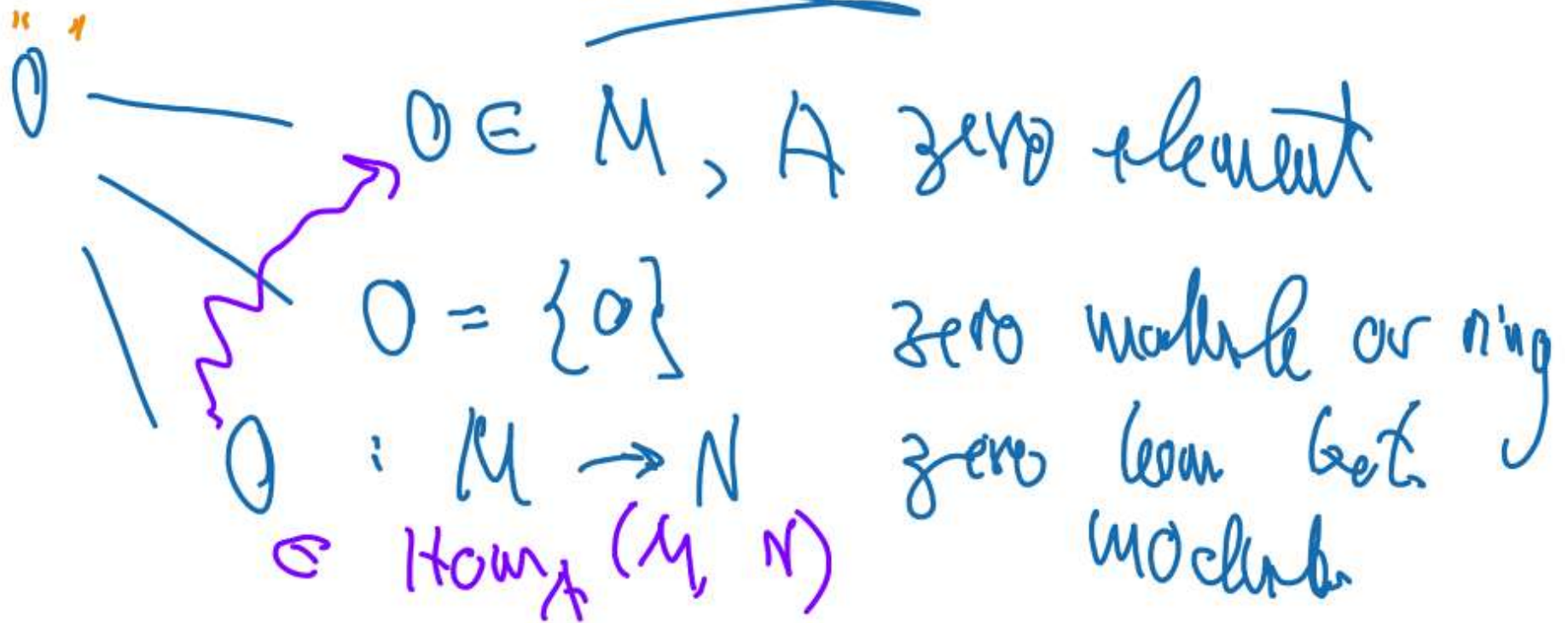


# Homological Algebra

lecture 10, 22 Dec 2021



comp:  $K$  is base comm. ring  
all linear structures & ops

Lem. 7.11

Let  $M \in \underline{\text{Mod}}(A)$ .

TFNE:

(i)  $M=0$

$A$  &  $B$  are  $K$ -linear  
central  $K$ -ring

(ie.  $M$  is the zero module)

(ii)  $\text{id}_M = 0$  as  $A$ -lin hom  $M \rightarrow M$

(ie. ~~as~~ as  $A$ -lin hom  $M \rightarrow M$ )  
 $\text{End}_A(M)$   
 $\cong$   
 $K$ -ring

pf:  
ex 25

Prop. Let  $F: \underline{\text{Mod}}(A) \rightarrow \underline{\text{Mod}}(B)$   
 be some linear functor.  
 If  $M \in \underline{\text{Mod}}(A)$  is  $M=0$   
 then  $\forall (N) \in \underline{\text{Mod}}(A)$  is  $F(M) = 0$ .

pf: as is

aside: if  $A$  is comm.  
 then  $\underline{\text{Mod}}(A)$  is  
 an  $A$ -lin. cat.  
 $\underline{\text{Mod}}(A)$  is

Recall the ~~same~~ topic with (8.7)  
 is "exact lin. functors"  
 $\text{Cent}(A) = \text{lin. cat.}$

$$G = \text{Gal}(C/K) \\ \{id, \sigma\}$$

$$F: \text{Mod}(A) \rightarrow \text{Mod}(B)$$

some H-lin function

$$A := C[G]$$

gyp ring  
skew gyp ring:

$$B := G \ltimes C$$

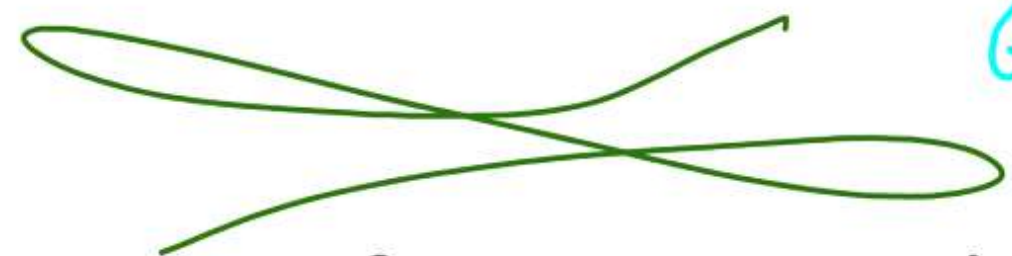
its  $C \triangleleft B \triangleleft \sigma$

