פתרונות

תרגיל 3

סטטיסטיקה מתמטית

MLE - אומד הנראות המקסימלי

1) מצא את MLE עבור פרמטר θ.

a) $B(1,p)$, $\theta=p$, $\theta=1-p$, $\theta=\frac{p}{1-p}$

b) $P(\lambda)$, $\theta=\lambda$, $\theta=\frac{1}{\lambda}$.

c) $U_d(N)$, $\theta=N$

d) $B(m,p)$, $\theta=p$, m ידוע

e) (m,p), $\theta=p$, m ידוע

2) מצא את MLE עבור θ.

a) $N(\mu,1)$, $\theta=\mu$

b) $N(0,\sigma^2)$, $\theta=\sigma^2$

c) $N(0,\sigma^2)$, $\theta=\sigma$

d) $N(\mu,\sigma^2)$, $\theta=(\mu,\sigma^2)$

3) מצא את MLE עבור θ.

a) $U(a,b)$, $\theta=a$, b ידוע.

b) $U(a,b)$, $\theta=b$, a ידוע.

c) $U(a,b)$, $\theta=(a,b)$

d) $U(-\theta,\theta)$.

e) $U(\theta,2\theta)$.

f) $U(\mu-a,\mu+a)$, $\theta=(\mu,a)$.

g) $U(\mu-1,\mu+1)$, $\theta=\mu$. (האומד אינו יחיד!)

h) $U(-a,3a)$, $\theta=a$

4). מצא אומד MME ואומד MLE.

a) $f(x,\theta)=\frac{x}{\theta^2}e^{-\frac{x}{\theta}}$, $x\geq 0$, $\theta>0$

b) $f(x,\theta)=e^{\theta-x}$, $x\geq\theta$, $-\infty<\theta<+\infty$

c) $f(x,\theta)=\frac{\theta^{2x}}{x!}e^{-\theta^2}$, $x=0,1,2,\dots$, $\theta>0$

d) $f(x,\theta)=2x\theta^2$, $0\leq x\leq\frac{1}{\theta}$, $\theta>0$

e) $f(x,\theta)=\frac{2x}{1-\theta^2}$, $\theta\leq x\leq 1$

f) $f(x,\theta)=\theta x^{\theta-1}$, $0\leq x\leq 1$

האם MME ו MLE מוצדקים עבור $\theta>0$?

g) $f(x,\theta)=\frac{1}{\theta}-\frac{|x|}{\theta^2}$, $|x|\leq\theta$, $\theta>0$

(MLE רק כאשר $n=1$)

h) $f(x,a)=\frac{a}{x^{a+1}}$, $1\leq x$, $a>0$

תרגיל 3

<u>תשובות ופתרונות</u>

1)

a) $\bar{X}$, $1-\bar{X}$, $\bar{X}(1-\bar{X})$

b) $\bar{X}=\hat{\lambda}$, $\widehat{\left(\frac{1}{\lambda}\right)}=\frac{1}{\bar{X}}$

c) $\tilde{X}=\max(X_1 \dots X_n)$, $\hat{N}=\tilde{X}$

d) $\hat{p}=\frac{1}{m}\bar{X}$

e) $\hat{p}=\frac{m}{\bar{X}}$

2)

a) $\hat{\mu}=\bar{X}$

b) $\widehat{\sigma^2}=\frac{1}{n}\sum_{i=1}^{n}X_i^2=M_2'$

c) $\hat{\sigma}=\sqrt{M_2'}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}X_i^2}$

d) $\widehat{\sigma^2}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$, $\hat{\mu}=\bar{X}$

$$=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar{X}^2$$

3)

a) , b) , c)

$\hat{a}=\underset{\sim}{X}=\min(X_1 \dots X_n)$

$\hat{b}=\tilde{X}=\max(X_1 \dots X_n)$

$\boxed{3 \; 8'27}$

d) (3

$$f(x,a) = \frac{1}{2a}, \quad |x| \le a \qquad \underline{X \sim U(-a, a)}$$

$$\underline{a > 0}$$

$$L(x_1 \dots x_n, a) = \left(\frac{1}{2a}\right)^n, \quad |x_i| \le a, \quad i = 1, 2, \dots, n$$

