Course Notes:

Homological Algebra

BGU, Fall 2021-22

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https://www.math.bgu.ac.il/~amyekut/teaching/2021-22/homol-alg/course_page.html

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0. Introduction

comment: Start of Lecture 1, 20 Oct 2021

Homological algebra is a generalization of *linear algebra over a field*. The *vector spaces* over a field $\mathbb{K}$ are replaced here by *modules* over a *ring* A, possibly *noncommutative*.

The main feature of linear algebra over a field $\mathbb{K}$ is that *every* $\mathbb{K}$*-module is free*, namely it has a basis. Thus the only invariant of a $\mathbb{K}$-module M is its *rank* (traditionally called the *dimension*), which is the size of a basis of M.

A homomorphism $\phi : M \to N$ of $\mathbb{K}$-modules can be described by a matrix, after we choose bases for M and N.

When A is a commutative ring that is not a field, this is not true. A-modules can be very complictaed.

For example, for $A = \mathbb{Z}$, a $\mathbb{Z}$-module M is just an *abelian group*. We all know that many abelian groups are not free; indeed, every nonzero finite abelian group M is not free.

When the ring A is not commutative, things can become even more complicated.

Homological algebra provides us with strong tools to describe what A-modules look like, and what their homomorphisms look like.

Homological algebra also has methods to describe how some algebraic objects are related to other objects of the same kind.

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Let me preview some results that we will prove using tools of homological algebra, either at the end of this course or in the subsequent course on *commutative algebra*.

Many of the concepts appearing in these statements will not be familiar to you. That's all right. Eventually – later today or in the coming months – everything will be defined and proved.

comment: I probably went too far with this preview – I hope that I will have time during this course to actually teach these things...

Given a commutative ring $\mathbb{K}$ and an integer $n \geq 1$, we denote by $\mathrm{Mat}_n(\mathbb{K})$ the ring of $n \times n$ matrices with entries in $\mathbb{K}$. This is a *central* $\mathbb{K}$*-ring*. Traditionally the name is "unital associative $\mathbb{K}$-algebra".

If A and B are central $\mathbb{K}$-rings, then so is their tensor product $A \otimes_{\mathbb{K}} B$. In case A itself is commutative, then $A \otimes_{\mathbb{K}} B$ is actually a central A-ring.

We now consider the fields $\mathbb{R}$ and $\mathbb{C}$. We say that a central $\mathbb{R}$-ring A is an $\mathbb{R}$-*form of the central* $\mathbb{C}$-*ring* $\operatorname{Mat}_2(\mathbb{C})$ if there is a $\mathbb{C}$-ring isomorphism

$$C \otimes_{\mathbb{R}} A \cong \operatorname{Mat}_2(\mathbb{C}). \tag{0.1}$$

I hope everybody heard about ring of *Hamilton quaternions*. It is a NC central $\mathbb{R}$-ring. As an $\mathbb{R}$-module it free with basis $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The multiplication satisfies $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{i} \cdot \boldsymbol{j} \cdot \boldsymbol{k} = -1$.

Let A be the $\mathbb{R}$-ring $\operatorname{Mat}_2(\mathbb{R})$, and let B be the $\mathbb{R}$-ring of quaternions. Observe that both A abd B are free $\mathbb{R}$-modules of rank 4, so they are *isomorphic as* $\mathbb{R}$-*modules*. But A and B are *not isomorphic as* $\mathbb{R}$-*rings*. Indeed, B is a *division ring*, namely every nonzero $b \in B$ is invertible; whereas there are nonzero matrices $a \in A$ s.t. $a^2 = 0$.

Theorem 0.2. *Let A be the* $\mathbb{R}$-*ring* $\operatorname{Mat}_2(\mathbb{R})$, *and let B be the* $\mathbb{R}$-*ring of quaternions. Then A and B are* $\mathbb{R}$-*forms of the* $\mathbb{C}$-*ring* $\operatorname{Mat}_2(\mathbb{C})$. *Furthermore, every* $\mathbb{R}$-*form of* $\operatorname{Mat}_2(\mathbb{C})$ *is isomorphic to A or to B.*

This classification theorem relies on *group cohomology*. Specifically we will need an analysis of the cohomologies

$$\mathrm{H}^1\big(G, \mathrm{PGL}_2(\mathbb{C})\big) \quad \text{and} \quad \mathrm{H}^2\big(G, \mathrm{GL}_1(\mathbb{C})\big). \tag{0.3}$$

Here G is the Galois group of the field extension $\mathbb{R} \to \mathbb{C}$. Note that the group $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^{\times}$ is abelian, but $\mathrm{PGL}_2(\mathbb{C})$ is not abelian.

