mathcal{O}_X|_{U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i} \tag{3.5.13}$$

for every i, etc.

The isomorphisms (3.5.13) and (3.5.11) together give the desired isomorphism of LR spaces

$$(g_i, \psi_i) : (X_i, \mathcal{O}_{X_i}) \xrightarrow{\simeq} (U_i, \mathcal{O}_X|_{U_i}).$$

These satisfy the condition stated in the theorem.

It is clear that $(X, \mathcal{O}_X)$ is a LR space.

Step 4. The uniqueness is basically clear from the construction $(*)$. □

Exercise 3.5.14. Fill the missing arguments $(*)$ in the proof of Theorem 3.5.1.

Corollary 3.5.15. *Assume that in Theorem 3.5.1 the LR spaces $(X_i, \mathcal{O}_{X_i})$ are all $\mathbb{K}$-schemes. Then the LR space $(X, \mathcal{O}_X)$ is a $\mathbb{K}$-scheme.*

Proof. Take a point $x \in X$. We need to produce an open neighborhood V of x in X s.t. $(V, \mathcal{O}_X|_V)$ is an affine scheme.

Let i be an index s.t. $x \in U_i$. Because X_i is a scheme, there is an open neighborhood V' of x in X_i s.t. $(V', \mathcal{O}_X|_{V'})$ is an affine scheme.

Let $V := g_i(V')$. Then

$$(V', \mathcal{O}_{X_i}|_{V'}) \cong (V, \mathcal{O}_X|_V)$$

in $\mathsf{LRSp}/\mathbb{K}$. □

Example 3.5.16. Let $\mathbb{K}$ be a nonzero ring. (If you prefer, you can assume that $\mathbb{K}$ is a field, or even an algebraically closed field.)

We define 1*-dimensional projective space*, also known as the *projective line* to be the following scheme $\mathbf{P}^1_{\mathbb{K}}$.

Let $U_0 := \operatorname{Spec}(\mathbb{K}[t_1])$ and $U_1 := \operatorname{Spec}(\mathbb{K}[t_0])$. So $U_0 \cong U_1 \cong \mathbf{A}^1_{\mathbb{K}}$, the 1-dimensional affine space over $\mathbb{K}$, see Exa 2.1.24.

Inside U_0 we have the affine open subscheme $U_{0,1} := \operatorname{Spec}(\mathbb{K}[t_1, t_1^{-1}])$, and inside U_1 we have the affine open subscheme $U_{1,0} := \operatorname{Spec}(\mathbb{K}[t_0, t_0^{-1}])$. The ring isomorphism

$$\psi_{0,1} : \mathbb{K}[t_0, t_0^{-1}] \xrightarrow{\simeq} \mathbb{K}[t_1, t_1^{-1}]\,,\ t_0 \mapsto t_1^{-1},$$

induces an isomorphism of affine schemes

$$(g_{0,1}, \psi_{0,1}) : U_{0,1} \xrightarrow{\simeq} U_{1,0}.$$

Because $|I| = 2$ the cocycle condition (C) is automatically satisfied.

By Thm 3.5.1 we can glue U_0 and U_1 along $(g_{0,1}, \psi_{0,1})$.

The resulting LRSp is the scheme $\mathbf{P}^1_{\mathbb{K}}$.

Exercise 3.5.17. Show that $\mathbf{P}^1_{\mathbb{K}}$ is not an affine scheme. (Hint: write $(X, \mathcal{O}_X) := \mathbf{P}^1_{\mathbb{K}}$. By a direct calculation, using the affine open covering $X = U_0 \cup U_1$, show that the ring $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$. Then use Cor 2.3.26.)

Exercise 3.5.18. Here we take $\mathbb{K} = \mathbb{R}$, the real numbers.

View the affine line $U_i(\mathbb{R}) = \mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ as a differentiable manifold, making it an object of $\mathsf{LRSp}/\mathbb{R}$.

