

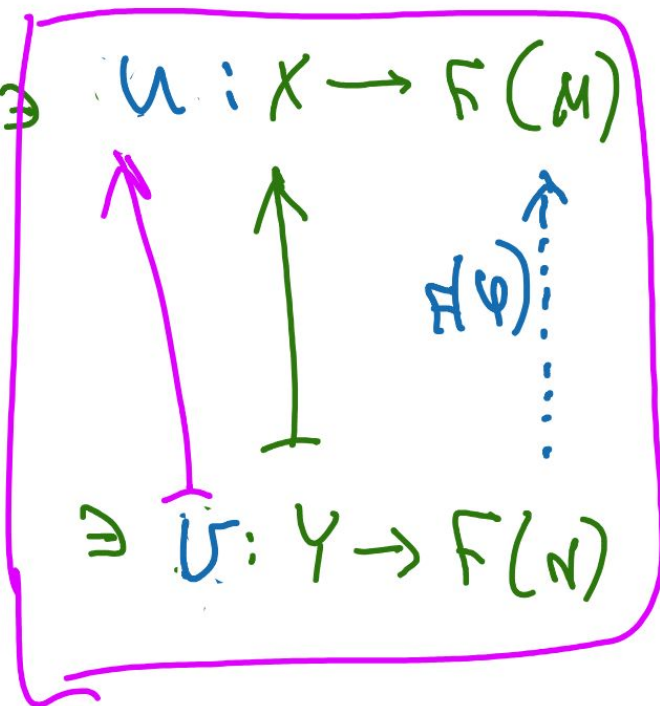
Homological Algebra

lecture 9, 15 Dec 2021

left side of the diagram

$$\text{Hom}_{\text{Set}}(X, F(M)) \ni \uparrow$$

$$\text{Hom}_{\text{Set}}(Y, F(N))$$

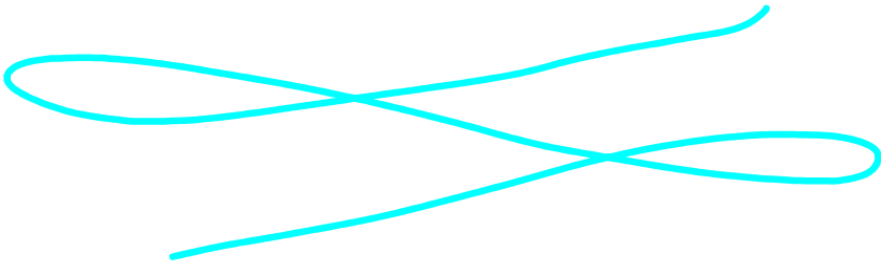
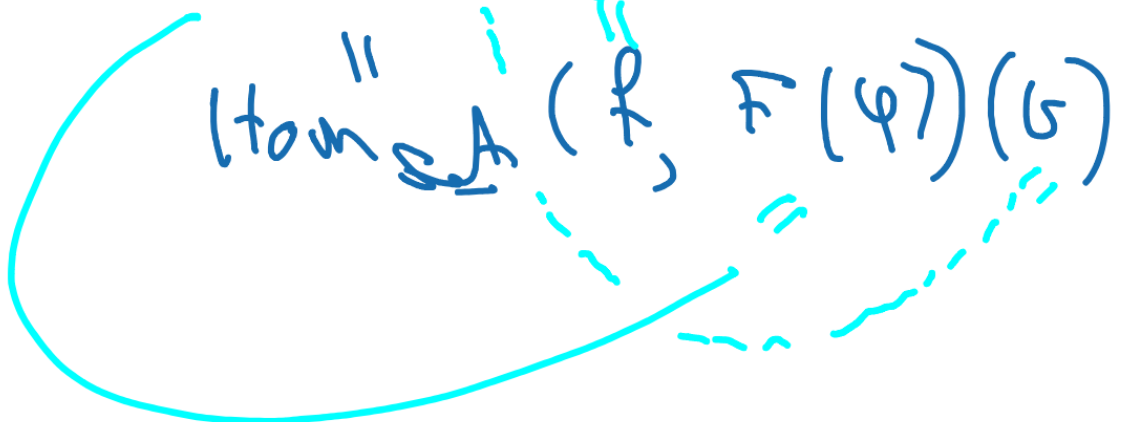


$$u(x) = F(Y)(v(f(x)))$$

$$= (F(Y) \circ v \circ f)(x)$$

$$\mathcal{U} = F(\psi) \circ \mathcal{U} \circ \mathcal{P} : X \rightarrow F(N)$$

