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סטטיסטיקה מתמטית — תרגיל 4

1) נלקח מדגם בגודל n מהתפלגות ברנולי $B(1,p)$. $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ ו- $T_n = \frac{n\bar{X}_n + 1}{n+2}$.
מצא את תוחלת ריבוע הטעות (MSE) לאומדים $\bar{X}_n$ ו- T_n עבור פרמטר p.
אם $n=8$ איזה אומד עדיף? אם $p \le \frac{1}{8}$ או $\frac{7}{8} \le p$;
אם $\frac{1}{7} \le p \le \frac{6}{7}$.

2) נלקח מדגם בגודל n מהתפלגות אחידה $U(0,\theta)$, $\tilde{X} = \max(X_1,\dots,X_n)$, $\underset{\sim}{X} = \min(X_1,\dots,X_n)$.

a) מצא אומד $\tilde{T}$ בלתי מוטה עבור θ מבוסס על $\tilde{X}$.

b) מצא אומד $\underset{\sim}{T}$ בלתי מוטה עבור θ מבוסס על $\underset{\sim}{X}$.

c) השווה את האומדים $n\underset{\sim}{X}$, $\tilde{X}$, $\underset{\sim}{T}$, $\tilde{T}$ וגם $2\bar{X}$ על-ידי MSE.

3) בדוק שהסטטיסטי $U = n \min_{1\le k \le n} X_k$ הוא אומד בלתי מוטה עבור פרמטר $\theta = \frac{1}{\lambda}$ של התפלגות $Exp(\lambda)$. חשב את $V(U)$ והשווה את האומד U עם $\bar{X}$.

4) מצא אומד בלתי מוטה עבור פרמטר $\theta = e^{\mu}$ של התפלגות נורמלית $N(\mu,1)$ מבוסס על ממוצע המדגם $\bar{X}$.
מצא MSE של האומד.

5. מצא אומד בלתי מוטה עבור פרמטר θ של אוכלוסייה בעלת צפיפות הבאה:

$$f(x,\theta)=\begin{cases}\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}, & x\geq 0\\ 0, & \text{אחרת}\end{cases}$$

מבוסס על ממוצע המדגם $\bar{X}$

חשב את MSE של האומד.

6. נלקח מדגם מהתפלגות

$$f(x,\gamma)=\begin{cases}\frac{2x}{\gamma^2}, & 0\leq x\leq\gamma\\ 0, & \text{אחרת}\end{cases}$$

a) חשב התפלגות של סטטיסטי $\tilde{X}=\max_{1\leq k\leq n} X_k$

b) חשב את $E\tilde{X}$ ו $V\tilde{X}$

c) מצא אומד בלתי מוטה U_1 עבור γ מבוסס על $\tilde{X}$.

d) מצא אומד בלתי מוטה U_2 עבור γ מבוסס על ממוצע המדגם $\bar{X}$

e) השווה את U_1 ו U_2 על-ידי MSE.

7. מצא את האומד T בלתי מוטה עבור פרמטר θ של התפלגות

$$f(x,\theta)=\frac{1}{\sqrt{2\pi\theta}}e^{-\frac{x^2}{2\theta}}$$

מבוסס על המומנט המדגמי השני $M_2'=\frac{1}{n}\sum_{k=1}^{n}X_k^2$

השווה את האומד T עם $\frac{n}{n+1}T$.

8. נלקח מדגם $(X_1,\ldots,X_n)$ מהתפלגות נורמלית $N(\mu,\sigma^2)$.

כדי לאמוד σ^2 משתמשים באומדים U מצורה הבאה:

a. $U=\frac{1}{n}\sum_i (X_i-\mu)^2$, (μ ידוע) , $MLE(\sigma^2)$

b. $U=\frac{1}{n}\sum_i X_i^2-\mu^2$ (μ ידוע) , $MME(\sigma^2)$

c. $U=\frac{1}{n}\sum_i (X_i-\bar{X})^2$ $MME(\sigma^2)=MLE(\sigma^2)$ (μ לא ידוע)

d. $U=S^2=\frac{1}{n-1}\sum_i (X_i-\bar{X})^2$

e. $U=\frac{1}{n+1}\sum_i (X_i-\bar{X})^2$

f. $U=\frac{1}{n+2}\sum_i (X_i-\bar{X})^2$

g. $U=\frac{1}{n+1}\sum_i (X_i-\mu)^2$ (μ ידוע).

