Homological Algebra

lecture 9, 15 Dec 2021

left side of the diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\underline{Set}}(X, F(M)) \ni & u : X \arrow[r] & F(M) \\
\mathrm{Hom}_{\underline{Set}}(Y, F(N)) \arrow[u] \ni & v : Y \arrow[r] \arrow[u, mapsto] & F(N) \arrow[u, dotted, "F(\varphi)"]
\end{tikzcd}
$$

$$u(x) = F(\varphi)(v(f(x))) \qquad F(\varphi)$$

$$= (F(\varphi) \circ v \circ f)(x)$$

$$u = F(\varphi) \circ v \circ f : X \longrightarrow F(N)$$

$$\parallel$$

$$\mathrm{Hom}_{\underline{s\mathcal{A}}}(f, F(\varphi))(v)$$

lect 9

7. Exact Linear Functor

A & B are commutative rings.

A sequence in $\underline{\mathrm{Mod}}(A)$ complex if $\varphi^{i}\circ\varphi^{i-1}=0$

$\mathrm{Im}(\varphi_{i-1}) \subseteq \mathrm{Ker}(\varphi_i)$

is a diagram

(7.2) $$\underline{S} = \left(\cdots M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \cdots \right)$$

$$H^i(\underline{S}) := \frac{\mathrm{Ker}(\varphi^i)}{\mathrm{Im}(\varphi^{i-1})}$$

in $\underline{\mathrm{Mod}}(A)$. This diag. can extend infinitely in either direction;

If $\underline{\underline{S}}$ has a $\begin{cases} \text{last} \\ \text{first} \end{cases}$ object, say $\begin{cases} M_{i_1}. \\ M_{i_0} \end{cases}$

An obj. in $\underline{\underline{S}}$ which is neither first or last, is called internal

Def 7.3 (exactness)

1. $\underline{\underline{S}}$ is said to be exact at some internal object M_i if

$$\mathrm{Im}(\varphi_{i-1}) = \mathrm{Ker}(\varphi_i)$$

as submodules of M_i

(2) $\mathcal{S}$ is called an exact sequ. if it's exact at all its internal objects.

Def 7.4 A short exact sequence is is an exact seq. of this shape:

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0 \tag{7.5}$$

Example Take a module M and a submodule $M' \subseteq M$. Define $M'' := M/M'$.

$$E := \left(0 \longrightarrow M' \xrightarrow[\text{inclusion}]{\varepsilon} M \xrightarrow[\text{can. proj.}]{\pi} M'' \longrightarrow 0 \right)$$

This is a short exact sequence

(isom of ex. seq) $\boxed{\text{exercise}}$

Why "exact" ?

Example X is an n-dimensional differentiable manifold (e.g. $\boxed{\mathbb{R}^n}$, with its C^∞ $\mathbb{R}$-valued function) (type C^∞ over $\mathbb{R}$)

For every $0 \leq p \leq n$ we have the deg. p differential forms:

$X = \mathbb{R}^n$

$t_1, \dots, t_n$

$$\Omega^p(X) = \bigoplus \mathcal{O}(X) \cdot dt_{i_1} \wedge \dots \wedge dt_{i_p}$$

$0 \leq i_1 < \dots < i_p \leq n$

$$\Omega^0(X) = \mathcal{O}(X) \qquad C^\infty(X) \ ?$$

diff. functions $f : X \to \mathbb{R}$

$$d^p : \Omega^p(X) \to \Omega^{p+1}(X) \qquad d^p \circ d^{p-1} = 0$$

$\mathbb{R}^2$:

$$d^0(f) = f_1 \cdot dt_1 + f_2 \cdot dt_2$$

$$f_1 = \frac{\partial}{\partial t_1}(f) \qquad \frac{\partial f_2}{\partial t_2}$$

$$d^1(f_1 \cdot dt_1 + f_2 \cdot dt_2) =$$

$$\left(d'(f_1) \wedge dt_1 + f_1 \cdot d(\cancel{d(t)}) \right)$$

$$+ \left(d(f_2) \wedge dt_2 + f_2 \cdot d(\cancel{d(t_2)}) \right)$$

$$= \frac{\partial f_1}{\partial t_2} \cdot dt_2 - \frac{\partial f_2}{\partial t_1} \cdot dt_1$$

$$\boxed{d^1 \circ d^0 = 0}$$

$\omega \in \Omega^p(X)$ s.t. $d(\omega) = 0$

closed

$\omega = d(\rho)$, $\rho \in \Omega^{p-1}$

exact

Poincaré Lemma

Consider $\begin{pmatrix} X = \mathbb{R}^2 \\ n = 2 \end{pmatrix}$ the seq.

$$(*) \quad 0 \to \mathbb{R} \longrightarrow \boxed{\underset{\text{"}\Omega^0(X)\text{"}}{\mathcal{O}(X)} \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^1} \Omega^2(X)} \longrightarrow 0$$

augmented de Rham complex

exact

If X is simply connected

the $(*)$ is an exact sequence.

Since $d^p \circ d^{p-1} = 0$

$\Rightarrow$ $\mathrm{Im}(d^{p-1}) \subseteq \mathrm{Ker}(d^p) \subseteq \Omega^p(X)$

$$\Rightarrow \quad H^p_{DR}(X) := \frac{\mathrm{Ker}(d^p)}{\mathrm{Im}(d^{p-1})}$$

p-th de Rham cohomology of X.

$$\boxed{X = \mathbb{R}^2 - \{0\}}$$

$$H^0_{DR}(X) = \mathbb{R}$$

$$H^2(X) = 0 \quad (?)$$

$$H^1(X) \cong \mathbb{R}$$

$d(\theta)$, θ angle

complex in Mod (A):

$$\underline{M} = \left(\cdots 0 \cdots M_{i-1} \xrightarrow{d^{i-1}} M_i \xrightarrow{d^i} M_{i+1} \xrightarrow{d^{i+1}} M_{i+1} \cdots 0 \cdots \right)$$

$$\boxed{d^i \circ d^{i-1} = 0}$$

exaction

$\Updownarrow$

$H^i(\ ) =$

$$H^i(\underline{M}) := \frac{\operatorname{Ker}(d^i)}{\operatorname{Im}(d^{i-1})}$$

i-th cohom. of $\underline{M}$