Homological Algebra

Lecture 11, 29.12.21

$\Bbbk$ base comm ring
A & B central $\Bbbk$-rings, all linear ops. are $\Bbbk$-linear.

start @ 12:15

Def. Suppose we have lin. functors

$F, G : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$.

Define the linear functor

$$\boxed{F \oplus G : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)}$$

to be: for $M \in \underline{\mathrm{Mod}}(A)$

ob: $(F \oplus G)(M) := F(M) \oplus G(M) \in \underline{\mathrm{Mod}}(B)$

morph: for $\left(M_0 \xrightarrow{\varphi} M_1\right)$ in $\underline{\mathrm{Mod}}(A)$

$$\begin{array}{ccc} (F\oplus G)(\varphi) : (F \oplus G)(M_0) = F(M_0) & \oplus & G(M_0) \\ \downarrow F(\varphi) & & \downarrow G(\varphi) \\ (F\oplus G)(M_1) = F(M_1) & \oplus & G(M_1) \end{array}$$

$F(\varphi) \oplus G(\varphi)$ in $\underline{\mathrm{Mod}}(B)$

Exer. Prove that $F \oplus G$ is indeed a $\Bbbk$-linear functor.

Solution of Exer. 7.20

idea: $F := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), -)$

We will prove later the F is left exact

Counterexample showing that F is not right exact:

$$\underline{D} := \left(0 \to \mathbb{Z}/(2) \xrightarrow{\varphi} \mathbb{Z}/(4) \xrightarrow{\psi} \mathbb{Z}/(2) \to 0\right)$$

$[1] \mapsto [2]$, $[1] \mapsto [1]$

$\underline{\mathrm{Ab}} = \underline{\mathrm{Mod}}(\mathbb{Z})$

$A = B := \mathbb{Z}$

$n \in \mathbb{Z}$,

$(n) := \mathbb{Z} \cdot n \subseteq \mathbb{Z}$

$\mathbb{Z}/(n)$ is a finite comm ring (also an ab. grp)

$$F(N) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), N) \ni \varphi$$

$$= \{ n \in N \mid 2 \cdot n = 0 \} \ni n := \varphi(1)$$

$2 \cdot N \subseteq N$

for $n \geq 2$ and $i \in \mathbb{Z}$ write $[i] := (i \bmod n) = i + (n) \in \mathbb{Z}/(n)$

So:

$$F(\underline{D}) = \left(0 \to \underbrace{\mathbb{Z}/(2)}_{= F(\mathbb{Z}/(2))} \xrightarrow[\cong]{F(\varphi)} \underbrace{2 \cdot \mathbb{Z}/(4)}_{= F(\mathbb{Z}/(4))} \xrightarrow[= 0]{F(\psi)} \mathbb{Z}/(2) \to 0\right)$$

$\mathbb{Z}/(2) \ni [1] \mapsto [2] \in 2 \cdot \mathbb{Z}/(4) \cong \mathbb{Z}/(2) \ni [1]$

$[2] \mapsto [0] \in \mathbb{Z}/(2) = F(\mathbb{Z}/(2))$, $[1] \in \mathbb{Z}/(2)$

$\mathbb{Z}/(2) \cong F(\mathbb{Z}/(2))$

$$\mathbb{Z}/(4) \xrightarrow{\psi} \mathbb{Z}/(2)$$

$$[1] \longmapsto [1]$$

$$(*)\quad [2] \longmapsto 2\cdot[1] = [2] = [0] = 0$$

not exact here

break @ 13:18 until 13:25

back only

exact

counterexample: $A = B := \mathbb{Z}$

$$\underline{\underline{E}} = \left(0 \to \mathbb{Z}/(3) \xrightarrow{\varphi} \mathbb{Z}/(9) \xrightarrow{\psi} \mathbb{Z}/(3) \to 0\right)$$

exact

$$[1] \longmapsto [3] \qquad [1] \longmapsto [1]$$

$$3\cdot[1] \longmapsto [0] = 3\cdot[1]$$

$$G(\underline{\underline{E}}) = \left(0 \longrightarrow \mathbb{Z}/(3) \xrightarrow[0]{G(\varphi)} \mathbb{Z}/(3) \xrightarrow[\cong]{G(\psi)} \mathbb{Z}/(3) \to 0\right)$$

not exact, exact ($\neq 0$), exact

$$[1] \longmapsto 3\cdot[1] = [3] = 0 \qquad [1] \longmapsto [1]$$

(2) use CRT:

$$\mathbb{Z}/(36) \cong \mathbb{Z}/(9) \times \mathbb{Z}/(4)$$

as rings: $\times$; as ab. group: $\oplus$

$$G : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B), \quad A = \mathbb{Z},\ B = \mathbb{Z}$$

$$\boxed{G := \mathbb{Z}/(3) \otimes_{\mathbb{Z}} (-)}$$

We will prove that G is right

Now: $G \oplus F$ is neither left exact nor right exact

$$\underline{\underline{D}} \oplus \underline{\underline{E}} = \begin{bmatrix} \underline{\underline{D}} \\ \underline{\underline{E}} \end{bmatrix} = \left(\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/(2) & \xrightarrow{\varphi} & \mathbb{Z}/(4) & \xrightarrow{\psi} & \mathbb{Z}/(2) & \to & 0 \\ 0 & \to & \mathbb{Z}/(3) & \to & \mathbb{Z}/(9) & \to & \mathbb{Z}/(3) & \to & 0 \end{array}\right)$$

$$[1] \longmapsto [2] \qquad [1] \longmapsto [1]$$

$$[1] \longmapsto [3] \qquad [1] \longmapsto [1]$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What is $(F \oplus G)(\underline{\underline{D}} \oplus \underline{\underline{E}})$?

