Homological Algebra

lecture 10, 22 Dec 2021

"0"

$0 \in M, A$ zero element

$0 = \{0\}$ zero module or ring

$0 : M \to N$ zero hom bet. modules

$\in \mathrm{Hom}_A(M, N)$

conv: k is base comm. ring

all linear structures & opers

A & B are k-linear central k-ring

Lem. 7.11

Let $M \in \underline{\mathrm{Mod}}(A)$.

TFAE: (i) $M = 0$ (i.e. M is triv zero module)

(ii) $id_M = 0$ as A-lin hom $M \to M$

(i.e. as elts of ring $\mathrm{End}_A(M)$)

k-ring

Pf: exer

Prop. Let $F : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$ be some linear functor.

If $M \in \underline{\mathrm{Mod}}(A)$ is $M = 0$

then $F(M) \in \underline{\mathrm{Mod}}(A)$ is $F(M) = 0$.

Pf: Exer

Aside: if A is comm. then $\underline{\mathrm{Mod}}(A)$ is an A-lin. cat.

[$\underline{\mathrm{Mod}}(A)$ is $\mathrm{Cent}(A)$-lin cat.]

Recall the [illegible] ~~[illegible]~~ topic w/in (§7) is "exact lin. functors"

$G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$
$\{id, \sigma\}$

$F : \underline{\mathrm{Mod}}(A) \longrightarrow \underline{\mathrm{Mod}}(B)$

some $\mathbb{K}$-lin functor

$G \curvearrowright \mathbb{C}$

$A := \mathbb{C}[G]$ over

grp ring comm

skew grp ring:

$\lambda \cdot \sigma = \sigma \cdot \lambda$

$B := G \ltimes \mathbb{C}$

λC

(λ, μ)

$\lambda, \mu \in \mathbb{C}$

as $\mathbb{C}$-lin... its $\mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot \sigma$

$\boxed{\lambda \cdot \sigma = \sigma(\lambda) \cdot id_{\mathbb{C}}}$

$\lambda \cdot id_{\mathbb{C}} + \mu \cdot \sigma \in B$

$$\mathrm{Ab}_- \quad \boxed{\mathrm{Cent}(B) = ?}$$

$$\{id_{\mathbb{C}}, \sigma\} = G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

$$\sigma(\lambda) = \bar{\lambda}$$

new exercise

$A := \mathbb{k}[G]$, G some group

(1) A is comm. iff G is abelian.

$\underline{\mathrm{Mod}}(A)$ is an $\mathbb{k}$-lin. cat. if $\mathbb{C}, \mathbb{R}, \mathrm{Ab}$

(2) Let $M \in \underline{\mathrm{Mod}}(A)$.

show that M has a left action by the group G. The action is $\mathbb{K}$-linear.

"equiv."

(3) Let $M^G := \{ m \in M \mid g(m) = m \ \forall g \in G \}$

invariants $\mathbb{K}$-module

$$f := (\ )^G : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(\mathbb{K})$$

$\mathbb{K}$-lin.

[expand info of f. exact exercise]

"canonical" $\overset{i}{\cong}$
$\mathbb{k}$-isom.

$$\boxed{M^G \cong \mathrm{Hom}_\Lambda(\mathbb{k}, M)}$$

$\Lambda = \mathbb{k}[G]$

G acts on $\mathbb{k}$ by the triv. act:

$\mathbb{k} \ni \lambda \qquad g(\lambda) := \lambda.$

Λ-mod hom, $\Lambda = \mathbb{k}[G] \xrightarrow{\text{augmentation}} \mathbb{k}$

$\mathbb{k}$-ring hom

$G \ni g \longmapsto 1$

i-th right derived functor

$\mathrm{Hom}_\Lambda(-,-) \rightsquigarrow \boxed{\mathrm{Ext}^i_\Lambda(-,-)}$

$\rightsquigarrow \ldots, \; i=0$

$\boxed{(-)^G \cong \mathrm{Hom}_{\lambda}(k, -)}$

$\rightsquigarrow$ $\boxed{H^i(G, -)} \cong \boxed{\mathrm{Ext}^i_{k[G]}(k, -)}$

i-th right der functor of $(-)^G$

group cohomology

end of 1st hour

~~brk to 3:25~~

Exact Lin. Functors

A & B are central k-rings.

- so $\underline{Mod}(A)$ & $\underline{Mod}(B)$ are k-linear cats.
- $F : \underline{Mod}(A) \to \underline{Mod}(B)$ k-lin. functor.

exact seq. in $\underline{Mod}(A)$ & $\underline{Mod}(B)$

$\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi) \Rightarrow \psi \circ \varphi = 0$

- $$\underline{\underline{E}} = \left(0 \xrightarrow{0} M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0 \right)$$

short ex. seq. in $\underline{Mod}(A)$.

$$F(\underline{\underline{E}}) = \left(\underset{\substack{\| \\ 0}}{F(0)} \xrightarrow{\overset{0}{\overset{\|}{F(\varphi)}}} F(M') \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} \cdots \to F(M'') \xrightarrow{\underset{\substack{\| \\ 0}}{F(0)}} \underset{\substack{\| \\ 0}}{F(0)} \right)$$

$F(\psi) \circ F(\varphi) = F(\psi \circ \varphi) = F(0) = 0$

a seq. in $\underline{\mathrm{Mod}}(B)$.

$(+)$ $F(\underline{\underline{E}}) = \left(0 \to F(M') \xrightarrow{F(\varphi)} M \xrightarrow{F(\psi)} F(M'') \to 0 \right)$ exact, exact, an exact

(if $\underline{\underline{M}}$ is is complex of A-mods $\Rightarrow$ $F(\underline{\underline{M}})$ is a cplx of B..)

A-B-fun

Def $\underline{\mathrm{Def}}$ functor $F : \underline{\mathrm{Mod}}(A) \to$

Mod (B) is called

see (∗)

- left exact if
- ~~right exact~~
- right exact if

exact: both rt & lt

lud