סטטיסטיקה מתמטית | תרגיל 2

אומד שיטת המומנטים..(MME)

1) מצא את MME עבור ההתפלגויות הבאות
ופרמטר θ (לא ידוע)

a). $B(m,p)$, $\theta=(m,p)$ b). $N(0,\sigma^2)$, $\theta=\sigma$

c). $\Gamma(\alpha,\lambda)$, $\theta=(\alpha,\lambda)$ d). $N(\mu,\sigma^2)$, $\theta=(\mu,\sigma^2)$

e). $U(a,b)$, $\theta=(a,b)$ f). $U(a,b)$, $\theta=b-a$

g). $U(a,2a)$, $\theta=a$ h). $U(-a,a)$, $\theta=a$

i). $\chi^2_m=\Gamma\left(\frac{m}{2},\frac{1}{2}\right)$, $\theta=m$ j). $NB(m,p)$. $\theta=(m,p)$

2) מצא את MME עבור θ כאשר

a) $$f(x,\theta)=\frac{1}{\theta}-\frac{|x|}{\theta^2}\ ,\quad |x|\le\theta .$$

b). $\text{lognormal}(\mu,\sigma^2)$,, $\theta=(\mu,\sigma^2)$

c) $$f(x,\theta)=\theta\, x^{\theta-1}\ ,\quad 0<x\le 1\ ,\quad Beta(\theta,1)$$

d) $$f(x,\theta)=\frac{2(\theta-x)}{\theta^2}\ ,\quad 0\le x\le\theta$$

מומנט המדגם
מסדר r

$$M'_r=\frac{1}{n}\left(X_1^r+X_2^r+\dots+X_n^r\right)$$

$$r=1,2,3,\dots$$

$$M'_1=\bar X$$

Table I DISCRETE DISTRIBUTIONS

Name of parametric family of distributions	Discrete density functions $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Discrete uniform $U_d(N)$	$f(x) = \frac{1}{N} I_{\{1,\ldots,N\}}(x)$	$N = 1, 2, \ldots$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\mu_2' = \frac{N(N+1)^2}{4}$; $\mu_4' = \frac{(N+1)(2N+1)(3N^2+3N-1)}{30}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli $B(1, p)$	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	$0 \le p \le 1$ $(q = 1-p)$	p	pq	$\mu_r' = p$ for all r	$q + pe^t$
Binomial $B(n, p)$	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\ldots,n\}}(x)$	$0 \le p \le 1$; $n = 1, 2, 3, \ldots$ $(q = 1-p)$	np	npq	$\mu_3 = npq(q-p)$; $\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$	$(q + pe^t)^n$
Hypergeometric $H(M, K, n)$	$f(x) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} I_{\{0,1,\ldots,n\}}(x)$	$M = 1, 2, \ldots$; $K = 0, 1, \ldots, M$; $n = 1, 2, \ldots, M$	$n\frac{K}{M}$	$n\frac{K}{M}\frac{M-K}{M}\frac{M-n}{M-1}$	$\mathscr{E}[X(X-1)\cdots(X-r+1)] = r!\frac{\binom{K}{r}\binom{n}{r}}{\binom{M}{r}}$	not useful
Poisson $P(\lambda)$	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$	$\lambda > 0$	λ	λ	$\kappa_r = \lambda$ for $r = 1, 2, \ldots$; $\mu_3 = \lambda$; $\mu_4 = \lambda + 3\lambda^2$	$\exp[\lambda(e^t - 1)]$
Geometric $G(p)$	$f(x) = pq^{x-1} I_{\{1,2,\ldots\}}(x)$	$0 < p \le 1$ $(q = 1-p)$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\mu_3 = \frac{q+q^2}{p^3}$; $\mu_4 = \frac{q+7q^2+q^3}{p^4}$	$\frac{pe^t}{1-qe^t}$
Negative binomial $NB(m, p)$	$f(x) = \binom{x-1}{m-1} p^m q^{x-m} I_{\{m, m+1, \ldots\}}$	$0 < p \le 1$	$\frac{M}{P}$	$\frac{rq}{p^2}$	$\mu_3 = \frac{r(q+q^2)}{p^3}$; $\mu_4 = \frac{r[q+(3r+4)q^2+q^3]}{p^4}$	

$$VS^2 = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n-1}\sum X_i^2 - \frac{n}{n-1}\overline{X}^2$$

$$B(a, b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

$$\Gamma(n) = (n-1)!$$

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Table 2 CONTINUOUS DISTRIBUTIONS