Inside $U_i(\mathbb{R})$ we have the open subspace $U_{i,j}(\mathbb{R})$.

We now apply the gluing procedure in Example 3.5.16, namely the gluing isomorphism is

$$(g_{0,1}, \psi_{0,1}) : U_{0,1}(\mathbb{R}) \xrightarrow{\simeq} U_{1,0}(\mathbb{R})$$

in the category $\mathsf{LRSp}/\mathbb{R}$.

Of course what we get is the projective line $\mathbf{P}^1_{\mathbb{R}}(\mathbb{R})$, as an object of $\mathsf{LRSp}/\mathbb{R}$.

Can you describe it is more familar terms? (Hint: it is a differentiable manifold.)

Exercise 3.5.19. Here we take $\mathbb{K} = \mathbb{C}$, the complex numbers.

View the affine line $U_i(\mathbb{C}) = \mathbf{A}^1_{\mathbb{C}}(\mathbb{C})$ as a complex manifold (a Riemann surface), making it an object of $\mathsf{LRSp}/\mathbb{C}$.

We now apply the gluing procedure in Example 3.5.16, namely the gluing isomorphism is

$$(g_{0,1}, \psi_{0,1}) : U_{0,1}(\mathbb{C}) \xrightarrow{\simeq} U_{1,0}(\mathbb{C})$$

in the category $\mathsf{LRSp}/\mathbb{C}$.

Of course what we get is the projective line $\mathbf{P}^1_{\mathbb{C}}(\mathbb{C})$, as an object of $\mathsf{LRSp}/\mathbb{C}$.

Can you describe it is more familar terms? (Hint: it is a complex manifold.)

End of Lecture 9

Lecture 10, 26 May 2021

Today we are going to continue with gluing of locally ringed spaces, doing it for schemes and for differentiable manifolds in parallel.

First I want to recall the Theorem 3.5.1 on gluing, with slightly modified notation (inspired by [Liu, Lemma 3.33]).

Theorem 3.5.20 (Gluing Locally Ringed Spaces)**.** *Let $\big\{(X_i, \mathcal{O}_{X_i})\big\}_{i \in I}$ be a collection of objects of* $\mathsf{LRSp}/\mathbb{K}$ *indexed by an ordered set I.*

We are given these data:

- *For every $i < j$ in I there are open subsets $X_{i,j} \subseteq X_i$ and $X_{j,i} \subseteq X_j$.*
- *For every $i < j$ in I there is an isomorphism*
$$(g_{i,j}, \psi_{i,j}) : (X_{i,j}, \mathcal{O}_{X_i}|_{X_{i,j}}) \xrightarrow{\simeq} (X_{j,i}, \mathcal{O}_{X_j}|_{U_{j,i}})$$
in $\mathsf{LRSp}/\mathbb{K}$.

The condition is this:

(C) *For every $i < j < k$ in I there is equality*
$$(g_{j,k}, \psi_{j,k}) \circ (g_{i,j}, \psi_{i,j}) = (g_{i,k}, \psi_{i,k})$$
of isomorphisms
$$\big(X_{i,j} \cap X_{i,k}, \mathcal{O}_{X_i}|_{X_{i,j} \cap X_{i,k}}\big) \xrightarrow{\simeq} \big(X_{k,i} \cap X_{k,j}, \mathcal{O}_{X_k}|_{X_{k,i} \cap X_{k,j}}\big).$$

Then there is an object $(X, \mathcal{O}_X)$ in $\mathsf{LRSp}/\mathbb{K}$*, with an open covering $X = \bigcup_{i \in I} U_i$, and with an isomorphism*
$$(g_i, \psi_i) : (X_i, \mathcal{O}_{X_i}) \xrightarrow{\simeq} (U_i, \mathcal{O}_X|_{U_i})$$
for every $i \in I$, satisfying
$$(g_i, \psi_i) = (g_j, \psi_j) \circ (g_{i,j}, \psi_{i,j})$$
for every $i < j$, as isomorphisms
$$\big(X_{i,j}, \mathcal{O}_{X_i}|_{X_{i,j}}\big) \xrightarrow{\simeq} \big(U_i \cap U_j, \mathcal{O}_X|_{U_i \cap U_j}\big).$$

Moreover, the object $(X, \mathcal{O}_X)$, with its collection of isomorphisms $\{(g_i, \psi_i)\}_{i \in I}$, is unique up to a unique isomorphism.