בכל המקרים מצא EU ו VU.

מצא את תוחלת ריבוע הטעות

$$MSE_U(\sigma^2)=R_U(\sigma^2)$$

והשווה את האומדים.

פתרון

$$MSE = R_T(\theta) = V_\theta T + (E_\theta T - \theta)^2$$ מבחן 4

1) $B(1, p)$, $\theta = p$, $\bar{X}_n$, $T_n = \dfrac{n\bar{X}_n + 1}{n+2}$

$\boxed{T_n \text{ [illegible]}}$

$E\bar{X}_n = p$, $V\bar{X}_n = \dfrac{p(1-p)}{n}$

$ET_n = \dfrac{np+1}{n+2}$, $VT_n = \dfrac{n}{(n+2)^2} p(1-p)$

$$R_{T_n}(p) = \frac{n\,p(1-p)}{(n+2)^2} + \left(\frac{np+1}{n+2} - p\right)^2 = \frac{(n-4)p(1-p)+1}{(n+2)^2}$$

$u = p(1-p)$, $n = 8$

$$R_{\bar{X}_n} = \frac{u}{n} = \frac{u}{8}, \quad R_{T_n} = \frac{(n-4)u+1}{(n+2)^2} = \frac{4u+1}{100}$$

$u < \frac{2}{17} \Leftrightarrow \frac{u}{8} < \frac{4u+1}{100}$, $0 \le u \le \frac{7}{64}$, $0 \le p \le \frac{1}{8}$ [illegible]

$u > \frac{2}{17} \Leftrightarrow \frac{u}{8} > \frac{4u+1}{100}$, $u \ge \frac{6}{49}$, $\frac{1}{7} \le p \le \frac{6}{7}$, [illegible]

2) $U(a, b)$

$$F_{\bar{X}}(t) = \begin{cases} 0, & t \le a \\ \left(\frac{x-a}{b-a}\right)^n, & a \le t < b \\ 1, & t \ge b \end{cases}$$

$$F_{\underset{\sim}{X}}(t) = \begin{cases} 0, & t < a \\ 1 - \left(\frac{b-x}{b-a}\right)^n, & a \le t < b \\ 1, & t \ge b \end{cases}$$

$\boxed{a = 0}$

$$f_{\tilde{X}}(t) = \frac{n(x-a)^{n-1}}{(b-a)^n} I_{[a,b]}(x),$$

$$f_{\underset{\sim}{X}}(t) = \frac{n(b-x)^{n-1}}{(b-a)^n} I_{[a,b]}(x),$$

$$E\tilde{X} = \int_0^{\theta} x \, \frac{n x^{n-1}}{\theta^n} = \frac{n}{\theta^n} \, \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\theta \qquad \text{a}$$

$$\tilde{T} = \frac{n+1}{n}\tilde{X} \qquad E\tilde{T} = \theta$$

$$E\underset{\sim}{X} = \frac{1}{\theta^n}\int_0^{\theta} x \; n(\theta - x)^{n-1}\,dx = \frac{1}{\theta^n}\left[-x(\theta-x)^n \Big|_0^{\theta} + \right. \qquad \text{b}$$

$$\left. + \int_0^{\theta} (\theta - x)^n\,dx \right] = -\frac{1}{\theta^n} \cdot \frac{(\theta-x)^{n+1}}{n+1}\Big|_0^{\theta} = \frac{\theta}{n+1}$$

$$\underset{\sim}{T} = (n+1)\underset{\sim}{X} \qquad E\underset{\sim}{T} = \theta$$

$$V\bar{X} = \frac{\theta^2}{12n} = R_{\bar{X}}(\theta) \,, \qquad E\bar{X} = \frac{\theta}{2} \,, \; E(2\bar{X}) = \theta \qquad \text{c}$$

