

Homological Algebra

Lecture 11, 29.12.21

1k base comm ring start of 12:15
 A & B central
 1k-ring, all linear ops. are K -linear.

Def. Suppose we have lin. functors

$$F, G: \text{Mod}(A) \rightarrow \text{Mod}(B)$$

the linear functor

$$F \oplus G: \text{Mod}(A) \rightarrow \text{Mod}(B)$$

to be: for $M \in \text{Mod}(A)$

$$(F \oplus G)(M) := F(M) \oplus G(M) \in \text{Mod}(B)$$

do:

Morph: for $(M_0 \xrightarrow{\varphi} M_1)$ in $\text{Mod}(A)$

$$(F \oplus G)(\varphi) := (F \oplus G)(M_0) = F(M_0) \oplus G(M_0) \xrightarrow{F(\varphi) \downarrow \quad G(\varphi) \downarrow} F(M_1) \oplus G(M_1) = (F \oplus G)(M_1)$$

in $\text{Mod}(B)$

Prove that $F \oplus G$ is indeed a K -linear functor.

Solution of Ex. 7.20

$$F := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), -)$$

We will prove later that F is left exact. Counterexample showing that F is not right exact:

$$D := (0 \rightarrow \mathbb{Z}/(2) \xrightarrow{\varphi} \mathbb{Z}/(4) \rightarrow \mathbb{Z}/(2) \rightarrow 0)$$

$\downarrow \quad \downarrow \quad \downarrow$
 $[1] \quad [1] \quad [1]$

$$\text{Ab} = \text{Mod}(\mathbb{Z})$$

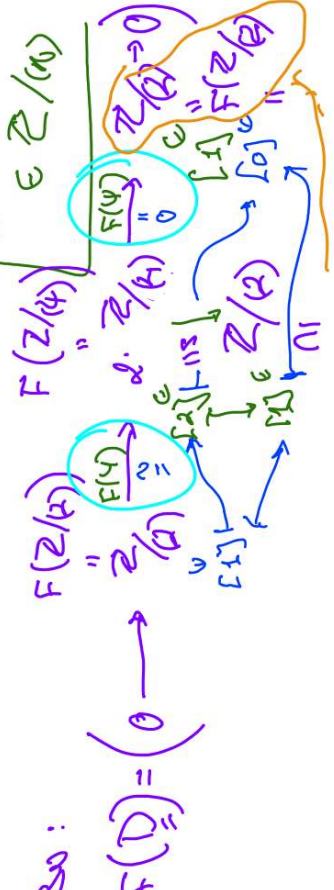
$$A = B = \mathbb{Z}$$

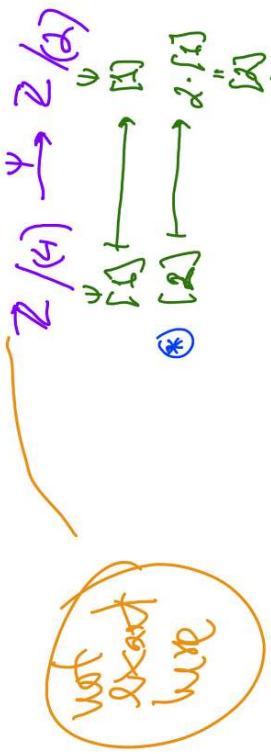
$$M \in \mathbb{Z},$$

$$N := \mathbb{Z} \oplus \mathbb{Z}$$

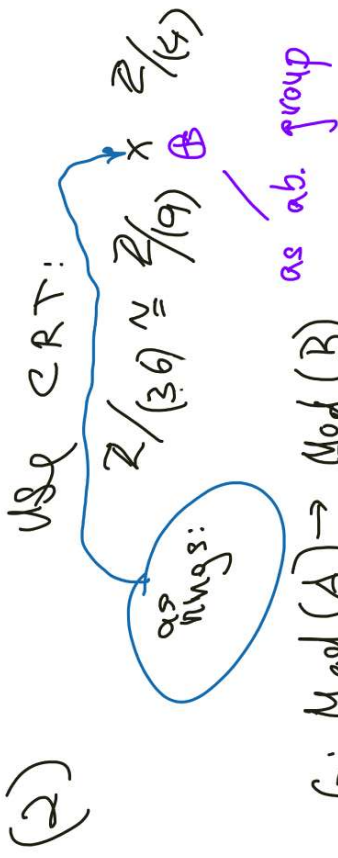
$\mathbb{Z}/(n)$ is a finite comm ring (also an ab. grp)

