An introductory talk about basic notions of Geometric Group Theory, with emphasis on asymptotic invariants of groups.
I plan to talk about the following asymptotic invariants of groups:

1. Volume growth.
2. Gromov hyperbolicity.
3. Asymptotic dimension.

I’ll start by introducing basic notions of the theory.
What is Geometric Group Theory?

A simple definition of Geometric group theory is that it is the study of groups as geometric objects.

Thinking about groups this way was popularized by Gromov who revolutionized the subject of infinite groups. This is a new field, so there are many fundamental theorems waiting to be discovered, and it is a rich field, lying at a juncture between algebra and topology, where a great variety of methods from other branches of mathematics can be utilized. Geometric group theory draws upon techniques from, and solves problems in the theory of 3-manifolds, hyperbolic geometry, combinatorial group theory, Lie groups...

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Let $G$ be a finitely generated group.

Let $S$ be a finite generator set of $G$ and let

$$S^{-1} = \{ g \in G : g^{-1} \in S \}.$$ 

For an element $g$ of $G$ let

$$|g| = \min \{ n : \exists \{s_1,s_2,\ldots,s_n\} \subset S \cup S^{-1} \ g = s_1 s_2 \cdots s_n \}.$$ 

be the minimum length of a word in alphabet $S \cup S^{-1}$ that represents $g$ in $G$.

A map $d : G \times G \to [0, \infty)$ defined by the formula

$$d(g, h) = |g^{-1} h|$$

is a metric on $G$. We call it the **word length metric**.
A Cayley graph of $\langle G, S \rangle$

Recall from the previous slide:

\[ d(g, h) = |g^{-1}h|, \quad \text{where } |g| = \min \{ n: \exists \{s_1, s_2, \ldots, s_n\} \subset S \cup S^{-1} \quad g = s_1 s_2 \cdots s_n \}. \]

Consider an undirected graph with a vertex set $G$ and a set of edges

\[ E = \left\{ \{g_1, g_2\} \subset G: g_1^{-1}g_2 \in S \cup S^{-1} \text{ and } g_1 \neq g_2 \right\}. \]

Realize it as a simplicial complex with uniform geodesic metric, i.e. with the maximal metric in which every edge is isometric to the unit interval $[0, 1]$. We call it a **Cayley graph** of $G$.

The distance from a vertex $g$ to a vertex $h$ in the Cayley graph of $G$ is equal to

\[ \min \{ n: \text{there exists a path from } g \text{ to } h \text{ with } n + 1 \text{ vertices} \}. \]

It coincides with the word length metric on $G$, as every path in the graph corresponds to a word in alphabet $S \cup S^{-1}$.
A Cayley graph of $\mathbb{Z}^2$

With a standard generator set \( \{ a = (1, 0), b = (0, 1) \} \).
A Cayley graph of $\mathbb{F}_2$ (a free group on two generators)

With a standard generator set \{a, b\}.
A Cayley graph of $\mathbb{F}_3$

With a standard generator set \{\(a, b, c\)\}. 
A Cayley graph of $\mathbb{Z}_3 \ast \mathbb{Z}_6$

Let $G$ and $H$ be groups.

We say that a word \( g_1 h_1 g_2 h_2 \cdots g_n h_n \)
in alphabet $G \sqcup H$ is **reduced**, if
\[
g_i \in G, h_i \in H, g_i \neq e \text{ for } i > 1 \text{ and } h_i \neq e \text{ for } i < n.
\]

A **free product** $G \ast H$ of $G$ and $H$ is a group whose elements are reduced words in alphabet $G \sqcup H$, with multiplication defined to be a concatenation followed by cancellation that transforms the concatenated word to a reduced form.

If $S$ is a generator set of $G$ and $T$ is a generator set of $H$, then $S \cdot e \sqcup e \cdot T$ is a generator set of $G \ast H$. 
A Cayley graph of $\mathbb{Z}_3 \ast \mathbb{Z}_6$

With a generator set $\{a, b\}$, where $a$ generates $\mathbb{Z}_3$ and $b$ generates $\mathbb{Z}_6$. 
A Cayley graph of Baumslag-Solitar group $BS(2, 1)$

The Baumslag-Solitar group $B(m, n)$ is given by the group presentation

$$\langle a, b \mid a^n = ba^mb^{-1} \rangle.$$

A Cayley graph of $B(2, 1)$ with a generator set $\{a, b, a^{-1}, b^{-1}\}$:

(in fact, this is a picture of a discrete model of a hyperbolic plane)
Gromov-Hausdorff convergence

A Gromov-Hausdorff distance between metric spaces $X$ and $Y$ is defined to be the infimum of the Hausdorff distances between isometric embeddings of $X$ and $Y$ into other metric spaces.

A metric space $(X, d)$ seen from the infinity is the Gromov-Hausdorff limit of a sequence $(X, d/n)$ of metric spaces.