$$V(2\bar{X}) = R_{2\bar{X}}(\theta) = \frac{\theta^2}{3n}$$

$$V\tilde{X} = \int_0^{\theta} x^2 \, \frac{n x^{n-1}}{\theta^n}\,dx - (E\tilde{X})^2 = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$R_{\tilde{X}}(\theta) = \frac{n\theta^2}{(n+2)(n+1)^2} + \left(\theta - \frac{n}{n+1}\theta\right)^2 =$$

$$= \frac{n + n+2}{(n+2)(n+1)^2}\theta^2 = \frac{2\theta^2}{(n+1)(n+2)}$$

$$R_{\tilde{T}}(\theta) = \frac{(n+1)^2}{n^2} V\tilde{X} = \frac{\theta^2}{n(n+2)}$$

$$E\underset{\sim}{X}^2 = \frac{n}{\theta^n}\int_0^{\theta} x^2 (\theta - x)^{n-1}\,dx \underset{x=\theta u}{=} \frac{n}{\theta^n}\cdot\theta^n\cdot\theta^2 \int_0^1 u^2(1-u)^{n-1}\,du =$$

$$= n\theta^2 B(3, n) = n\theta^2 \frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)} = n\theta^2 \frac{2\cdot(n-1)!}{(n+2)!} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$V\underset{\sim}{X} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} =$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$\boxed{V\tilde{X} = V\underset{\sim}{X}}$$

פרדק

$$R_{n\tilde{X}}(b) = V(n\tilde{X}) + (b - E(n\tilde{X}))^2 =$$

$$= \frac{n^3 b^2}{(n+2)(n+1)^2} + \left(b - \frac{n}{n+1} b\right)^2 = \frac{(n^3+n+2)b^2}{(n+2)(n+1)^2}$$

$$R_{\tilde{T}}(b) = V((n+1)\tilde{X}) = \frac{n}{n+2} b^2 \xrightarrow[n\to\infty]{} b^2 \neq 0$$

$$R_{n\tilde{X}}(b) > R_{\tilde{T}}(b) > R_{2\bar{X}}(b) > R_{\tilde{X}}(b) > R_{\tilde{T}^*}(b)$$ מסקנה

לכל $n>1$

לכל $b>0$

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$U = n \min X_k$, $\theta = \frac{1}{\lambda}$, $X_k \sim Exp(\lambda)$

$F_{\tilde{X}}(t) = 1 - (1 - F_{X_k}(t))^n$, $F_{X_k}(t) = 1 - e^{-\lambda t}$, $t \geq 0$

$F_{\tilde{X}}(t) = 1 - e^{-n\lambda t}$, $t \geq 0$, $\tilde{X} \sim Exp(n\lambda)$

$VU = n^2 \frac{1}{n^2\lambda^2} = \frac{1}{\lambda^2}$, , $EU = n \cdot \frac{1}{n\lambda} = \frac{1}{\lambda} = \theta$

$R_U(\lambda) = \frac{1}{\lambda^2}$

$E\bar{X} = \frac{1}{\lambda}$, $V\bar{X} = \frac{1}{n\lambda^2}$

$R_{\bar{X}}(\lambda) = \frac{1}{n\lambda^2} < \frac{1}{\lambda^2} = R_U(\lambda)$ לכל $\lambda > 0$

לכל $n>1$

$$\bar{X} \sim N\left(\mu, \tfrac{1}{n}\right) \qquad \theta = e^{\mu} \qquad X \sim N(\mu, 1). \quad \underline{\underline{(4}}$$

$$m_{\bar{X}}(t) = e^{\mu t + \frac{1}{2}\frac{1}{n}t^2} = Ee^{\bar{X}}.$$

$$Ee^{\bar{X}} = e^{\mu + \frac{1}{2n}} \quad , \quad T = e^{-\frac{1}{2n}} e^{\bar{X}}$$

$$ET = e^{\mu} = \theta \quad ,$$ T כ"א עבור θ

$$VT = ET^2 - (ET)^2 = E\left(e^{-\frac{1}{2n}} e^{\bar{X}}\right)^2 - e^{2\mu}$$