$$\max |x_i| \le a$$

$$y = \min a = \max |x_i|$$

$$\max L = L(x_1 \dots x_n, y)$$

$$\hat{a} = Y = \max(|X_1|, |X_2|, \dots, |X_n|)$$

$\boxed{\text{b (3}}$

$$X \sim U(a, 2a)$$

$$\underline{a > 0}$$

$$f(x,a) = \frac{1}{a}, \quad a \le x \le 2a$$

$$L(x_1, \dots, x_n, a) = \frac{1}{a^n}, \quad a \le \underset{\sim}{x} \le \tilde{x} \le 2a$$

$$\frac{\tilde{x}}{2} \le a \le \underset{\sim}{x}$$

$$\min a = \frac{\tilde{x}}{2}$$

$$\hat{a} = \frac{1}{2}\tilde{X} \qquad \text{MLE}$$

$$\hat{b} = \max(X_1, \dots, X_n), \qquad X \sim U(a, b)$$

$$\underline{b > a}$$

$$\hat{b} = \tilde{X}$$

$$\hat{a} = \min(X_1 \dots X_n) = \underset{\sim}{X}$$

(3

(a

(b

(c

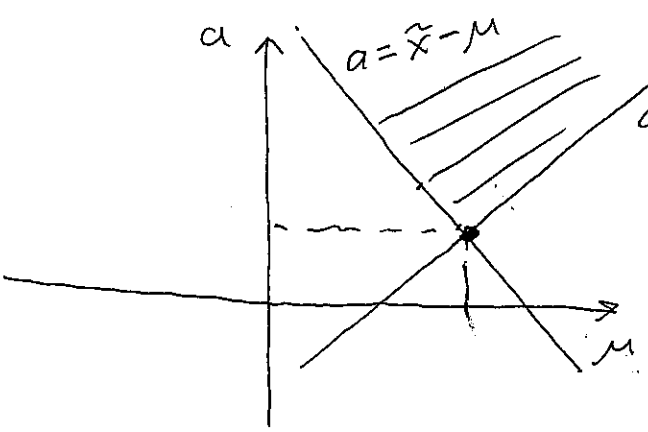
(f (3 $\boxed{\text{תרגיל } 3}$

$$f(x,\mu,a)=\frac{1}{2a}\ ;\ \mu-a\le x\le\mu+a$$

$$\theta=(\mu,a) \qquad X\sim U(\mu-a,\mu+a)$$

$$L(x_1,\dots x_n,\mu,a)=\left(\frac{1}{2a}\right)^n,\ x_i\in[\mu-a,\mu+a]$$
$$i=1,2,\dots,n$$

$$\mu-a\le \underset{\sim}{x}\le\tilde{x}\le\mu+a\ ,\ \begin{cases}\tilde{x}=\max_i x_i\\ \underset{\sim}{x}=\min x_i\end{cases}$$



בוחרים a מינימלי

ז"א $\left(\frac{1}{2a}\right)^n$ מקסימלי

$$\begin{cases}a=\mu-\underset{\sim}{x}\\ a=\tilde{x}-\mu\end{cases} \qquad \begin{cases}\underset{\sim}{x}=\mu-a\\ \tilde{x}=\mu+a\end{cases}$$

$$a=\frac{\tilde{x}-\underset{\sim}{x}}{2}\ ,\ \mu=\frac{\tilde{x}+\underset{\sim}{x}}{2}$$

$$\hat{\theta}=(\hat{\mu},\hat{a}) \qquad \hat{\mu}=\frac{\tilde{X}+\underset{\sim}{X}}{2}$$

$$\tilde{X}=\max(X_1\dots X_n) \qquad \hat{a}=\frac{\tilde{X}-\underset{\sim}{X}}{2}$$

$$\underset{\sim}{X}=\min(X_1,\dots X_n)$$

$$MLE(\theta)=\frac{\tilde{X}}{2} \qquad MME(\theta)=\frac{\bar{X}}{2}$$.(e .(3

$U(-a, 3a)$ (h (3 $\boxed{3\,\delta' z 7 \rho}$

$\underline{\theta = a}$, $a > 0$

$$f(x, a) = \frac{1}{4a}, \quad -a \le x \le 3a$$

$$L(x_1, \dots, x_n, a) = \left(\frac{1}{2a}\right)^n, \quad -a \le x_k \le 3a$$

$$k = 1, 2, \dots, n$$

$$\underset{\sim}{x} = \min(x_1 \dots x_n)$$

$$\tilde{x} = \max(x_1, \dots, x_n)$$

$$-a \le \underset{\sim}{x} \le \tilde{x} \le 3a$$

$$\begin{cases} a \ge \frac{1}{3}\tilde{x} \\ a \ge -\underset{\sim}{x} \end{cases} \qquad a \ge \max\left(\frac{\tilde{x}}{3}, -\underset{\sim}{x}\right)$$

