Homological Algebra

Lecture 13, 12 Jan 2021

8. Morphisms of Functors

Ab categories
MO

Def Suppose $\underline{C}$ & $\underline{D}$ are categories, and

$$F, G : \underline{C} \longrightarrow \underline{D}$$

are functors. A morphism of functors (aka natural transformation)

$$\boxed{\eta : F \longrightarrow G}$$

is a collection

$$\eta = \{\eta_C\}_{C \in Ob(\underline{C})}$$

of morphism

$$\boxed{\eta_C : F(C) \longrightarrow G(C)}$$

in $\underline{D}$ satisfying the condition: for every morphism $f : C_0 \longrightarrow C_1$ in $\underline{C}$, the diagram

$$(*) \quad \begin{array}{ccc} F(C_0) & \xrightarrow{F(f)} & F(C_1) \\ \eta_{C_0}\downarrow & & \downarrow \eta_{C_1} \\ G(C_0) & \xrightarrow{G(f)} & G(C_1) \end{array}$$

in $\underline{D}$ is commutative.

Example ($\mathbb{K} = A$) A is comm. $A = B$. Let $\varphi : L \longrightarrow M$ be a hom. in $\underline{\mathrm{Mod}}(A)$.

Consider the A-lin. functors:

$$F, G : \underline{\mathrm{Mod}}(A) \longrightarrow \underline{\mathrm{Mod}}(A)$$

$$\boxed{F := L \otimes_A (-)}$$

$$G := M \otimes_A (-).$$

For every $N \in \mathrm{Mod}(A)$ consider the hom

$$\begin{array}{ccc} F(N) & \xrightarrow{\eta_N} & G(N) \\ \|\!\downarrow & & \downarrow\!\| \\ L \otimes_A N & \xrightarrow{\varphi \otimes \mathrm{id}_N} & M \otimes_A N \end{array}$$

we know ...

$l \otimes n \longmapsto \varphi(l) \otimes n$

$$\boxed{\eta_N := \varphi \otimes \mathrm{id}_N}$$

Claim: $\eta = \{\eta_N\}_{N \in \underline{\mathrm{Mod}}(A)}$ is a morphism of functors $F \to G$.

Pf: We need to check that (*) is commutative for every $\psi : N_0 \to N_1$ in $\underline{\mathrm{Mod}}(A)$. $(= \underline{C})$

$l \otimes n \longmapsto l \otimes \psi(n)$

$$\begin{tikzcd}
L \otimes_A N_0 = F(N_0) \arrow[r, "F(\psi)"] \arrow[d, "\eta_{N_0}"'] \arrow[dr, phantom, "(*)"] & F(N_1) = L \otimes_A N_1 \arrow[d, "\eta_{N_1}"] \\
M \otimes_A N_0 = G(N_0) \arrow[r, "G(\psi)"', "\mathrm{id}_M \otimes \psi"] & G(N_1) = M \otimes_A N_1
\end{tikzcd}$$

$\mathrm{id}_L \otimes \psi$, $\varphi \otimes \mathrm{id}_{N_0}$, $\varphi \otimes \mathrm{id}_{N_1}$

in $\underline{\mathrm{Mod}}(A)$ $(= \underline{D})$

$\varphi(l) \otimes n \longmapsto \varphi(l) \otimes \psi(n)$

$\square$

end 1st hour

break to 13:21

2nd hour

9. Equivalence of Categories

Def Let $\underline{C}$ & $\underline{D}$ be categories and $F, G : \underline{C} \to \underline{D}$ functors, and $\eta : F \to G$ a morphism of functors.

We say that η is an isomorphism of functors (aka natural isom.) if for object $C \in \underline{C}$ the morphism

$$\boxed{\eta_C : F(C) \to G(C)}$$

$\exists!\ \zeta_C$ in $\underline{D}$

in $\underline{D}$ is an isomorphism.

Prop Suppose $\eta : F \to G$ (as above) is an isom. of functors. Then there is a morph. of functors

$$\xi : G \to F$$

st. $\quad \eta \circ \xi = id_G \qquad \underline{\text{pf}} \quad \xi = \{\xi_c\}$

$$\xi \circ \eta = id_F \qquad \xi_c := (\eta_c)^{-1}.$$

Exercise

Given cat's $\underline{C}$ & $\underline{D}$

and functors $F_i : \underline{C} \to \underline{D}$, $i = 0, 1, \ldots$

and morphisms of functors

$$\eta_i : F_i \longrightarrow F_{i+1}$$

They can be composed:

$$\eta_1 \circ \eta_2 : F_0 \to F_2$$

etc; and for every $F : \underline{C} \to \underline{D}$

there the $\overbrace{}^{\text{id. morph.}}$ $id_F : F \to F$

which does η : $\forall\ c \in \underline{C}$,

$(id_F)_c : F(c) \xrightarrow{id_{F(c)}} F(c_0)$ in $\underline{D}$

$\eta_c := id_{F(c)}$

(in $\underline{D}$)