$$= E\, e^{2\bar{X}} \cdot e^{-\frac{1}{n}} - e^{2\mu} = ? \qquad 2\bar{X} \sim N\left(2\mu, \tfrac{4}{n}\right)$$

$$\left(m_{2\bar{X}}(t) = e^{2\mu t + \frac{1}{2}\cdot\frac{4}{n}\cdot t^2} \quad , E e^{2\bar{X}} = e^{2\mu + \frac{2}{n}} \right)$$

$$VT = e^{-\frac{1}{n}} e^{2\mu + \frac{2}{n}} - e^{2\mu} = e^{2\mu}\left(e^{\frac{1}{n}} - 1\right)$$

$$E\bar{X} = \sqrt{2}\,\theta \cdot \frac{1}{2}\sqrt{\pi}$$

$$T = \frac{\sqrt{2}}{\sqrt{\pi}}\bar{X}, \quad ET = \theta$$

$$VT = \frac{2}{\pi}\frac{1}{n}VX,$$

$$VX = \frac{\Gamma(2)}{\frac{1}{2\theta^2}} - \left(\frac{\sqrt{\pi}}{\sqrt{2}}\theta\right)^2$$

$$VX = 2\theta^2 - \frac{\pi}{2}\theta^2$$

$$R_T(\theta) = VT = \theta^2\left(\frac{4}{\pi} - 1\right)\frac{1}{n}$$

$X \sim Weibull\left(\frac{1}{2\theta^2}, 2\right)$ (5)

$a = \frac{1}{2\theta^2}, \; b = 2$

$$EX = \frac{\Gamma\left(1+\frac{1}{b}\right)}{a^{\frac{1}{b}}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\frac{1}{2\theta^2}}}$$

$$VX = \frac{\Gamma\left(1+\frac{2}{b}\right)}{a^{\frac{2}{b}}} - (EX)^2$$

$, f(x,\gamma) = \frac{2x}{\gamma^2} I_{[0,\gamma]}(x)$. (6)

$$EX = \frac{2}{\gamma^2}\int_0^\gamma x^2\,dx = \frac{2}{3}\gamma$$

$$EX^2 = \frac{2}{\gamma^2}\int_0^\gamma x^3\,dx = \frac{\gamma^2}{2}, \quad VX = \frac{\gamma^2}{18}$$

$$U_2 = \frac{3}{2}\bar{X} \qquad EU_2 = \gamma, \quad VU_2 = \frac{\gamma^2}{8n}.$$

$$F_X(x) = \frac{x^2}{\gamma^2} \cdot I_{[0,\gamma]}(x)$$

$$F_{\tilde{X}}(x) = \frac{x^{2n}}{\gamma^{2n}} I_{[0,\gamma]}(x)$$

$$f_{\tilde{X}}(x) = 2n\frac{x^{2n-1}}{\gamma^{2n}} I_{[0,\gamma]}(x)$$

$$E\tilde{X} = \frac{2n}{\gamma^{2n}}\int_0^\gamma x^{2n}\,dx = \frac{2n}{2n+1}\gamma$$

$$U_1 = \frac{2n+1}{2n}\tilde{X}.$$

$$E\tilde{X}^2 = \frac{2n}{\gamma^{2n}} \int_0^{\gamma} x^{2n+1}\,dx = \frac{2n}{2n+2}\gamma^2$$

$$V\tilde{X} = \gamma^2\left(\frac{2n}{2n+2} - \frac{(2n)^2}{(2n+1)^2}\right) = \frac{n\gamma^2}{(n+1)(2n+1)^2}$$

$$VU_1 = \frac{(2n+1)^2}{(2n)^2}\cdot V\tilde{X} = \frac{\gamma^2}{4n(n+1)}$$

$$R_{U_2}(\gamma) = \frac{\gamma^2}{8n} > \frac{\gamma^2}{4n(n+1)} = R_{U_1}(\gamma)$$

$\underline{n>1}$

$$E\left(\frac{1}{n}\sum_k X_k^2\right) = \sigma^2, \quad T = \frac{1}{n}\sum_k X_k^2$$