$$F(N) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/(2), N) \cong \begin{cases} n \in \mathbb{Z} & (2 \cdot n = 0) \\ 2 \cdot \mathbb{Z} \oplus \mathbb{Z} & (2 \cdot n \neq 0) \end{cases}$$





break @ 13:18
until 13:25
back early



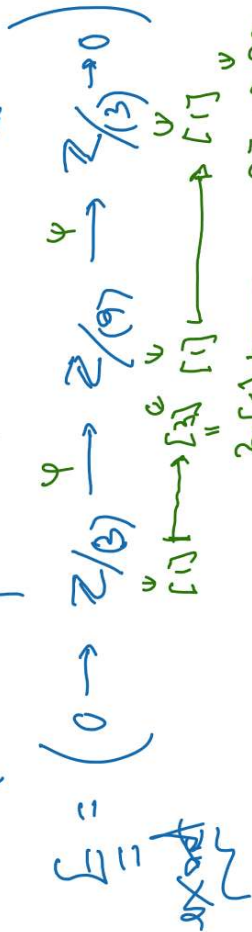
$$G: \text{Mod}(\mathbb{Z}/(A)) \rightarrow \text{Mod}(\mathbb{Z}/(B))$$

$$G := \mathbb{Z}/(2) \otimes_{\mathbb{Z}} (-)$$

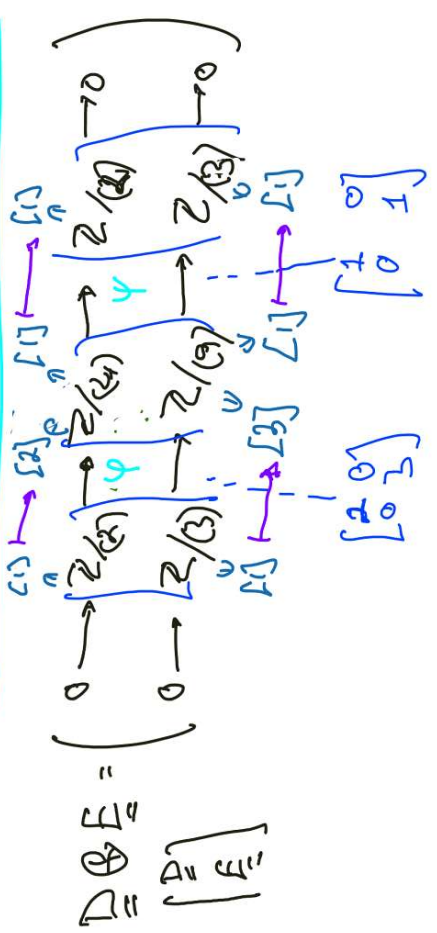
We will prove that G is ~~not~~ right

exact.