it is: $= \begin{bmatrix} F \\ G \end{bmatrix}\left(\begin{bmatrix} \underline{\underline{D}} \\ \underline{\underline{E}} \end{bmatrix}\right)$

$$\left(\begin{array}{c} 0 \to \begin{bmatrix} \mathbb{Z}/(2) \\ \mathbb{Z}/(3) \end{bmatrix} \xrightarrow{\begin{smallmatrix} \cong & \\ & 0 \end{smallmatrix}} \begin{bmatrix} \mathbb{Z}/(2) \\ \mathbb{Z}/(3) \end{bmatrix} \xrightarrow{\begin{smallmatrix} 0 & \\ & \cong \end{smallmatrix}} \begin{bmatrix} \mathbb{Z}/(2) \\ \mathbb{Z}/(3) \end{bmatrix} \to 0 \end{array}\right)$$

$2 \cdot [\mathbb{Z}/(4)] \cong \mathbb{Z}/(2)$

not exact — exact — not exact

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{Z})$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad 2 \cdot (\mathbb{Z}/(4)) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\cong$ $\mathbb{Z}/4 \to \mathbb{Z}/2$

$$\boxed{\begin{array}{ccc} (\mathbb{Z}/(2)) \times (\mathbb{Z}/(3)) & \cong & \mathbb{Z}/(6) \\ ([1],[1]) & \leftarrow\!\shortmid & [1] \\ (\mathbb{Z}/(4)) \times (\mathbb{Z}/(9)) & \cong & \mathbb{Z}/(36) \\ \cup & & \cup \\ ([1],[1]) & \leftarrow\!\shortmid & [1] \end{array}}$$

ring isomorphism

φ: $([1],[1]) \mapsto ([2],[3])$, $[1] \mapsto [30]$

$\underline{\underline{C}} \cong \underline{\underline{D}} \oplus \underline{\underline{E}}$:

$[2] = [30]$ in $\mathbb{Z}/(4)$
$[3] = [30]$ in $\mathbb{Z}/(9)$

$$\left(0 \to \mathbb{Z}/(6) \xrightarrow{\varphi} \mathbb{Z}/(36) \xrightarrow{\psi} \mathbb{Z}/(6) \to 0\right)$$

$[1] \longmapsto [30] = -[6]$, $\quad [1] \longmapsto [1]$

$(F \oplus G)(\underline{\underline{D}} \oplus \underline{\underline{E}})$

$$\left(0 \to \mathbb{Z}/(6) \to \mathbb{Z}/(18) \to \mathbb{Z}/(6) \to 0\right)$$

fishy ?

$$\left(0 \to \mathbb{Z}/(6) \xrightarrow{6\cdot(-)} \mathbb{Z}/(36) \xrightarrow{1\cdot(-)} \mathbb{Z}/(6) \to 0\right)$$

$\underline{\underline{D}} \oplus \underline{\underline{E}}$

$[1] \longmapsto [6]$ $\quad [1] \longmapsto [1] \ne 0$

exact

end of live lecture.

$\Downarrow$ correction below.

$\mathrm{Ker}(\psi) = 6 \cdot \frac{\mathbb{Z}}{36}$ gen. by $[6]$

$\mathrm{Im}(\varphi) = 30 \cdot \frac{\mathbb{Z}}{(36)}$ gen by $[30] = -[6]$

so $\underline{\mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)}$

$\mathrm{Ker}(\varphi) = 0 \qquad \mathrm{Im}(\psi) = \mathbb{Z}/(6)$

Now what is the functor

$H : \underline{Ab} \to \underline{Ab}$ corresponding to $F \oplus G$?

The sequence $H(\underline{e})$ is this:

$$\left(0 \to \mathbb{Z}/(6) \xrightarrow{H(\varphi)} \mathbb{Z}/(6) \xrightarrow{H(\psi)} \mathbb{Z}/(6) \to 0\right)$$

$$[1] \longmapsto [3] \qquad [1] \longmapsto [4] = -[2]$$

calc: $(F \oplus G)(\varphi) = ([1],[1]) = ([1],[0])$

$$\underset{\mathbb{Z}/(2) \oplus \mathbb{Z}/(3)}{([1],[0])} \longleftarrow \underset{\mathbb{Z}/(6)}{[3]}$$

$(F \oplus G)(\psi)([1],[1]) = ([0],[1])$

$$\underset{\mathbb{Z}/(2) \oplus \mathbb{Z}/(3)}{([0],[1])} \longleftarrow \underset{\mathbb{Z}/(6)}{[4]}$$

not injective $\mathrm{Ker}(H(\varphi)) = 3 \cdot (\mathbb{Z}/6) \neq 0$

not surj. $\mathrm{Im}(H(\psi)) = 2 \cdot (\mathbb{Z}/(6)) \subsetneq \mathbb{Z}/(6)$

$\mathrm{Ker}(H(\psi)) = 3 \cdot (\mathbb{Z}/(6))$

$\mathrm{Im}(H(\varphi)) = 3 \cdot \mathbb{Z}/(6)$ "