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Uniform or rectangular $U(a, b)$	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$-\infty < a < b < \infty$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\mu_r = 0$ for r odd; $\mu_r = \frac{(b-a)^r}{2^r(r+1)}$ for r even	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal $N(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x-\mu)^2/2\sigma^2]$	$-\infty < \mu < \infty$, $\sigma > 0$	μ	σ^2	$\mu_r = 0$, r odd; $\mu_r = \frac{r!}{(r/2)!}\frac{\sigma^r}{2^{r/2}}$, r even; $\kappa_r = 0$, $r > 2$	$\exp[\mu t + \frac{1}{2}\sigma^2 t^2]$
Exponential $Exp(\lambda)$	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\mu_r' = \frac{\Gamma(r+1)}{\lambda^r}$	$\frac{\lambda}{\lambda - t}$ for $t < \lambda$
Gamma $\Gamma(r, \lambda)$	$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$, $r > 0$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$	$\mu_j' = \frac{\Gamma(r+j)}{\lambda^j \Gamma(r)} = \frac{\alpha(\alpha+1)\cdots(\alpha+r-1)}{\lambda^r}$	$\left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$
Beta $Beta(a, b)$	$f(x) = \frac{1}{B(a, b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$	$a > 0$, $b > 0$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$	$\mu_r' = \frac{B(r+a, b)}{B(a, b)}$	not useful
Cauchy	$f(x) = \frac{1}{\pi\beta\{1 + [(x-\alpha)/\beta]^2\}}$	$-\infty < \alpha < \infty$, $\beta > 0$	Does not exist	Does not exist	Do not exist	Characteristic function is $e^{i\alpha t - \beta\lvert t\rvert}$
Lognormal	$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp[-(\log_e x - \mu)^2/2\sigma^2] I_{(0,\infty)}(x)$	$-\infty < \mu < \infty$, $\sigma > 0$	$\exp[\mu + \frac{1}{2}\sigma^2]$	$\exp[2\mu + 2\sigma^2] - \exp[2\mu + \sigma^2]$	$\mu_r' = \exp[r\mu + \frac{1}{2}r^2\sigma^2]$	not useful
Double exponential	$f(x) = \frac{1}{2\beta} \exp\left(-\frac{\lvert x-\alpha\rvert}{\beta}\right)$	$-\infty < \alpha < \infty$, $\beta > 0$	α	$2\beta^2$	$\mu_r = 0$ for r odd; $\mu_r = r!\,\beta^r$ for r even	$\frac{e^{\alpha t}}{1 - (\beta t)^2}$

(continued)

$$B(a, b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

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Table 2 CONTINUOUS DISTRIBUTIONS (*continued*)