Then we had Corollary 3.5.15, now repeated:

Corollary 3.5.21. *Assume that in Theorem 3.5.1 the LR spaces $(X_i, \mathcal{O}_{X_i})$ are all $\mathbb{K}$-schemes. Then the LR space $(X, \mathcal{O}_X)$ is a $\mathbb{K}$-scheme.*

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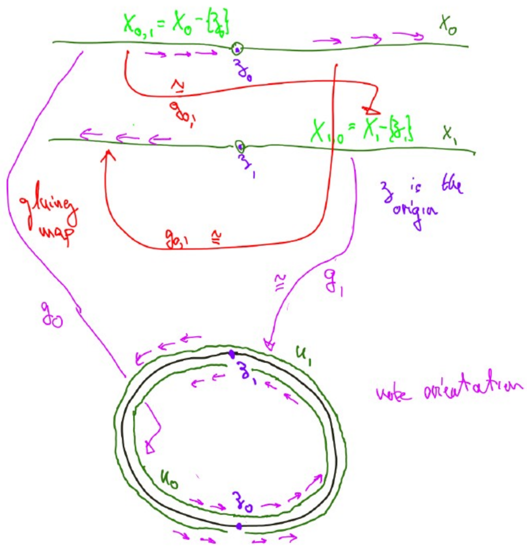
In Exercise 3.5.18 you were asked to glue 2 copies of the affine line $\mathbf{A}^1_{\mathbb{K}}$, to obtain the projective line $\mathbf{P}^1_{\mathbb{K}}$.

Here the real picture, i.e. gluing two copies of $\mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$ as an object of $\mathsf{LRSp}/\mathbb{R}$. See picture (3.5.22). As a manifold it is the circle:
$$\mathbf{P}^1_{\mathbb{R}}(\mathbb{R}) \cong \mathbf{S}^1.$$

Here is a pathological construction.

(3.5.22)


Example 3.5.23. Here again we take $\mathbb{K} = \mathbb{R}$, the real numbers.

View the affine line $X_i(\mathbb{R}) = \mathbf{A}^1_{\mathbb{R}}(\mathbb{R})$, $i = 0, 1$, as a differentiable manifold, making it an object of $\mathsf{LRSp}/\mathbb{R}$.

Inside $X_i(\mathbb{R})$ we have the open subspace $X_{i,j}(\mathbb{R})$, the complement of the origin.

We now apply another gluing procedure: the gluing isomorphism is

$$(g_{0,1}, \psi_{0,1}) : X_{0,1}(\mathbb{R}) \xrightarrow{\simeq} X_{1,0}(\mathbb{R})$$

is the identity.

We get a locally ringed space, an object of the category $\mathsf{LRSp}/\mathbb{R}$.

But this is not a manifold, since it is *not Hausdorff*.

Doing this very same construction in $\mathsf{LRSp}/\mathbb{R}$ we get a legitimate scheme.

But is it a *nonseparated scheme*. (We won't have time to learn this.)

See picture (3.5.24).

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In the next example we will get the projective plane $\mathbf{P}^2_{\mathbb{K}}$ be gluing 3 copies of $\mathbf{A}^2_{\mathbb{K}}$. The notation will be different from that of Exer 3.5.18.