$\mathcal{N}(0, \sigma^2)$

$\theta = \sigma^2$

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$$ET = \theta$$

$$VT = \frac{1}{n} VX^2, \quad X \sim \mathcal{N}(0, \sigma^2)$$

$$VX^2 = EX^4 - (EX^2)^2 = \mu_4' - (\mu_2')^2 = \left(\frac{4!}{2}\frac{\sigma^4}{2^2} - \sigma^2\right)$$

$$= 3\sigma^4 - \sigma^4 = 2\sigma^4$$

$$VT = \frac{2\sigma^4}{n} = R_T(\sigma^2)$$

$$T_1 = \frac{n}{n+1}T$$

$$R_{T_1}(\sigma^2) = VT_1 + (\sigma^2 - ET_1)^2 =$$

$$= \frac{n^2}{(n+1)^2}\cdot\frac{2\sigma^4}{n} + \left(\sigma^2 - \frac{n}{n+1}\sigma^2\right)^2 = \frac{(2n+1)\sigma^4}{(n+1)^2}$$

$$\forall n \quad \frac{2n+1}{(n+1)^2} < \frac{2}{n}$$

$$2n^2 + n < 2n^2 + 4n + 2$$

$$n < 3n + 2$$

$$\underline{R_{T_1}(\sigma^2) < R_T(\sigma^2)}$$

U	EU	VU	$MSE(U)$	
$\frac{1}{n}\sum_k (X_k-\mu)^2$ MLE μ ידוע	σ^2 ח"ה	$\frac{2\sigma^4}{n}$	$\frac{2\sigma^4}{n}$	$\frac{nU}{\sigma^2}\sim\chi^2_n$
$\frac{1}{n}\sum_k X_k^2-\mu^2$ μ ידוע MME	σ^2 ח"ה	$\frac{2\sigma^4}{n}+\frac{4\mu^2\sigma^2}{n}$	$\frac{2\sigma^4}{n}+\frac{4\mu^2\sigma^2}{n}$	
$\frac{1}{n}\sum_k (X_k-\bar{X})^2$ MME=MLE	$\frac{n-1}{n}\sigma^2$	$\frac{2(n-1)}{n^2}\sigma^4$	$\frac{2n-1}{n^2}\sigma^4$	$\frac{nU}{\sigma^2}\sim\chi^2_{n-1}$
$S^2=\frac{1}{n-1}\sum_k (X_k-\bar{X})^2$	σ^2 ח"ה	$\frac{2\sigma^4}{n-1}$	$\frac{2\sigma^4}{n-1}$	$\frac{n-1}{\sigma^2}U\sim\chi^2_{n-1}$
$\frac{1}{n+1}\sum_k (X_k-\bar{X})^2$	$\frac{n-1}{n+1}\sigma^2$	$\frac{2(n-1)}{(n+1)^2}\sigma^4$	$\frac{2\sigma^4}{n+1}$	$\frac{n+1}{\sigma^2}U\sim\chi^2_{n-1}$
$\frac{1}{n+2}\sum_k (X_k-\bar{X})^2$	$\frac{n-1}{n+1}\sigma^2$	$\frac{2(n-1)}{(n+2)^2}\sigma^4$	$\frac{2n+7}{(n+2)^2}\sigma^4$	$\frac{n+2}{\sigma^2}U\sim\chi^2_{n-1}$
$\frac{1}{n+1}\sum_k (X_k-\mu)^2$ μ ידוע	$\frac{n}{n+1}\sigma^2$	$\frac{2n}{(n+1)^2}\sigma^4$	$\frac{2n+1}{(n+1)^2}\sigma^4$	$\frac{n+1}{\sigma^2}U\sim\chi^2_n$