This material belongs to topic 10 in the syllabus, and I hope we will reach it.

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comment: The next discussion – from here until formula (0.9) – was not done in class. Please read it, but not too carefully, since it is just a vague description of things to come.

In the course "Commutative Algebra" we will prove the next theorem.

Theorem 0.4. *Let A be a noetherian commutative ring, and let M be a finitely generated A-module. The following three conditions are equivalent*:

(i) *The A-module M is a projective.*
(ii) *For every maximal ideal $\mathfrak{m} \subseteq A$, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is free.*
(iii) *The A-module M is flat.*

This theorem belongs to topic 9.

For Theorem 0.4 we shall require the following tools from homological algebra. Given a ring A, left A-modules M, N and an integer $q \geq 0$, there is the *q-th Ext group*

$$\operatorname{Ext}^q_A(M, N). \tag{0.5}$$

It is an abelian group, and it depends *functorially* on M and N.

For $q = 0$ there is a *functorial isomorphism*

$$\operatorname{Ext}^0_A(M, N) \cong \operatorname{Hom}_A(M, N). \tag{0.6}$$

The other tool needed for the proof is this: given a right A-module L and an integer $q \geq 0$, there is an abelian group

$$\operatorname{Tor}^A_q(L, N), \tag{0.7}$$

called the *q-th Tor group*. It depends functorially on L and N.

For $q = 0$ there is a functorial isomorphism

$$\operatorname{Tor}^A_0(L, N) \cong L \otimes_A N. \tag{0.8}$$

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There is a connection between group cohomology and Ext.

Given a group G we can form its *group ring* $A := \mathbb{Z}[G]$.

If M is an abelian group with an action of G on it, then M is a left A-module.

For every $q \in \mathbb{N}$ there is a functorial isomorphism of abelian groups

$$\mathrm{H}^q(G, M) \cong \operatorname{Ext}^q_A(\mathbb{Z}, M). \tag{0.9}$$

Here $\mathbb{Z}$ is given the trivial G action.

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To end the introduction, let me say that $\operatorname{Ext}^q_A(-,-)$ is the q-th right derived functor of $\operatorname{Hom}_A(-,-)$; and $\operatorname{Tor}^A_q(-,-)$ is the q-th left derived functor of $(-) \otimes_A (-)$. These are the derived functors appearing in topic 9.

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Here is a the syllabus for this course. It is tentative: I will change material, and the order of presentation, as I go along. Most of the material is in these older course notes: [Yek1], [Yek2], [Yek4] and [Yek5],

(1) **Review of prior material**. On rings, ideals and modules (including noncommutative rings).
(2) **Categories and functors**. Emphasis on linear categories and functors. (This topic will be introduced gradually, as we go along.)
(3) **Universal constructions**. Free modules, products, direct sums, polynomial rings.
(4) **Tensor products**. Definition, construction and properties.
(5) **Exactness**. Exact sequences and functors.
(6) **Special modules**. Projective, injective and flat modules.
(7) **Complexes of modules**. Operations on complexes, homotopies, the long exact cohomology sequence.
(8) **Resolutions**. Projective, flat and injective resolutions.

(9) **Left and right derived functors**. Applications to commutative algebra.
(10) **Further applications of derived functors**. Classification problems, extensions.
(11) **Morita Theory**. Equivalences of module categories and invertible bimodules.

Some of the material might move to the subsequent course "Commutative Algebra".

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Here are a few words on **administration**.

(1) Read the handout.
(2) You are required to *register* only if you want to get credit for the course.
(3) I expect all the students (registered or not) to *attend all lectures*. In case you must be absent, please send me an *email in advance*.
(4) The *homework* will be assigned as material labeled “exercise” during the lecture (and in the notes). This is often complementary material. You should do it all and submit to me in writing each week. I will usually just indicate in my list who submitted the homework (but sometimes, randomly, I will look at it).
(5) If you want help with homework, or to discuss some other math, you can send me an email. A zoom meeting can also be arranged (by email).

1. Review of Rings, Ideals and Modules

comment: From here on in this lecture the pace is much slower – probably too slow.

Most of this review material should be familiar to you, so I will go over it quickly.

A *ring* is a mathematical structure $(A, 0, 1, +, \cdot)$ consisting of:

- A set A.
- Distinguished elements $0, 1 \in A$, called *zero* and *one* (or the *unit*).
- Binary oprations $+$ and $\cdot$, called addition and multiplication.