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Weibull $W(\alpha, \beta)$	$f(x) = abx^{b-1}\exp[-ax^b]I_{(0,\infty)}(x)$	$a > 0$ $b > 0$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1}) - \Gamma^2(1+b^{-1})]$	$\mu_r' = a^{-r/b}\Gamma\left(1+\frac{r}{b}\right)$	$\mathscr{E}[X^t] = a^{-t/b}\Gamma\left(1+\frac{t}{b}\right)$
Logistic	$F(x) = [1 + e^{-(x-\alpha)/\beta}]^{-1}$	$-\infty < \alpha < \infty$ $\beta > 0$	α	$\frac{\beta^2\pi^2}{3}$		$e^{\alpha t}\pi\beta t \csc(\pi\beta t)$
Pareto $Par(\theta, x_0)$	$f(x) = \frac{\theta x_0^\theta}{x^{\theta+1}} I_{(x_0,\infty)}(x)$	$x_0 > 0$ $\theta > 0$	$\frac{\theta x_0}{\theta - 1}$ for $\theta > 1$	$\frac{\theta x_0^2}{(\theta-1)^2(\theta-2)}$ for $\theta > 2$	$\mu_r' = \frac{\theta x_0^r}{\theta - r}$ for $\theta > r$	does not exist
Gumbel or extreme value	$F(x) = \exp(-e^{-(x-\alpha)/\beta})$	$-\infty < \alpha < \infty$ $\beta > 0$	$\alpha + \beta\gamma$, $\gamma \approx .577216$	$\frac{\pi^2\beta^2}{6}$	$\kappa_r = (-\beta)^r\psi^{(r-1)}(1)$ for $r \geq 2$, where $\psi(\cdot)$ is digamma function	$e^{\alpha t}\Gamma(1-\beta t)$ for $t < 1/\beta$
t distribution	$f(x) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \frac{1}{(1+x^2/k)^{(k+1)/2}}$	$k > 0$	$\mu = 0$ for $k > 1$	$\frac{k}{k-2}$ for $k > 2$	$\mu_r = 0$ for $k > r$ and r odd; $\mu_r = \frac{k^{r/2}B((r+1)/2, (k-r)/2)}{B(\frac{1}{2}, k/2)}$ for $k > r$ and r even	does not exist
F distribution	$f(x) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} \times \frac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}} I_{(0,\infty)}(x)$	$m, n = 1, 2, \ldots$	$\frac{n}{n-2}$ for $n > 2$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$ for $n > 4$	$\mu_r' = \left(\frac{n}{m}\right)^r \frac{\Gamma(m/2+r)\Gamma(n/2-r)}{\Gamma(m/2)\Gamma(n/2)}$ for $r < \frac{n}{2}$	does not exist
Chi-square distribution	$f(x) = \frac{1}{\Gamma(k/2)}\left(\frac{1}{2}\right)^{k/2} x^{k/2-1}e^{-(1/2)x}I_{(0,\infty)}(x)$	$k = 1, 2, \ldots$	k	$2k$	$\mu_j' = \frac{2^j\Gamma(k/2+j)}{\Gamma(k/2)}$	$\left(\frac{1}{1-2t}\right)^{k/2}$ for $t < 1/2$

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פתרונות

תרגיל 2

(1

(a $B(m, p)$ $\quad$ $\theta = (m, p)$.

MME

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k = M_1'$$

$$M_2' = \frac{1}{n}\sum_{k=1}^{n} X_k^2$$

$$\mu_1' = mp$$

$$\mu_2' = mpq + (mp)^2 = \mu_1' q + (\mu_1')^2$$

$$\mu_1' q = \mu_2' - (\mu_1')^2$$

$$p = 1 - q = 1 - \frac{\mu_2' - (\mu_1')^2}{\mu_1'} = \frac{\mu_1' - \mu_2' + (\mu_1')^2}{\mu_1'}$$

$$m = \frac{\mu_1'}{p} = \frac{(\mu_1')^2}{\mu_1' - \mu_2' + (\mu_1')^2}$$

$$\hat{p} = \frac{\bar{X} - M_2' + (\bar{X})^2}{\bar{X}}, \quad \hat{m} = \frac{(\bar{X})^2}{\bar{X} - M_2' + (\bar{X})^2}$$

(b $N(0, \sigma^2)$, $\sigma = \theta$

$$\hat{\sigma}^2 = M_2' - (\bar{X})^2$$

$$\hat{\sigma} = \sqrt{M_2' - (\bar{X})^2}$$

(c $\Gamma(\alpha, \lambda)$, $\theta = (\alpha, \lambda)$, $\mu_1' = \frac{\alpha}{\lambda}$, $\mu_2' = \frac{\alpha(\alpha+1)}{\lambda^2}$

$$\frac{\mu_2'}{\mu_1'} = \frac{\alpha+1}{\lambda} = \mu_1' + \frac{1}{\lambda}, \quad \frac{1}{\lambda} = \frac{\mu_2' - (\mu_1')^2}{\mu_1'}$$

$$\lambda = \frac{\mu_1'}{\mu_2' - (\mu_1')^2}, \quad \alpha = \lambda \mu_1' = \frac{(\mu_1')^2}{\mu_2' - (\mu_1')^2}$$

$$\hat{\lambda} = \frac{\bar{X}}{M_2' - (\bar{X})^2}, \quad \hat{\alpha} = \frac{(\bar{X})^2}{M_2' - (\bar{X})^2}$$

(d $N(\mu, \sigma^2)$ $\quad$ $\theta = (\mu, \sigma^2)$

$$\hat{\mu} = \bar{X}$$

$$\hat{\sigma}^2 = M_2' - (\bar{X})^2$$

$\therefore$ $\theta=(a,b)$, $U(a,b)$ (e (1 קרב'ל 2

$$\mu_1' = \frac{a+b}{2}$$

$$\mu_2' = \frac{a^2+ab+b^2}{3}$$

$$\begin{cases} \hat{a} = \bar{X} - \sqrt{3}\sqrt{M_2' - (\bar{X})^2} \\ \hat{b} = \bar{X} + \sqrt{3}\sqrt{M_2' - (\bar{X})^2} \end{cases}$$