$$\lambda \cdot id + \mu \cdot \sigma \in B$$

$$\lambda \cdot \sigma = \sigma \cdot \lambda$$

$$\lambda \cdot \sigma = \sigma(\lambda) \cdot id$$

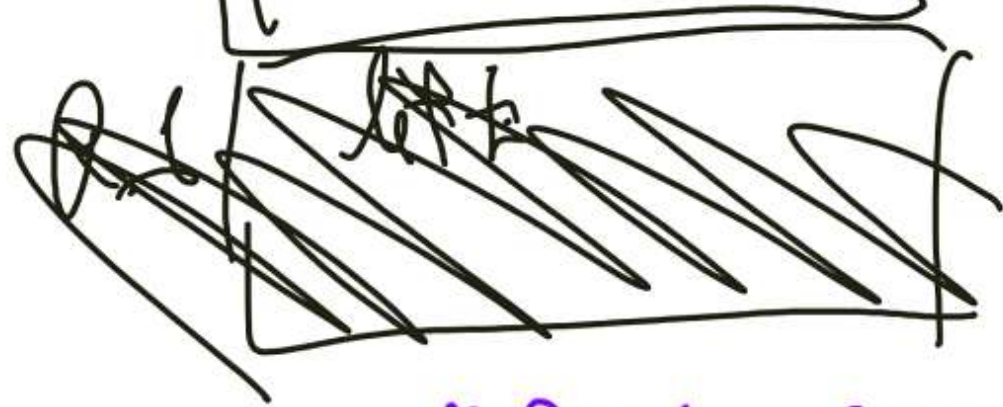
$$\mu \in C$$



Q6.  $\text{Cent}(B) = ?$

$$\{id_e, \sigma\} = G = \text{Gal}(e/\mathbb{F})$$

$$\sigma(\lambda) = -\lambda$$



new exercise

$$A \cong K[G]$$

~~$G = \text{Gal}(K/\mathbb{F})$~~   
 $G$  some group

- ①  $A$  is comm. iff  $G$  is abelian
- ②  $A$  is a d.l.h. cat. iff  $\exists M \in \text{Mod}(A)$  s.t.  $M \cong M^{\otimes 2}$

show that  $M$  has <sup>an</sup> action ~~by~~ <sup>left</sup> by the group  $G$ . This action is  $K$ -linear.

(3)

Let  $M^G := \{ m \in M \mid g(m) = m \ \forall g \in G \}$

↑ invariants      ↗- module

"equiv."

$$F := (\ )^G : \underline{\text{Mod}}(A) \rightarrow \underline{\text{Mod}}(K)$$

↗- fun.

exp and into of  $F$ . <sup>an</sup> exact exercise

"canonical" <sup>i</sup>  
 $\mathbb{K}$ -isom.

$$U^G \cong \text{Hom}_A(\mathbb{K}, M)$$

$$A = \mathbb{K}[G]$$

$G$  acts on  $\mathbb{K}$  by the triv. rep:

$$\mathbb{K} \ni \lambda \xrightarrow{g(\lambda) := \lambda} \text{Hom}_A(A = \mathbb{K}[G]) \xrightarrow{\text{augmentation}} \mathbb{K}$$

$i$ -th right derived functor  
 $\text{Hom}_A(-, \mathbb{K})$

$$G \ni g \longmapsto 1$$


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$$\text{Ext}^i(-, -)$$

$\dots, i=0$

$$(-)^G \text{ or } \text{Hom}_k(k, -)$$

$$\text{Ext}_{k[x]}^i(k, -)$$

$$H^i(G, -)$$

with input of function

$$f \in (-)^G$$

group cohomology

~~group cohomology~~  
 end of test hour  
 bye to 3:20



# Exact Lin. Functors

$A$  &  $B$  are central  $K$ -rings.

• So  $\underline{\text{Mod}}(A)$  &  $\underline{\text{Mod}}(B)$  are  $K$ -linear sr. cats.

•  $F: \underline{\text{Mod}}(A) \rightarrow \underline{\text{Mod}}(B)$   
 $K$ -lin. functor.

Exact seq. in  $\underline{\text{Mod}}(A)$  &  $\underline{\text{Mod}}(B)$

• 
$$\text{III} = \left( 0 \rightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \rightarrow 0 \right)$$
  
short ex. seq. in  $\underline{\text{Mod}}(A)$ .  
 $\text{Im}(\varphi) = \text{Ker}(\psi) \Rightarrow \psi \circ \varphi = 0$

$$F(\underline{E}) = \left( \begin{array}{c} F(0) \xrightarrow{F(\alpha)} F(M') \xrightarrow{F(\psi)} F(M) \xrightarrow{F(\psi)} F(0) \\ \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ 0 \quad \quad \quad F(M'') \xrightarrow{F(\beta)} F(0) \end{array} \right)$$

$F(\psi) \circ F(\psi) = F(\psi \circ \psi) = F(0) = 0$

a seq. in  $\text{Mod}(B)$ .

$$F(\underline{E}) = \left( \begin{array}{c} 0 \rightarrow F(M') \xrightarrow{F(\psi)} M \xrightarrow{F(\psi)} F(M'') \rightarrow 0 \\ \text{exact} \quad \quad \quad \text{exact} \end{array} \right)$$

an exact

(if  $\underline{N}$  is complex of  $A$ -mods  $\Rightarrow F(\underline{N})$  is complex of  $B$ -mods)



Def.  $\checkmark$  functor  $F: \text{Mod}(A) \rightarrow \text{Mod}(B)$

Mod(2) is called

- see (f)
- left exact if
  - ~~right exact~~
  - right exact if

exact: both if & ff

Mod