Remark 3.5.25. The construction I gave in class used the punctured affine space $Y := \mathbf{A}^3_{\mathbb{K}} - \{z\}$ and its affine open covering $Y = U_0 \cup U_1 \cup Y_2$, where

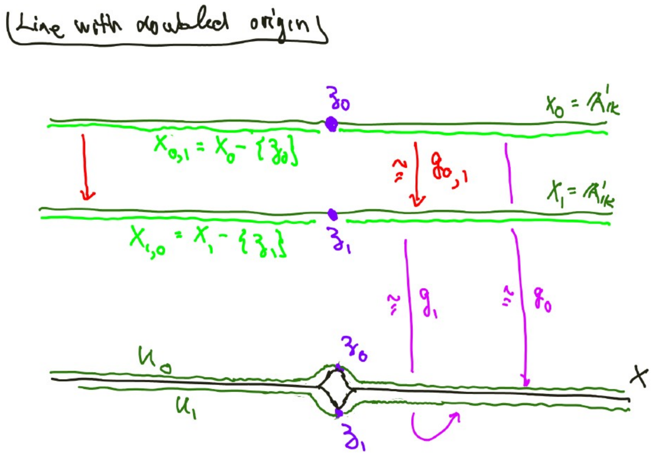
$$Y_i := \operatorname{NZer}_Y(t_i) \cong \operatorname{Spec}(\mathbb{K}[t_0, t_1, t_2, t_i^{-1}]).$$

In the example below all the work is algebraic: we use many subrings of the Laurent polynomial ring

$$A_{\text{loc}} := \mathbb{K}[t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}].$$

It might be a good idea to compare the two approaches.

(3.5.24)



Example 3.5.26. We start with the polynomial ring

$$A := \mathbb{K}[t_0, t_1, t_2].$$

This is a graded ring with $\deg(t_i) = 1$.

Consider its localization

$$A_{\mathrm{loc}} := A_{t_0 \cdot t_1 \cdot t_2} = \mathbb{K}[t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}].$$

It is called the ring of Laurant polynomials, and is is also graded.

We will be interested in several subrings of

$$B := (A_{\mathrm{loc}})_0,$$

the ring of degree 0 elements of A_{loc}.

For every $0 \leq i \leq 2$ let $j < k$ be the other indices, and define the elements

$$s_{i,j} := t_j \cdot t_i^{-1},\ s_{i,k} := t_k \cdot t_i^{-1}$$

in B, and the subring

$$B_i := \mathbb{K}[s_{i,j}, s_{i,k}] \subseteq B.$$

For instance,

$$B_0 = \mathbb{K}[s_{0,1}, s_{0,2}],$$

and multiplication by t_0 is a $\mathbb{K}$-ring isomorphism (forgetting the grading)

$$B_0 \xrightarrow{\simeq} \mathbb{K}[t_1, t_2],\ s_{0,1} \mapsto t_1,\ s_{0,2} \mapsto t_2.$$

Define the affine scheme

$$X_i := \operatorname{Spec}(B_i) \cong \mathbf{A}^2_{\mathbb{K}}.$$

For $i < j$ let

$$B_{i,j} := B_i \cdot B_j \subseteq B.$$

A small calculation shows that

$$B_{i,j} = (B_i)_{s_{i,j}} = (B_j)_{s_{j,i}},$$

principal localizations. This uses the fact that

$$s_{j,i} = t_i \cdot t_j^{-1} = (t_j \cdot t_i^{-1})^{-1} = (s_{i,j})^{-1} \in B.$$

Moreover
$$B_{i,j} = \mathbb{K}[s_{i,j}, s_{i,k}, s_{j,i}, s_{j,k}].$$
Define
$$X_{i,j} := \operatorname{Spec}(B_{i,j}) \subseteq X_i$$
and
$$X_{j,i} := \operatorname{Spec}(B_{i,j}) \subseteq X_j.$$
Define
$$B_{0,1,2} := B_0 \cdot B_1 \cdot B_2 \subseteq B.$$
In fact there is equality:
$$B_{0,1,2} = B.$$
A calculation of principal open sets shows that
$$X_{i,j} \cap X_{i,k} = \operatorname{Spec}(B)$$
for every i.