Table 1 DISCRETE DISTRIBUTIONS

Name of parametric family of distributions	Discrete density functions $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}] = m_X(t)$
Discrete uniform $U_d(N)$	$f(x) = \frac{1}{N} I_{\{1,\ldots,N\}}(x)$	$N = 1, 2, \ldots$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\mu_3' = \frac{N(N+1)^2}{4}$ $\mu_4' = \frac{(N+1)(2N+1)(3N^2+3N-1)}{30}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli $B(1, p)$	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	$0 \le p \le 1$ $(q = 1-p)$	p	pq	$\mu_r' = p$ for all r	$q + pe^t$
Binomial $B(n, p)$	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\ldots,n\}}(x)$	$0 \le p \le 1$ $n = 1, 2, 3, \ldots$ $(q = 1-p)$	np	npq	$\mu_3 = npq(q-p)$ $\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$	$(q+pe^t)^n$
Hypergeometric $H(n, M, K)$	$f(x) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} I_{\{0,1,\ldots,n\}}(x)$	$M = 1, 2, \ldots$ $K = 0, 1, \ldots, M$ $n = 1, 2, \ldots, M$	$n\frac{K}{M}$	$n\frac{K}{M}\frac{M-K}{M}\frac{M-n}{M-1}$	$\mathscr{E}[X(X-1)\cdots(X-r+1)] = r!\frac{\binom{K}{r}\binom{n}{r}}{\binom{M}{r}}$	not useful
Poisson $P(\lambda)$	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$	$\lambda > 0$	λ	λ	$\kappa_r = \lambda$ for $r = 1, 2, \ldots$ $\mu_3 = \lambda$ $\mu_4 = \lambda + 3\lambda^2$	$\exp[\lambda(e^t - 1)]$
Geometric $G(p)$	$f(x) = p\, q^{x-1} I_{\{1,2,\ldots\}}$	$0 < p \le 1$ $(q = 1-p)$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\mu_3 = \frac{q+q^2}{p^2}$ $\mu_4 = \frac{q+7q^2+q^3}{p^4}$	$\frac{pe^t}{1-qe^t}$
Negative binomial $NB(r, p)$	$f(x) = \binom{x-1}{r-1} p^r q^{x-r} I_{\{r,r+1,\ldots\}}$		$\frac{r}{p}$	$\frac{rq}{p^2}$	$\mu_3 = \frac{r(q+q^2)}{p^3}$ $\mu_4 = \frac{r[q+(3r+4)q^2+q^3]}{p^4}$	$\left(\frac{pe^t}{1-qe^t}\right)^r$

Table 2 CONTINUOUS DISTRIBUTIONS

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu'_r = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}] = M_X(t)$
Uniform or rectangular $U(a, b)$	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$-\infty < a < b < \infty$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\mu_r = 0$ for r odd; $\mu_r = \frac{(b-a)^r}{2^r(r+1)}$ for r even	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal $N(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x-\mu)^2/2\sigma^2]$	$-\infty < \mu < \infty$, $\sigma > 0$	μ	σ^2	$\mu_r = 0$, r odd; $\mu_r = \frac{r!}{(r/2)!}\frac{\sigma^r}{2^{r/2}}$, r even; $\kappa_r = 0$, $r > 2$	$\exp[\mu t + \frac{1}{2}\sigma^2 t^2]$
Exponential $Exp(\lambda)$	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\mu'_r = \frac{\Gamma(r+1)}{\lambda^r}$	$\frac{\lambda}{\lambda - t}$ for $t < \lambda$
Gamma $\Gamma(r, \lambda)$	$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$, $r > 0$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$	$\mu'_j = \frac{\Gamma(r+j)}{\lambda^j \Gamma(r)}$	$\left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$
Beta $Beta(a, b)$	$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$	$a > 0$, $b > 0$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$	$\mu'_r = \frac{B(r+a, b)}{B(a, b)}$	not useful
Cauchy	$f(x) = \frac{1}{\pi\beta\{1 + [(x-\alpha)/\beta]^2\}}$	$-\infty < \alpha < \infty$, $\beta > 0$	Does not exist	Does not exist	Do not exist	Characteristic function is $e^{i\alpha t - \beta\lvert t\rvert}$
Lognormal	$f(x) = \frac{1}{x\sqrt{2\pi}\sigma} \exp[-(\log_e x - \mu)^2/2\sigma^2] I_{(0,\infty)}(x)$	$-\infty < \mu < \infty$, $\sigma > 0$	$\exp[\mu + \frac{1}{2}\sigma^2]$	$\exp[2\mu + 2\sigma^2] - \exp[2\mu + \sigma^2]$	$\mu'_r = \exp[r\mu + \frac{1}{2}r^2\sigma^2]$	not useful
Double exponential	$f(x) = \frac{1}{2\beta} \exp\left(-\frac{\lvert x-\alpha\rvert}{\beta}\right)$	$-\infty < \alpha < \infty$, $\beta > 0$	α	$2\beta^2$	$\mu_r = 0$ for r odd; $\mu_r = r!\,\beta^r$ for r even	$\frac{e^{\alpha t}}{1 - (\beta t)^2}$