Examples:

1. A bounded space seen from the infinity is a point.
2. $\mathbb{Z}^2$ seen from the infinity is $\mathbb{R}^2$. 
Dependence on a generator set

A Cayley graph of $\mathbb{Z}$ with a generator set $\{1\}$:

A Cayley graph of $\mathbb{Z}$ with a generator set $\{2, 3\}$:
Quasi-isometries

A map $f$ from a metric space $X$ into a metric space $Y$ is a **quasi-isometric embedding** if there exist positive real numbers $C$ and $D$ such that for every two points $x_1$ and $x_2$ in $X$ we have

$$\frac{1}{C} d(x_1, x_2) - D \leq d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) + D.$$

We say that $f$ is a **quasi-isometry**, if additionally its image is $R$-dense in $Y$ for some $R < \infty$, i.e. if

$$\exists_{R < \infty} \forall y \in Y \exists_{x \in X} d(f(x), y) < R.$$

The Cayley graphs from the previous slide are quasi-isometric.
Examples and non-examples of quasi-isometries

Examples of quasi-isometries:
1. An inclusion $\mathbb{Z}^2 \subset \mathbb{R}^2$ (both spaces with euclidean metrics)
2. A function $f: \mathbb{R} \to \mathbb{Z}$ given by the formula $f(x) = \lfloor x \rfloor$.
3. A constant map $X \to \{0\}$, for any bounded space $X$.

Examples of maps that are not quasi-isometries:
1. $n \to n^2$.
2. $n^2 \to n^3$.

Examples of non-quasi-isometric metric spaces:
1. Euclidean and hyperbolic plane.

Examples of quasi-isometric groups:
1. Any two finite groups.
2. $\mathbb{Z}$ and $\mathbb{Z}_2 \ast \mathbb{Z}_2$.
3. Any two "virtually $G$" groups.
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Independence of a generator set

**Lemma.** Cayley graphs corresponding to different generator sets of a group are quasi-isometric.

Notation:

$S, T$ - two finite generator sets of a group $G$.

$|g|_S$ and $|g|_T$ - minimal word lengths for $S$ and $T$.

$d_S(g, h) = |g^{-1}h|_S$ and $d_T(g, h) = |g^{-1}h|_T$ - word length metrics on $G$. 
A geometric action of a group $G$ on a metric space $X$ is a group action such that:

1. $G$ acts on $X$ by isometries,
2. the action is properly discontinuous, i.e. for every compact set $K \subset X$ the set $\{g \in G : g(K) \cap K \neq \emptyset\}$ is finite,
3. the quotient $X/G$ is compact in the quotient topology.

Švarc-Milnor Lemma. If a group $G$ acts geometrically on a proper geodesic space $X$, then it is finitely generated and quasi-isometric to $X$. 
Asymptotic invariants:

Volume growth
Volume growth

Let $X$ be a metric space. We let $\text{Vol}_X(t)$ be the supremum of cardinalities of closed $t$-balls in $X$. We say that the map $\text{Vol}_X : \mathbb{R}_+ \to \mathbb{N} \cup \{\infty\}$ is the \textbf{volume growth} of $X$.

We say that $X$ is of \textbf{bounded geometry} if $\text{Vol}_X(t) < \infty$ for each $t$.

Let $f, g : \mathbb{R}_+ \to \mathbb{N}$. We write that

a) $f \preceq g$, if $f(r) \leq C_1 (g(C_2 r + 1) + 1)$ for some constants $C_1, C_2$ and all $r$;

b) $f \sim g$, if $f \preceq g$ and $g \preceq f$;

c) $f \prec g$, if $f \preceq g$ but it is not true that $f \sim g$.

\textbf{Proposition.} If $X$ and $Y$ are quasi-isometric and of bounded geometry, then $\text{Vol}_X \sim \text{Vol}_Y$. 
We have

\[ \text{Vol}_{\mathbb{Z}^n}(t) \sim t^n. \]

If \( n \neq m, \) then \( t^n \not\sim t^m, \) hence \( \mathbb{Z}^n \) is not quasi-isometric to \( \mathbb{Z}^m \) for \( n \neq m. \)
Volume growth of $BS(2, 1)$

We have

$$\text{Vol}_{BS(2, 1)}(t) \sim 2^t.$$ 

Note that for each $a, b > 1$ we have $a^t \sim b^t$ and $t^a \not\sim b^t$.

There are groups with intermediate volume growth; so called Grigorchuk groups.
Gromov’s polynomial growth theorem

**Gromov theorem.** A group $G$ has polynomial volume growth if and only if it is virtually nilpotent.

A group $G$ has **polynomial volume growth**, if $\operatorname{Vol}_G(t) \preceq t^n$ for some $n$.

A group $G$ is **virtually nilpotent**, if it contains a nilpotent group of finite index.

A group $G$ is **nilpotent**, if its lower central series terminate in a trivial group.

**Lower central series** of a group $G$ is a sequence $G_1 = G$ and $G_{n+1} = [G_n, G] = \{g^{-1}h^{-1}gh: g \in G_1, h \in G\}$. 