Define the isomorphism of schemes
$$g_{i,j} : X_{i,j} \xrightarrow{\simeq} X_{j,i}$$
to be the identity of $\operatorname{Spec}(B_{i,j})$.

Then
$$g_{1,2} \circ g_{0,1} = g_{0,2}$$
are the identity of $\operatorname{Spec}(B)$.

According to Thm 3.5.20 there is a scheme
$$X = \mathbf{P}^2_{\mathbb{K}}$$
with an open covering
$$\mathbf{P}^2_{\mathbb{K}} = U_0 \cup U_1 \cup U_2,$$
with an isom
$$g_i : X_i \xrightarrow{\simeq} U_i$$
of schemes,

such that
$$g_i = g_j \circ g_{i,j} : X_{i,j} \xrightarrow{\simeq} U_i \cap U_j \subseteq X.$$

Exercise 3.5.28. Try to perform the gluing of Exa 3.5.26 in $\mathsf{LRSp}/\mathbb{R}$. You will get a compact manifold $\mathbf{P}^2_{\mathbb{R}}(\mathbb{R})$. Can you identify it as some other familiar manifold?

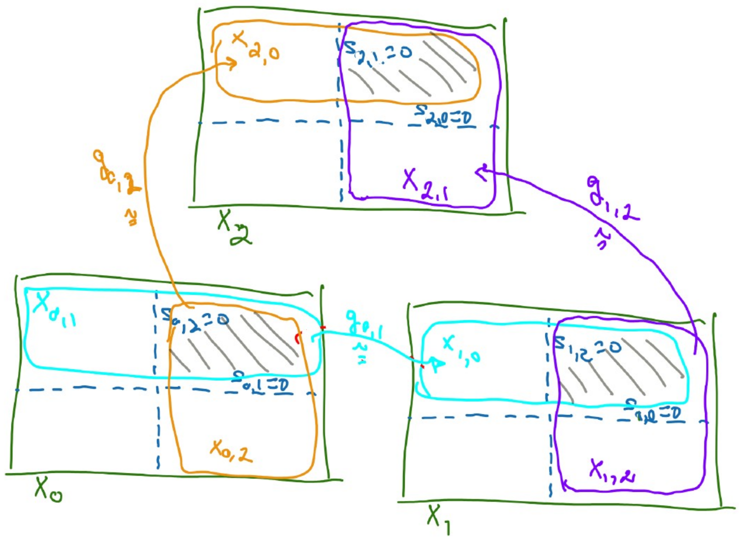
It will be an easier task after the next lecture – we will prove that there is a surjective map of schemes
$$p : Y \to \mathbf{P}^2_{\mathbb{K}},$$
where Y is the punctured space from Remark 3.5.25.

When $\mathbb{K}$ is a field, The group $\mathrm{GL}_1(\mathbb{K})$ acts on the set $Y(\mathbb{K})$, and its orbits are the fibers of
$$p : Y(\mathbb{K}) \to \mathbf{P}^2(\mathbb{K}).$$

We will return to this example / exercise next week.

$\boxed{\text{End of Lecture 10}}$

(3.5.27)


$X_{2,0}$
$s_{2,1}=0$
$s_{2,0}=0$
$X_{2,1}$
X_2
$g_{0,2}$
$\cong$
$g_{1,2}$
$\cong$
$X_{0,1}$
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$s_{0,1}=0$
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$g_{0,1}$
$\cong$
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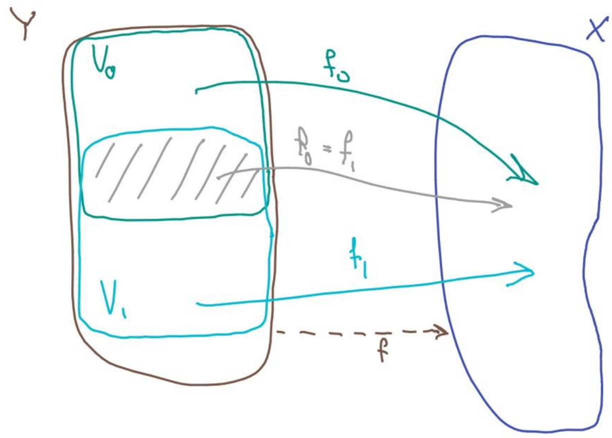
$s_{1,0}=0$
$X_{1,2}$
X_1