The axioms are:

- ▹ The sytem $(A, 0, +)$ is an abelian group. (This is called the *additive group* of A.)
- ▹ Multiplication is associative.
- ▹ Multiplication is distributive on both sides w.r.t. addition.
- ▹ The element 1 is neutral for multiplication.

We usually say that A is a ring, leaving the rest of the structure implicit.

A ring A is called *the zero ring* if $A = \{0\}$.

Exercise 1.1. Show that a ring A is the zero ring iff $1 = 0$ in A.

The ring A is called *commutative* if

$$b \cdot a = a \cdot b \quad \text{for all} \quad a, b \in A. \tag{1.2}$$

When we say that a ring A is *noncommutative* (NC), we mean that it is not necessarily commutative. This is a bit confusing.

We will often encounter noncommutative rings, such as the ring of matrices $A = \mathrm{Mat}_n(\mathbb{K})$ where $\mathbb{K}$ is a field and $n \geq 2$. (We already saw the case $n = 2$.)

Exercise 1.3. Let $\mathbb{K}$ be a nonzero commutative ring, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some integer $n \geq 1$. Show that A is commutative iff $n = 1$.

Remark 1.4. If any of the statements in this review is not clear to you, then try to prove it (or ask some other student, or look it in one of the basic references such as [Art] or [Jac]).

On the other hand, if an exercise seems too easy for you, and you are sure of the answer, then you don't have to solve it – just write "this is too easy for me" as your solution. It is your responsibility to know the answer!

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Let A be a ring. An element $a \in A$ is called *invertible* if there exists an element $b \in A$ satisfying $a \cdot b = b \cdot a = 1$. If such b exists, then it is unique, it is called the *inverse* of a, and is denoted by a^{-1}.

The set of invertible elements of A is denoted by $A^{\times}$. The system $(A^{\times}, 1, \cdot)$ is a group, called the *multiplicative group of A*.

A commutative ring A is called a *field* if it is a nonzero ring, and every nonzero element $a \in A$ is invertible. In other words, A is a field iff $A^{\times} = A - \{0\}$, the set of nonzero elements of A.

The NC analogue of a field is called a *division ring*. The same conditions, except that A is not necessarily commutative. The easiest (and historically first) division ring that's not commutative if the ring of quaternions, which we saw earlier.

Exercise 1.5. Try to prove that the ring B of quaternions is in fact a division ring. If it is hard then look it up in one of the reference. (I don't know a proof. If you find a nice proof, tell it to us in class.)

Exercise 1.6.

(1) Find the group $\mathbb{Z}^{\times}$.
(2) Let $\mathbb{K}$ be a nonzero commutative ring, and let $\mathbb{K}[t]$ be the polynomial ring in one variable over $\mathbb{K}$. Find the group $\mathbb{K}[t]^{\times}$.
(3) Let $A := \mathbb{Z}[\boldsymbol{i}] \subseteq \mathbb{C}$, the subring of $\mathbb{C}$ generated by $\boldsymbol{i} = \sqrt{-1}$. It is called the ring of *Gauss integers*. Find the group $A^{\times}$. (Hint: consider $|a|$.)
(4) Let $\mathbb{K}$ be a field, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some $n \geq 1$. Find the group $A^{\times}$.
(5) Let $\mathbb{K}$ be a nonzero commutative ring, and let $A := \mathrm{Mat}_n(\mathbb{K})$ for some $n \geq 1$. Find the group $A^{\times}$. (Hint: Cramer's formula.)

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A *subring* B of a ring A is a subset $B \subseteq A$ s.t.

- $0, 1 \in B$.
- $a, b \in B \Rightarrow a + b \in B$.
- $a, b \in B \Rightarrow a \cdot b \in B$.

Thus $(B, 0, 1, +, \cdot)$ is itself a ring.

Let A be a ring. A *left ideal* I in A is a subset $I \subseteq A$ s.t.

- $0 \in I$.
- $a, b \in I \Rightarrow a + b \in I$.
- $a \in A$ and $b \in I \Rightarrow a \cdot b \in I$.

Note that $(I, 0, +)$ is a subgroup of the additive group $(A, 0, +)$.

A *right ideal* of A is defined likewise, except that the last condition is $b \cdot a \in I$.

A *two-sided ideal* of A is a subset $I \subseteq A$ that's both a left and a right ideal.

Of course when A is a commutative ring, all three types of ideals are the same.

comment: End of Lecture 1

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