$\theta = b-a$, $U(a,b)$ (f

$$\hat{\theta} = 2\sqrt{3}\sqrt{M_2' - (\bar{X})^2}$$

$\theta = a$, $U(a,2a)$. (g

$$\mu_1' = \frac{3}{2}a \qquad \hat{\theta} = \frac{2}{3}\bar{X}$$

$\theta = a$, $U(-a,a)$. (h

$$\mu_1' = 0$$

$$\mu_2' = \frac{(2a)^2}{12} = \frac{a^2}{3} \qquad \hat{a} = \sqrt{3M_2'}$$

$\theta = n$, $\chi_n^2 \sim \Gamma\left(\frac{n}{2},\frac{1}{2}\right)$ (i

$$\mu_1' = n \qquad \hat{n} = \bar{X}$$

$NB(m,p)$ (j

$$\theta = (m,p)$$

$$\mu_1' = \frac{m}{p} \qquad \mu_2' = \frac{mq}{p^2} + \left(\frac{m}{p}\right)^2 = \frac{m(q+m)}{p^2}$$

$$\mu_2' = \mu_1'\left(\frac{m+q}{p}\right) = \mu_1'\left(\mu_1' + \frac{q}{p}\right)$$

$$\frac{q}{p} = \frac{\mu_2'}{\mu_1'} - \mu_1' , \quad m = p\mu_1' = \frac{(\mu_1')^2}{\mu_2' - (\mu_1')^2 + \mu_1'}$$

$$p = \frac{\mu_1'}{\mu_2' - (\mu_1')^2 + \mu_1'}$$

$$\hat{m} = \frac{\bar{X}^2}{M_2' - \bar{X}^2 + \bar{X}}$$

$$\hat{p} = \frac{\bar{X}}{M_2' - \bar{X}^2 + \bar{X}}$$

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תרגיל 2

2)

a)

$0<\theta,\ |x|\le\theta \qquad f(x,\theta)=\dfrac{\theta-|x|}{\theta^2}$

$\mu_1'=0 \qquad \mu_2'=2\int_0^\theta x^2\,\dfrac{\theta-x}{\theta^2}\,dx=\dfrac{2}{\theta^2}\left(\dfrac{x^3\theta}{3}-\dfrac{x^4}{4}\right)\Big|_0^\theta=$

$\mu_2'=\dfrac{\theta^2}{6} \qquad \hat\theta=\sqrt{6M_2'}=\sqrt{\dfrac{6}{n}\sum_{k=1}^{n}X_k^2}$

b)

$\theta=(\mu,\sigma^2)$ lognormal(μ,σ^2)

$\mu_1'=e^{\mu+\frac12\sigma^2} \qquad 2\mu+\sigma^2=2\ln\mu_1$

$\mu_2'=e^{2\mu+2\sigma^2} \qquad 2\mu+2\sigma^2=\ln\mu_2'$

$\sigma^2=\ln\mu_2'-2\ln\mu_1'$

$\mu=\dfrac{\ln\mu_1^2-\sigma^2}{2} \qquad \sigma^2=\ln\dfrac{\mu_2'}{(\mu_1')^2}$

$\mu=\dfrac{2\ln\mu_1^2-\ln\mu_2'}{2} \qquad \hat\sigma^2=\ln\dfrac{M_2'}{(\bar X)^2}$

$\hat\mu=\dfrac12\ln\dfrac{\bar X^4}{M_2'}$

c)

$\mu_1'=\dfrac{\theta}{\theta+1} \qquad 0\le x\le 1,\ f(x,\theta)=\theta x^{\theta-1}$

$\boxed{\text{Beta}(\theta,1)}$

$\theta=\dfrac{\mu_1'}{1-\mu_1'} \qquad \hat\theta=\dfrac{\bar X}{1-\bar X}$

d)

$0\le x\le\theta, \qquad f(x,\theta)=\dfrac{2}{\theta^2}(\theta-x)$

$\mu_1'=EX=\dfrac{2}{\theta^2}\int_0^\theta x(\theta-x)\,dx=\dfrac{\theta}{3} \qquad \hat\theta=3\bar X$