$$\text{Hom}_{\mathcal{A}}(P, F(\psi)(\mathcal{U}))$$

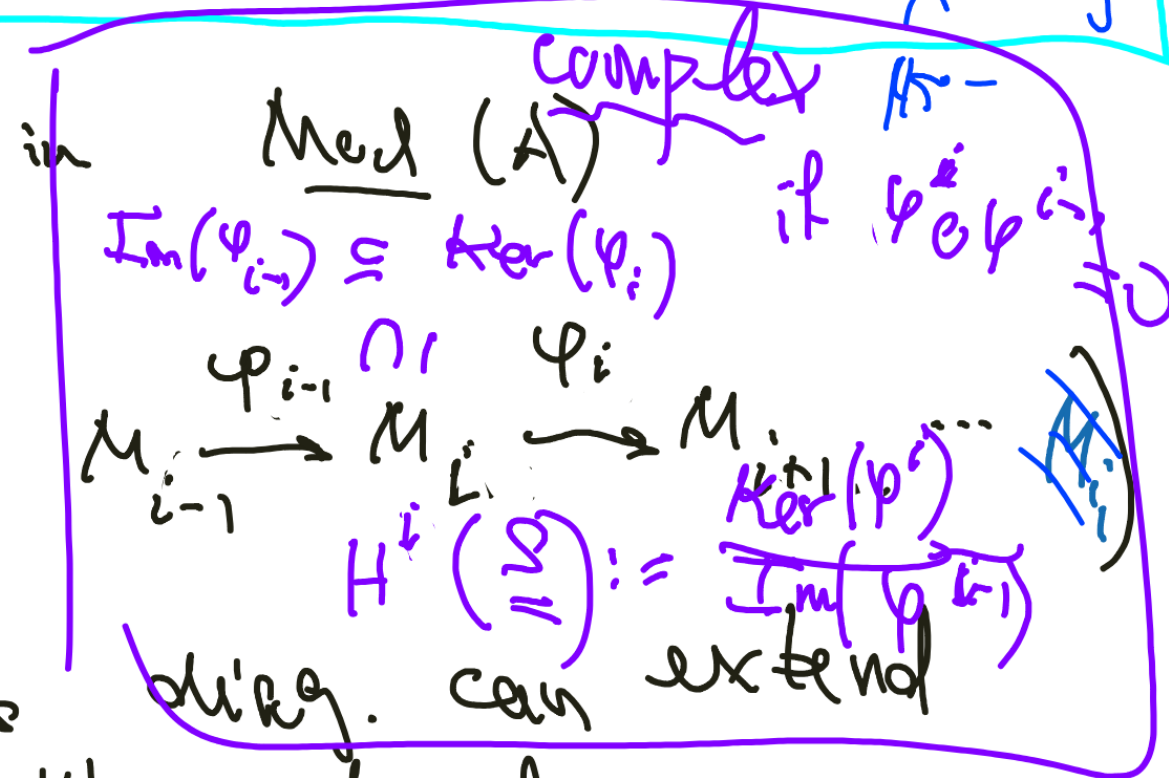


lect 9

7. Exact Linear Functor

A & B are commutative rings

(7.2) is a commutative diagram



is in $\text{Mod}(A)$. This can extend infinitely in either direction;

If \mathcal{S} has a last object,

Say $\begin{cases} M_{i-1} \\ M_i \end{cases}$

An obj. in \mathcal{S} which is neither first or last, is called interval

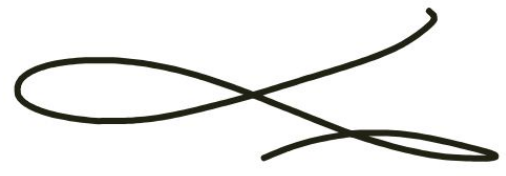
Def 7.3 (exactness)

(1) \mathcal{S} is said to be exact at some interval object M_i

if
$$\text{Im}(\psi_{i-1}) = \text{Ker}(\psi_i)$$

are submodules of M_i

\mathcal{C} is called an exact sequence
 if it's exact at all its
 internal objects.



Def 7.4 A short exact sequence
 is is an exact seq. of the shape:

(7.5) $0 \rightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \rightarrow 0$

Vertical lines with orange ticks are drawn under each object: 0 , M' , M , M'' , and 0 .

Example

Take a module M and a submodule $M' \subseteq M$.

Define $M'' = M/M'$.

$$\text{Ther} = \left(0 \rightarrow M' \xrightarrow{\text{inclusion}} M \xrightarrow{\pi} M'' \rightarrow 0 \right)$$

inclusion can. prof.

(Note: Orange arrows in the original image point from the text above to the corresponding terms in the sequence: from 'M' to the first M', from 'M' to the second M', from 'M'' to M'', and from '0' to the final 0. A double arrow points from 'M'' to the second M'.)

This is a ~~short~~ short exact sequence

(isom of ex. seq)

exercise

Why "exact" ?