(3.6.2)



Lecture 11, 2 June 2021

We have 3 meetings left (including today). The plan:

- Glue maps of schemes. Use this to understand projective space better.
- Fibered products of schemes.

3.6. **Gluing Maps of Schemes.** In Thm 3.5.1 we saw how to glue LR spaces. It is easier to do the following:

Theorem 3.6.1 (Gluing Maps of LR Spaces)**.** *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be objects of* $\mathsf{LRSp}/\mathbb{K}$*, let $Y = \bigcup_{i \in I} V_i$ be an open covering, and for every i let*

$$(f_i, \psi_i) : (V_i, \mathcal{O}_Y|_{V_i}) \to (X, \mathcal{O}_X)$$

be a map in $\mathsf{LRSp}/\mathbb{K}$.

We assume that this condition holds: for every $i, j \in I$ there is equality

$$(f_i, \psi_i)|_{V_i \cap V_j} = (f_j, \psi_j)|_{V_i \cap V_j}$$

of maps

$$(V_i \cap V_j, \mathcal{O}_Y|_{V_{i,j}}) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$.

Then there is a unique map

$$(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

in $\mathsf{LRSp}/\mathbb{K}$ *such that*

$$(f, \psi)|_{V_i} = (f_i, \psi_i)$$

for every i.

Proof. The existence and uniqueness of a map of sets $f : Y \to X$ satisfying $f|_{V_i} = f_i$ is clear. Since continuity is a local property of Y, the resulting f is continuous.

We need to produce the homomorphism of sheaves of rings

$$\psi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$$

on X. By the adjunction formula between f_* and f^{-1} (see [Ye4, Thm 4.3.9]), this amounts to producing a homomorphism of sheaves of rings

$$\psi : f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \tag{3.6.3}$$

on Y. Now we are given homomorphisms

$$\psi_i : f^{-1}(\mathcal{O}_X)|_{V_i} \to \mathcal{O}_Y|_{V_i}$$

that agree on double intersections. According to [Ye4, Thm 3.7.9] these can be glued uniquely to a homomorphism of sheaves of ring ψ as in (3.6.3), such that $\psi|_{V_i} = \psi_i$. On stalks it is a local homomorphism, beacause this is a local propoerty on Y. So (f, ψ) is a map of locally ringed spaces. □

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We now return to look at $\mathbf{P}^n_{\mathbb{K}}$, for $n = 2$, continuing Exa 3.5.26. (The higher dimensional cases are really the same, but more confusing to discuss and to visualize.)

Example 3.6.4. We have constructed $X = \mathbf{P}^2_{\mathbb{K}}$ as the gluing of three affine schemes:

$$X = \mathbf{P}^2_{\mathbb{K}} = \bigcup_{i=0,1,2} U_i$$

where

$$U_i = \operatorname{Spec}(B_i)$$

and B_i is a subring of

$$B_{0,1,2} := (A_{0,1,2})_0,$$

the degree 0 subring of the graded ring

$$A_{0,1,2} = \mathbb{K}[t_0^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}].$$