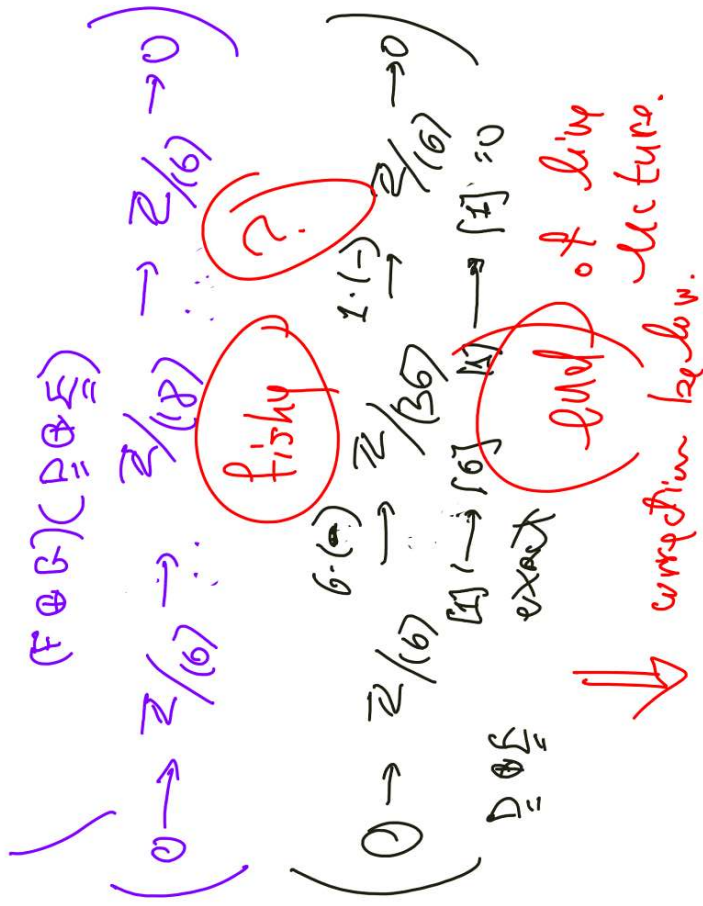
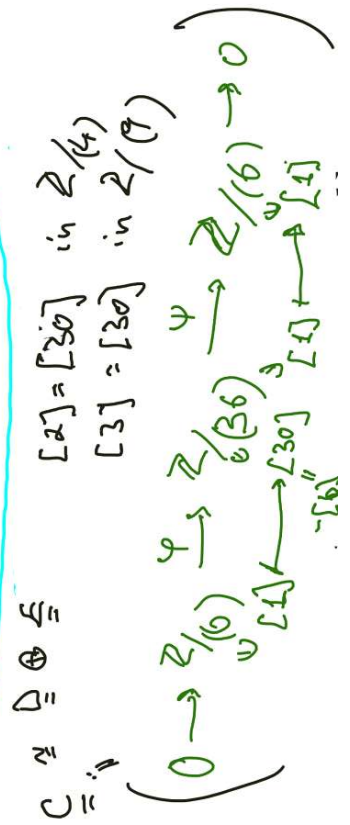
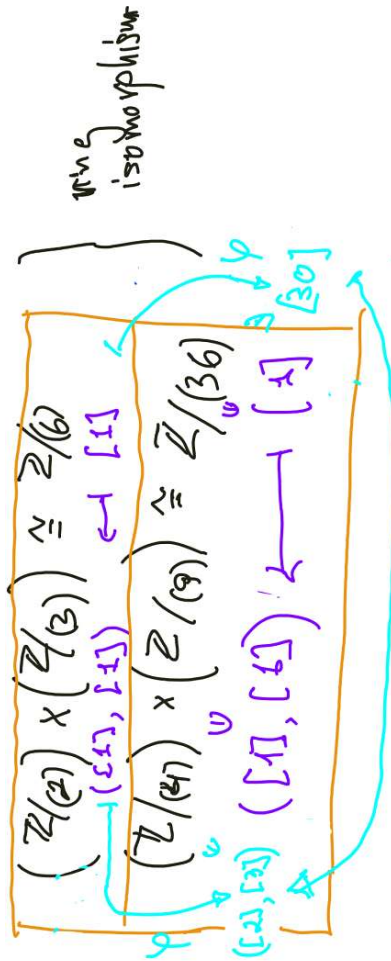
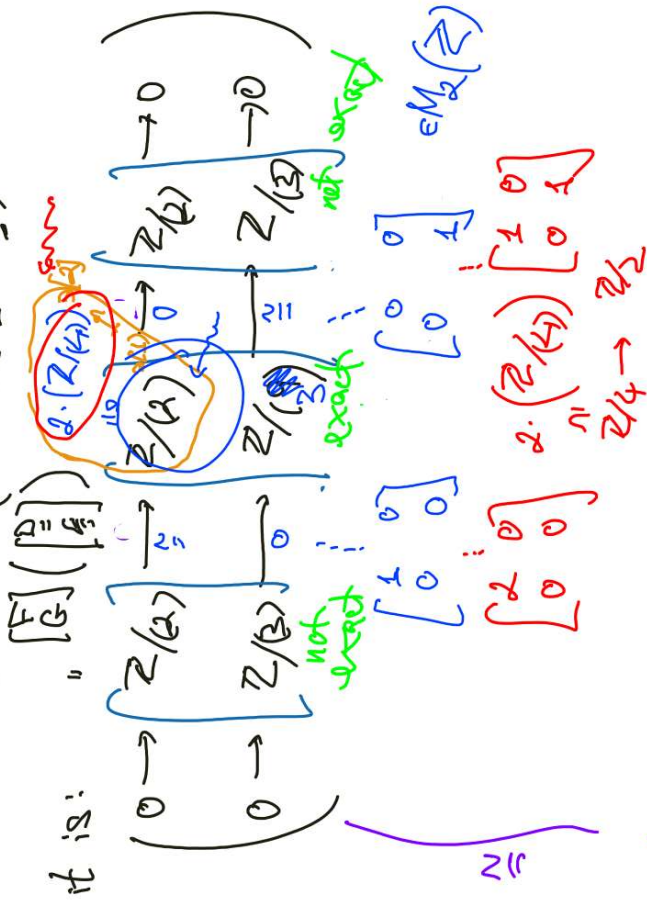
counter example: ~~not~~ $A = B = \mathbb{Z}$