Example X is an n -dimensional
differentiable manifold

(e.g. \mathbb{R}^n , with its C^∞ \mathbb{R} -valued functions) $\left(\begin{array}{c} \text{type } C \\ \mathbb{R} \end{array} \right)$

For every $0 \leq p \leq n$ we have the
deg. p differential forms:

$$\Omega^p(X) = \bigoplus_{1 \leq i_1 < \dots < i_p \leq n} \mathcal{P}(X) \cdot dt_{i_1} \wedge \dots \wedge dt_{i_p}$$

$X = \mathbb{R}^n$
 t_1, \dots, t_n

$$\Omega^0(X) = \mathcal{O}(X)$$

$C^\infty(X)$?

diff. functions $f: X \rightarrow \mathbb{R}$

$$d^p : \Omega^p(X) \rightarrow \Omega^{p+1}(X)$$

$$d^p \circ d^p = 0$$

$$\mathbb{R}^2 :$$

$$d^0(f) =$$

$$f_1 dt_1 + f_2 dt_2$$

$$f_1 = \frac{\partial}{\partial t_1} (f)$$

$$\frac{\partial f_2}{\partial t_2}$$

$$d^1(f_1 dt_1 + f_2 dt_2) = \cancel{df_1 dt_1} + \cancel{df_2 dt_2}$$

$$\begin{aligned}
 & \left(\underbrace{d^0(f_1)}_{\rightarrow} \cdot dt_1 + f_1 \cdot \cancel{d(d(t_1))} \right) \\
 & + \left(\underbrace{d(f_2)}_{\rightarrow} \cdot dt_2 + f_2 \cdot \cancel{d(d(t_2))} \right) \\
 \Rightarrow & \frac{\partial f_1}{\partial t_2} \cdot dt_2 - \frac{\partial f_2}{\partial t_1} \cdot dt_1
 \end{aligned}$$

$$\boxed{\|d^1 \circ d^0 = 0\|}$$

$$w \in \Omega^p(x) \text{ s.t. } d(w) = 0$$

\uparrow

closed

$$\omega = d(\rho), \quad \rho \in \Omega^{p-1}$$

exact

Poincaré Lemma

Consider

$$\begin{pmatrix} X = \mathbb{R}^2 \\ n=2 \end{pmatrix}$$

the seq.

(*)

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X) \rightarrow 0$$

augmented de Rham complex
exact

If X is simply connected
the $(*)$ is an exact sequence.

Since $d^p \circ d^{p-1} = 0$

$$\Rightarrow \text{Im}(d^{p-1}) \subseteq \text{Ker}(d^p) \subseteq \Omega^p(X)$$

$$\Rightarrow H_{DR}^p(X) = \frac{\text{Ker}(d^p)}{\text{Im}(d^{p-1})}$$

p -th de Rham
cohomology of X ,

$$X = \mathbb{R}^2 - \{0\}$$

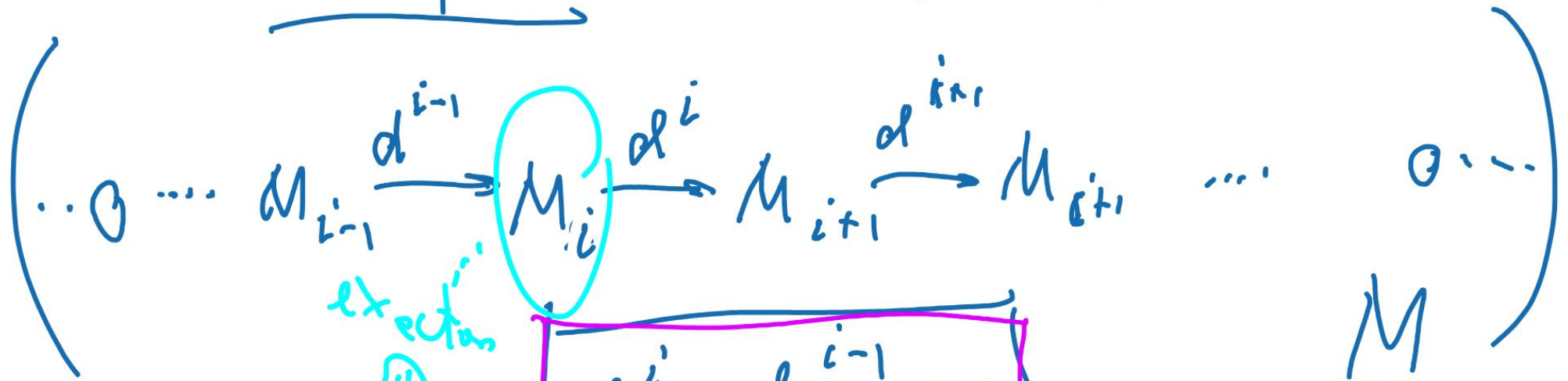
$$H^0_{DR}(X) = \mathbb{R}$$

$$H^2(X) = 0 \quad (?)$$

$$H^1(X) \cong \mathbb{R}$$

$d(\theta) \rightarrow \theta$ angle

complex in $\text{Mod}(A)$:



exact
 \uparrow
 $H^i(\underline{M}) =$

$$d^i \circ d^{i-1} = 0$$

$$H^i(\underline{M}) := \frac{\text{Ker}(d^i)}{\text{Im}(d^{i+1})}$$

i -th cobon. of \underline{N}

$$\underline{M}$$