The notation is a bit different from what we had before. Today we take

$$A := \mathbb{K}[t_0, t_1, t_2],$$

$$A_i := A[t_i^{-1}] = \mathbb{K}[t_0, t_1, t_2, t_i^{-1}],$$

etc. So

$$A_{i,j} = A_i \cdot A_j = \mathbb{K}[t_0, t_1, t_2, t_i^{-1}, t_j^{-1}] \subseteq A_{0,1,2}.$$

and

$$B_{i,j} = (A_{i,j})_0 = B_i \cdot B_j \subseteq B_{0,1,2}.$$

The punctures 3-dim affine space

$$Y = \mathbf{A}^3_{\mathbb{K}} - \{z\}$$

is also the union of three affine schemes:

$$Y = \bigcup_{i=0,1,2} V_i$$

where

$$V_i = \operatorname{Spec}(A_i) = \operatorname{NZer}_Y(t_i).$$

for every i the inclusion of rings $B_i \to A_i$ gives a map of affine schemes

$$f_i : V_i \to U_i.$$

These agree on the double intersections

$$V_i \cap V_j = \mathrm{Spec}(A_{i,j}) = \mathrm{NZer}_Y(t_i \cdot t_j).$$

According to Thm 3.6.1 there is a map of schemes

$$f : Y \to X = \mathbf{P}^2_{\mathbb{K}}.$$

Exercise 3.6.5. We use the notation of the example above. Consider a $\mathbb{K}$-ring C.

(1) Show that the points of the set $V_i(C) \subseteq Y(C)$ are the triples (c_0, c_1, c_2) in C such that c_i is invertible. (Hint: identify $Y(C)$ as a subset of $\mathbf{A}^2_{\mathbb{K}}(C) = C^3$.)

(2) For $i = 0, 1, 2$, show that the points of the set $U_i(C) \subseteq \mathbf{P}^2_{\mathbb{K}}(C)$ can be uniquely described as triples (c_0, c_1, c_2) in C such that $c_i = 1$.

(3) Describe the map
$$f_C : Y(C) \to \mathbf{P}^2_{\mathbb{K}}(C)$$
using items (1)-(2).

(4) Show that the group $\mathrm{GL}_1(C)$ acts on the set $Y(C)$, that this action commutes with the projection f_C, and that the fibers of f_C are the orbits of this group action.

Remark 3.6.6. I decided not to talk about the twisting sheaf $\mathcal{O}_{\mathbf{P}^n_{\mathbb{K}}}(1)$ and the universal property of $\mathbf{P}^n_{\mathbb{K}}$.

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3.7. **Fibered Products.**

The fibered product is a refinement of the notion of product.

The definition is abstract – it makes sense in any category.

Definition 3.7.1. Let $f : Y \to X$ and $g : Z \to X$ be morphisms in a category C. A *fibered product of Y and Z over X* is an object $W \in \mathsf{C}$, with morphisms $p : W \to Y$ and $q : W \to Z$, satisfying these two conditions:

(i) $f \circ p = g \circ q$.

(ii) Given an object W', and morphisms $p' : W' \to Y$ and $q' : W' \to Z$, such that $f \circ p' = g \circ q'$, there exists a unique morphism $r : W' \to W$ such that $p' = p \circ r$ and $q' = q \circ r$.

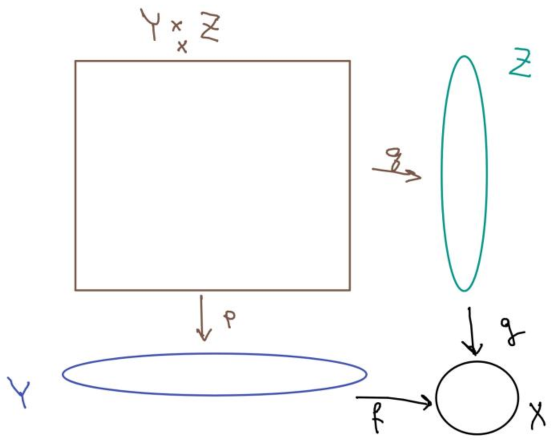
This is illustrated in the next commutative diagram.

$$\begin{tikzcd}
W' \arrow[dr, dashed, "r"] \arrow[drr, dashed, bend left, "q'"] \arrow[ddr, dashed, bend right, "p'"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"] & Z \arrow[d, "g"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

See also the picture (3.7.2).