Def Let $\underline{C}$ & $\underline{D}$ be cat's and let $F : \underline{C} \to \underline{D}$ be a functor. We call F an equivalence

of categories if there exists a functor

$$G : \underline{D} \to \underline{C}$$

and isomorphisms of functors

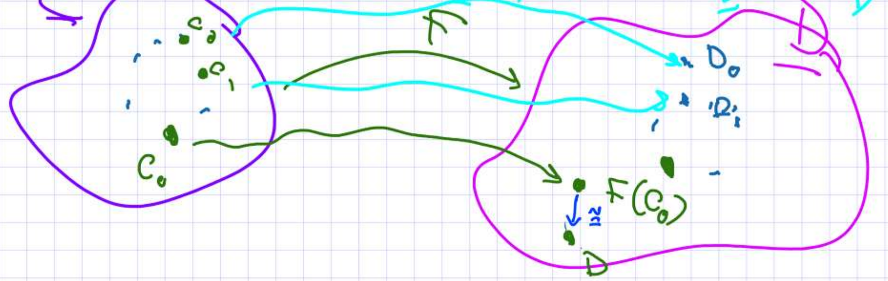
$$\eta : Id_{\underline{C}} \xrightarrow{\sim} G \circ F$$

$$\xi : Id_{\underline{D}} \xrightarrow{\sim} F \circ G$$

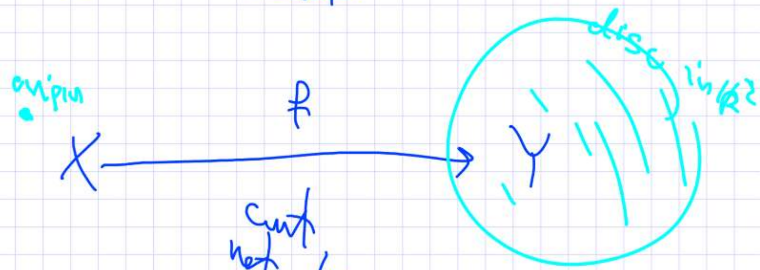
the id. functor of the cat $\underline{C}$

Thm $F: \underline{C} \to \underline{D}$ is an (equiv. of cat's) iff (F is essentially surjective (on objects) and fully faithful.)

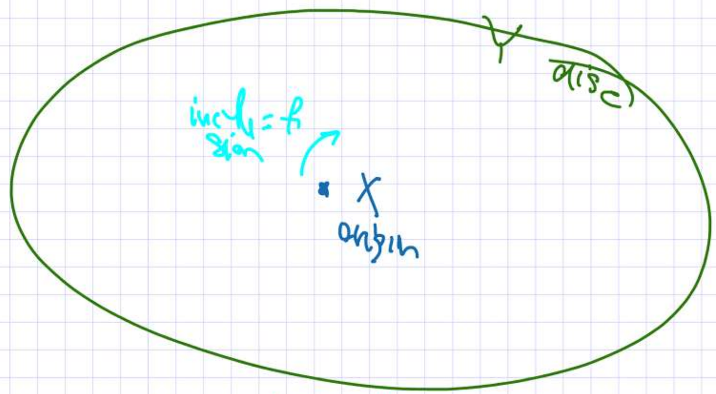
$F: \mathrm{Hom}_{\underline{C}}(C_0, C_1) \xrightarrow{\cong} \mathrm{Hom}_{\underline{D}}(D_0, D_1)$



topology



cont
not homeomorphism,
but homotopy equivalence



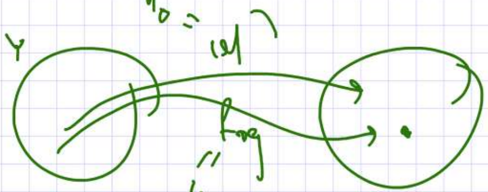
$f: X \to Y$ inclusion

$g: Y \to X$ constant.

$\boxed{g \circ f = id_X}$

$f \circ g = ?$

$f \circ g \underset{\text{homotopic}}{\sim} id_Y$



$h: [0,1] \times Y \to Y$ contin.

$t \in [0,1]$, $y \in Y$

$h_0 = id$

$h_1 = f \circ g$

$h(t, y) = (1-t) \cdot y$

end