MLE $\hat{a} = \max\left(\frac{1}{3}\max_{1\le k\le n} X_k, -\min_{1\le k\le n} X_k\right)$

MME $\hat{a} = \bar{X}$, $EX = a$

$$f(x, \theta) = \frac{2x}{1-\theta^2} I_{[\theta, 1]}(x)$$

(e (4

$$\mu_1' = EX = \frac{2}{3}\,\frac{1+\theta+\theta^2}{1+\theta}$$

$$MME(\theta) = \frac{1}{4}\left(3\bar{X} - 2 + \sqrt{9\bar{X}^2 + 12\bar{X} - 12}\right.$$

$$L = \frac{2^n \prod_i x_i}{(1-\theta^2)^n} I_{[\theta, 1]}(\underset{\sim}{x}) \cdot I_{[\theta, 1]}(\tilde{x})$$

$$(\theta \le \underset{\sim}{x} \le \tilde{x} \le 1)$$

$$MLE(\theta) = \underset{\sim}{X} = \min(X_1 \dots X_n)$$

$\boxed{3\ \text{תרגיל}}$ $\boxed{(ג\ 3)}$

? MLE $\quad \theta = \mu \quad \underline{X \sim U(\mu-1, \mu+1)}$

$$L(x_1 \dots x_n, \mu) = \frac{1}{2^n}, \quad \mu - 1 \le x_i \le \mu + 1$$

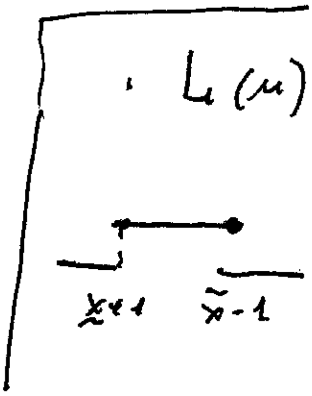
$$\tilde{x} - 1 \le \mu \le \underset{\sim}{x} + 1$$

כל אחד $\hat{\mu} \in [\tilde{x}-1, \underset{\sim}{x}+1]$ הוא MLE

<u>למשל</u>

$$\hat{\mu} = \tilde{X} - 1, \quad \hat{\mu} = \underset{\sim}{X} + 1$$

$$\hat{\mu} = \frac{1}{2}(\tilde{X} + \underset{\sim}{X})$$



$\boxed{(ב\ 4)}$

$$\mu_1' = EX = \int_0^\infty x\, e^{a-x}\, dx = a + 1 \qquad f(x,a) = e^{a-x}, \; x \ge a$$

$$\hat{a} = \bar{X} - 1 \quad \underline{\text{MME}} \qquad X = Y + a, \; Y \sim Exp(1)$$

$$L(x_1 \dots x_n, a) = \prod_{i=1}^{n} e^{a - x_i} = e^{na - s}, \quad x_i \ge a$$