It is clear that the fibered product (W, p, q) is unique, up to a unique isomorphism.

(3.7.2)



The notation for the fibered product is $Y \times_X Z$, leaving the morphisms implicit.

Existence is another story.

Example 3.7.3. In the category Set the fibered product is

$$Y \times_X Z = \{(y, z) \mid f(y) = g(z)\} \subseteq Y \times Z,$$

with the obvious projections p, q.

Here is another name for the same categorical construct.

Definition 3.7.4. A diagram

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

in the category C is called *cartesian* if $W = Y \times_X Z$.

The fibered product of schemes is much more complicated to construct. We will approach it in steps.

As a first step, let's try to produce the product of manifolds. (The product is the fibered product over the terminal object, of course.)

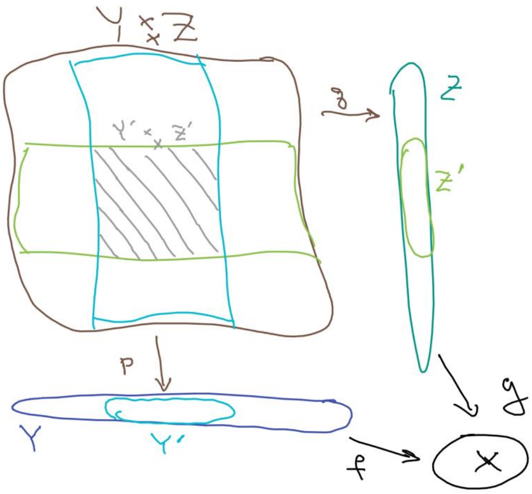
Example 3.7.5. Let Y and Z be manifolds. The underlying topological space of $Y \times Z$ is obvious. (More precisely, the obvious topology is the correct one; this is false for schemes!)

How do we produce an atlas in $Y \times Z$?

Suppose we have open coverings $Y = \bigcup_i V_i$ and $Z = \bigcup_i W_i$, such that each V_i is an open set in $\mathbf{A}^m(\mathbb{R})$, each W_i is an open set in $\mathbf{A}^n(\mathbb{R})$, and for very $(y, z) \in Y \times Z$ there is an index i such that $y \in V_i$ and $z \in W_i$.

Of course

(3.7.6)
$$Y \times Z = \bigcup_i (V_i \times W_i)$$

(3.7.8) 

is an open covering.

But each $V_i \times W_i$ is an open set in $\mathbf{A}^{m+n}(\mathbb{R})$. And

$$(V_i \times W_i) \cap (V_i \times W_i) = (V_i \cap V_j) \times (W_i \cap W_j),$$

also an open set in $\mathbf{A}^{m+n}(\mathbb{R})$, in two ways, which are related to each other by a diffeomorphism. We see that (3.7.6) is a differentiable atlas.

We will do something like that for LR spaces.

First a lemma.

Lemma 3.7.7. *Let $f : Y \to X$ and $g : Z \to X$ be maps in $\mathsf{LRSp}/\mathbb{K}$. Let $X' \subseteq X$, $Y' \subseteq Y$ and $Z' \subseteq Z$ be open subspaces, such that $f(Y') \subseteq X'$ and $g(Z') \subseteq X'$. Assume the fiber product*

$$W = Y \times_X Z$$

exists, with projections p, q. Let W' be the open subspace

$$W' := p^{-1}(Y') \cap q^{-1}(Z') \subseteq Y \times_X Z,$$

with induced projections $p' : W \to Y'$ and $q' : W \to Z'$. Then $W' = Y' \times_{X'} Z'$.

See Figure (3.7.8) for an illustration of this lemma. (There should be an open subspace $X' \subseteq X$.)

Exercise 3.7.9. Prove Lemma 3.7.7.

Next week we will prove a theorem with a sufficient condition for the existence of the fibered product in $\mathsf{LRSp}/\mathbb{K}$, and another theorem saying that this is always true for schemes.

To here Lecture 11

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