(continued)

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right]$$

$$ES^2 = \sigma^2 = VX_i$$

$$VS^2 = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$$

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-1}\,dx$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \qquad \Gamma(n) = (n-1)!$$

Table 2 CONTINUOUS DISTRIBUTIONS (*continued*)

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}] = m_X(t)$
Weibull	$f(x) = abx^{b-1}\exp[-ax^b]I_{(0,\infty)}(x)$	$a > 0$ $b > 0$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1}) - \Gamma^2(1+b^{-1})]$	$\mu_r' = a^{-r/b}\Gamma\left(1+\frac{r}{b}\right)$	$\mathscr{E}[X^t] = a^{-t/b}\Gamma\left(1+\frac{t}{b}\right)$
Logistic	$F(x) = [1 + e^{-(x-\alpha)/\beta}]^{-1}$	$-\infty < \alpha < \infty$ $\beta > 0$	α	$\frac{\beta^2\pi^2}{3}$		$e^{\alpha t}\pi\beta t\csc(\pi\beta t)$
Pareto	$f(x) = \frac{\theta x_0^\theta}{x^{\theta+1}} I_{(x_0,\infty)}(x)$	$x_0 > 0$ $\theta > 0$	$\frac{\theta x_0}{\theta - 1}$ for $\theta > 1$	$\frac{\theta x_0^2}{(\theta-1)^2(\theta-2)}$ for $\theta > 2$	$\mu_r' = \frac{\theta x_0^r}{\theta - r}$ for $\theta > r$	does not exist
Gumbel or extreme value	$F(x) = \exp(-e^{-(x-\alpha)/\beta})$	$-\infty < \alpha < \infty$ $\beta > 0$	$\alpha + \beta\gamma$, $\gamma \approx .577216$	$\frac{\pi^2\beta^2}{6}$	$\kappa_r = (-\beta)^r\psi^{(r-1)}(1)$ for $r \geq 2$, where $\psi(\cdot)$ is digamma function	$e^{\alpha t}\Gamma(1-\beta t)$ for $t < 1/\beta$
t distribution	$f(x) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)}\frac{1}{\sqrt{k\pi}}\frac{1}{(1+x^2/k)^{(k+1)/2}}$	$k > 0$	$\mu = 0$ for $k > 1$	$\frac{k}{k-2}$ for $k > 2$	$\mu_r = 0$ for $k > r$ and r odd $\mu_r = \frac{k^{r/2}B((r+1)/2, (k-r)/2)}{B(\frac{1}{2}, k/2)}$ for $k > r$ and r even	does not exist
F distribution	$f(x) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} \times \frac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}} I_{(0,\infty)}(x)$	$m, n = 1, 2, \ldots$	$\frac{n}{n-2}$ for $n > 2$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$ for $n > 4$	$\mu_r' = \left(\frac{n}{m}\right)^r \frac{\Gamma(m/2+r)\Gamma(n/2-r)}{\Gamma(m/2)\Gamma(n/2)}$ for $r < \frac{n}{2}$	does not exist
Chi-square distribution	$f(x) = \frac{1}{\Gamma(k/2)}\left(\frac{1}{2}\right)^{k/2} x^{k/2-1}e^{-(1/2)x}I_{(0,\infty)}(x)$	$k = 1, 2, \ldots$	k	$2k$	$\mu_j' = \frac{2^j\Gamma(k/2+j)}{\Gamma(k/2)}$	$\left(\frac{1}{1-2t}\right)^{k/2}$ for $t < 1/2$