Lecture 12 of Homologic. Alg

5 Jan 2022

$B := \mathbb{Z}/(36)$

$$\underline{E} := \boxed{0 \to \mathbb{Z}/(6) \xrightarrow{\varphi} \mathbb{Z}/(36) \xrightarrow{\psi} \mathbb{Z}/(6) \to 0}$$

$\underline{\mathrm{Mod}}(B)$

$\varphi(\bar{i}) = 6 \cdot \bar{i}$ $\qquad \psi(\bar{i}) = \underline{i}$

$F \circ E = G \circ F$

$F := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(18), -)$

$G := \mathbb{Z}/(12) \otimes_{\mathbb{Z}} (-)$

$\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}} B$ $\qquad \mathbb{Z}_{(3)} \otimes_{\mathbb{Z}} B$

CRT: with $B = \mathbb{Z}/(36) \xrightarrow[\text{ring isom}]{\exists !} \mathbb{Z}/(4) \times \mathbb{Z}/(9)$ in $\underline{\mathrm{Ring}}_c$

$\underline{E}_{(2)}$: $0 \to \mathbb{Z}/(2) \xrightarrow{2\cdot} \mathbb{Z}/(4) \to \mathbb{Z}/(2) \to 0$

$\underline{E}_{(3)}$: $0 \to \mathbb{Z}/(3) \xrightarrow{3\cdot} \mathbb{Z}/(9) \to \mathbb{Z}/(3) \to 0$

$F_{(2)} = \mathrm{Hom}_{\mathbb{Z}/(4)}(\mathbb{Z}/(2), -)$

$F_{(3)} = \mathrm{Hom}_{\mathbb{Z}/(9)}(\mathbb{Z}/(9), -)$

$G_{(2)}$ $\qquad G_{(3)}$

(2) $\quad (3)$ $\quad (5)$ $\quad (7)$ $\quad (11)$

$\mathrm{MSpec}(\mathbb{Z})$ $\qquad$ line

$\mathrm{MSpec}(\mathbb{C}[t]) \cong \mathbb{C}$

$(t-\lambda) \mapsto \lambda$

$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ $\quad \lambda \neq \mu$

$(4) \cap (2) \cap \mathbb{Z} \cup (3) \cup (9)$ $\quad$ max

$\lambda \approx 2$

$\mu \approx 3$ $\rightsquigarrow$ $\left[\begin{array}{c|c} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} & 0 \\ \hline 0 & \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \end{array}\right] \in M_4(\mathbb{C})$ $\cong M_2(M_2(\mathbb{C}))$

B "geom"

$(t-\lambda)$ max $\supseteq (t-\lambda)^2$

$\mathbb{C}[t]$

$(t-\mu) \supseteq (t-\mu)^3$

$M = \mathbb{C}^4 = M_1 \oplus M_2$

$M_1 \cong B/(t-\lambda)^2$ $\qquad M_2 \cong B/(t-\mu)^2$

break to 13:10

2nd hour

Now for genuine theory

Def Let $F : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$ be some linear functor. (contravariant)

We say that F is left exact (resp. right exact, res. exact) if for every short exact seq.

$\underline{\underline{E}} = \left(0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0\right)$

in $\underline{\mathrm{Mod}}(A)$, the seq.

(*) $0 \to F(M') \xrightarrow{F(\psi)} F(M) \xrightarrow{F(\varphi)} F(M'') \to 0$

right exact

left exact

exact

(**)

as seq. in $\underline{\mathrm{Mod}}(B)$

(***)

Rem The main point in (abelian, i.e. linear) homological is to make sense or give meaning to the failure of exactness of functor.

given a linear functor

$F : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$

we will construct its left right derived functors, which are lin. fu

i.e. $L_i F : \underline{\mathrm{Mod}}(A) \to \underline{\mathrm{Mod}}(B)$

$F = M \otimes_A (-)$

$L_i F = \mathrm{Tor}_i^A(M, -)$

$R^i F$

$F = \mathrm{Hom}_A(M, -)$

$R^i F = \mathrm{Ext}^i_A(M, -)$

(*) F is exact iff $L_i F = 0$ for all $i > 0$,

iff $R^i F = 0$ for all $i > 0$.

NEXT SEMESTER

splitting short exact sequences

in $\underline{\mathrm{Mod}}(A)$

$\tau \circ \varepsilon = id_{M'}$ $\quad$ τ? $\quad$ σ? $\quad$ $\pi \circ \sigma = id_{M''}$

exact $\underline{\underline{E}} = \left(0 \to M' \xrightarrow{\varepsilon} M \xrightarrow{\pi} M'' \to 0\right)$

$\downarrow \cong$ $\quad$ id $\quad$ $\cong$ $\quad$ t? $\quad$ $\downarrow$ id

splitting $\underline{\underline{E}}$:

$\underline{\underline{E}}^* = \left(0 \to M' \xrightarrow{\begin{bmatrix} id \\ 0 \end{bmatrix}} M' \oplus M'' \xrightarrow{[0, id]} M'' \to 0\right)$

exc have t, or a σ, or a τ, are equiv.

$A = \mathbb{Z}$

$0 \to \mathbb{Z}/(2) \xrightarrow{2\cdot(\,)} \mathbb{Z}/(4) \xrightarrow{\pi} \mathbb{Z}/(2) \to 0$

$M' \xrightarrow{\varphi} M \xrightarrow{\pi} M''$

does not split