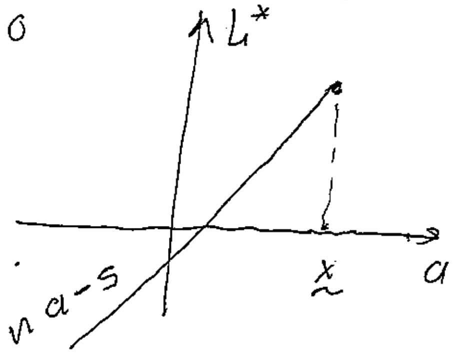
$s = x_1 + \dots + x_n$ כאשר

$$L^* = \ln L = na - s$$

$$x_i \ge a$$

$$\frac{dL^*}{da} = n \ne 0 \qquad \underset{\sim}{x} = \min_i x_i \ge a$$

L^*



$$a_{max} = \underset{\sim}{x}$$

$$\hat{a} = \underset{\sim}{X} \qquad \underline{\text{MLE}}$$

$$\underset{\sim}{X} = \min(X_1, \dots, X_n)$$

h) (4

$$f(x;a) = a\,x^{-1-a}, \quad x \geq 1$$

MME

$$\mu_1' = EX = \int_1^{+\infty} a\,x^{-a}\,dx = \frac{a}{a-1}$$

$\underline{a>1}$ p/c $\quad a = \frac{\mu_1'}{\mu_1' - 1}$

$$\hat{a} = \frac{\bar{X}}{\bar{X} - 1}$$

MLE

$$L(x_1 \ldots x_n, a) = \frac{a^n}{\prod_i x_i^{a+1}}, \quad x_i \geq 1$$

$$L^* = \ln L = n \ln a - (a+1) \sum_i \ln x_i$$

$$\frac{dL^*}{da} = \frac{n}{a} - \sum_i \ln x_i = 0$$

$$a = \frac{n}{\sum_i \ln x_i}$$

$$\hat{a} = \frac{1}{\frac{1}{n} \sum_i \ln X_i}$$

$f(x,\theta)=\theta x^{\theta-1}$ (f (4 תרגיל 3

$0\le x\le 1\ ,\ \theta>0$

$$EX=\int_0^1 \theta x\cdot x^{\theta-1}dx=\frac{\theta}{\theta+1}=\mu_1'$$ MME

MME $\hat\theta=\frac{\bar X}{\bar X-1}$ $\qquad \theta=\frac{\mu_1'}{1-\mu_1'}$

$$L(x_1\dots x_n\ \theta)=\theta^n\left(\prod_{k=1}^n x_k\right)^{\theta-1},$$ MLE

$$L^*=\ln L=n\ln\theta+(\theta-1)\sum_{k=1}^n \ln x_k$$

$$\frac{\partial L^*}{\partial\theta}=\frac{n}{\theta}+\sum_{k=1}^n \ln x_k=0\ ,\ \theta=\frac{-n}{\sum_{k=1}^n \ln x_k}$$

MLE $\hat\theta=\frac{-n}{\sum_{k=1}^n \ln X_k}$.

$\theta>0$ $\qquad f(x,\theta)=\frac{x}{\theta^2}e^{-\frac{x}{\theta}}\ ,\ x\ge 0$ (a (4

$$\mu_1'=\frac{2}{\frac{1}{\theta}}=2\theta \qquad X\sim\Gamma\left(2,\frac{1}{\theta}\right)$$

MME $\hat\theta=\frac{1}{2}\bar X$.

$$L=\frac{1}{\theta^{2n}}\prod_{k=1}^n x_k\ e^{-\frac{1}{\theta}\sum_{k=1}^n x_k}$$

$$L^*=\ln L=-2n\ln\theta+\sum_{k=1}^n \ln x_k-\frac{1}{\theta}\sum_{k=1}^n x_k$$

$$\frac{\partial L^*}{\partial\theta}=-\frac{2n}{\theta}+\frac{1}{\theta^2}\left(\sum_{k=1}^n x_k\right)=0\ ,\ \theta=\frac{1}{2n}\sum_{k=1}^n X_k$$

MLE $\hat\theta=\frac{\bar X}{2}$

g) (4 $\boxed{\text{תרגיל 3}}$

$$f(x,\theta) = \frac{1}{\theta} - \frac{|x|}{\theta^2} \quad , \; |x| \le \theta \; ,$$

MME $\quad \mu_1' = 0 \quad$ f זוגית.

$$\mu_2' = 2\int_0^\theta x^2\left(\frac{1}{\theta} - \frac{x}{\theta^2}\right)dx =$$

$$= \frac{2}{\theta}\cdot\frac{x^3}{3}\Big|_0^\theta - \frac{2}{\theta^2}\,\frac{x^4}{4}\Big|_0^\theta = \frac{2}{3}\theta^2 - \frac{\theta^2}{2} = \frac{\theta^2}{6}$$

$$\hat\theta = \sqrt{6M_2'} = \sqrt{6}\sqrt{\frac{1}{n}\sum_{k=1}^n X_k^2}$$

MLE $\quad n=1 \quad L(x,\theta) = \dfrac{\theta - |x|}{\theta^2} \; , \; |x| \le \theta$

$$\frac{\partial L}{\partial\theta} = -\frac{1}{\theta^2} + \frac{2|x|}{\theta^3} = 0 \quad , \quad \theta_{max} = 2|X|$$

$\hat\theta = 2|X_1| \quad , n=1 \quad$ MLE

$\hat\theta = \sqrt{6}\,|X_1| \quad , n=1 \quad$ MME

אם $a>1$ $\quad MME(a) = \dfrac{\bar X}{\bar X - 1}$ (h (4

אם $0 < a \le 1$ MME לא קיים

$$MLE(a) = \frac{n}{\sum_i \ln X_i}$$

$MME(\theta) = MLE(\theta) = \sqrt{\bar X}$ (c (4