now: $G \oplus F$ is ~~not~~ exact



What is $(F \oplus G)(D \oplus E)$?



$\text{Ker}(\varphi) = 6 \cdot \frac{\mathbb{Z}}{36}$ gen. by $[6]$
 $\text{Im}(\varphi) = 30 \cdot \frac{\mathbb{Z}}{36}$ gen. by $[30] = -[6]$
 so $\text{Im}(\varphi) = \text{Ker}(\varphi)$
 $\text{Ker}(\varphi) = 0$ $\text{Im}(\varphi) = \mathbb{Z}/(6)$

Now what is the functor
 $H: \mathcal{A} \rightarrow \mathcal{A}$ corresponding to
 $F \oplus G$?

The sequence $H(\underline{0})$ is thus:

$$0 \rightarrow \mathbb{Z}/(6) \xrightarrow{H(\varphi)} \mathbb{Z}/(6) \xrightarrow{H(\psi)} \mathbb{Z}/(6) \rightarrow 0$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ \mathbb{Z} & \xrightarrow{[1]} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{[4]} & \mathbb{Z} \end{matrix}$$

calc: $(F \oplus G)(\varphi) = ([1], [1]) = ([1], [0])$

$$([1], [0]) \longleftarrow [3]$$

$$\mathbb{Z}/(2) \oplus \mathbb{Z}/(6) \xrightarrow{\quad} \mathbb{Z}/(6)$$

$$(F \oplus G)(\psi) ([1], [1]) = ([0], [1])$$

$$([0], [1]) \longleftarrow [4]$$

$$\mathbb{Z}/(2) \oplus \mathbb{Z}/(6) \xrightarrow{\quad} \mathbb{Z}/(6)$$

not injective $\text{Ker}(H(\varphi)) = 3 \cdot (\mathbb{Z}/(6)) \neq 0$

not surj: $\text{Im}(H(\psi)) = 2 \cdot (\mathbb{Z}/(6)) \neq \mathbb{Z}/(6)$

$$\begin{aligned} \text{Ker}(H(\varphi)) &= 3 \cdot (\mathbb{Z}/(6)) \\ \text{Im}(H(\psi)) &= 3 \cdot (\mathbb{Z}/(6)) \end{aligned}$$