סטטיסטיקה מתמטית — תרגיל 5

1. חשב חסם ראו-קרמר (Rao-Cramer) של התפלגות עבור פרמטר θ,
2. בדוק: האומד T בלתי מוטה,
3. בדוק: האם T הוא יעיל (efficient). אם לא, חשב את היעילות של T.

$$\bar{X}=\frac{1}{n}\sum_k X_k$$
$$S^2=\frac{1}{n-1}\sum_k (X_k-\bar{X})^2$$

1) $B(1,p)$, $\theta=p$, $T=\bar{X}$

2) $B(1,p)$, $\theta=p(1-p)$, $T=S^2$

3) $N(\mu,\sigma^2)$, $\theta=\mu$, $T=\bar{X}$

4) $N(\mu,\sigma^2)$, $\theta=\sigma^2$, μ ידוע, $T=\frac{1}{n}\sum_k (X_k-\mu)^2$

5) $N(\mu,\sigma^2)$, $\theta=\sigma^2$, μ ידוע, $T=\frac{1}{n}\sum_k X_k^2-\mu^2$
(**רמז**: $VX_k^2=4\mu^2\sigma^2+2\sigma^4$ בדוק.)

6) $N(\mu,\sigma^2)$, $\theta=\sigma^2$, $T=S^2$

7) $\Gamma(m,\lambda)$, $\theta=\frac{1}{\lambda}$, $T=\frac{1}{m}\bar{X}$, m ידוע

8) $Beta(a,1)$, $\theta=a$, $T=-\frac{n-1}{\sum_k \ln X_k}$, $n>2$
(רמז $Z=-\sum_k \ln X_k \sim \Gamma(n,\lambda)$ בדוק)

9) $Exp(\lambda)$, $\theta=\frac{1}{\lambda}$, $T=n\cdot\min_{1\le k\le n} X_k$

10) $P(\lambda)$, $\theta=e^{-\lambda}$, $T=\frac{1}{n}\sum_{k=1}^{n} I_0(X_k)$

כאשר $I_0(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$

פתרונות

סטטיסטיקה מתמטית

תרגיל 5

1)

$B(1, p)$

$E\bar{X} = p$, $\theta = p$ $V\bar{X} = \frac{p(1-p)}{n} = \frac{1}{n \cdot \frac{1}{p(1-p)}}$

2)

$ES^2 = VX = p(1-p)$, $\tau(p) = p(1-p)$, $B(1, p)$

$RCB = \frac{(1-2p)^2}{n \cdot \frac{1}{p-p^2}}$ חסם R-C , $I(p) = \frac{1}{p(1-p)}$

$\tau'(p) = 1 - 2p = q - p$

$V(S^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$, $\sigma^4 = p^2 q^2$

$\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$

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3)

$E\bar{X} = \mu$, $RCB = \frac{\sigma^2}{n}$ $I(\mu) = \frac{1}{\sigma^2}$

$V\bar{X} = \frac{\sigma^2}{n}$

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4)

μ ידוע

$RCB = \frac{2\sigma^4}{n}$ $I(\sigma^2) = \frac{1}{2\sigma^4}$

$E\left(\frac{1}{n}\sum_k (X_k - \mu)^2\right) = \sigma^2$, $VT = \frac{\sigma^4}{n^2} V\left(\frac{nT}{\sigma^2}\right) = \frac{2n\sigma^4}{n^2} =$

$T = \frac{1}{n}\sum_k (X_k - \mu)^2$, $\boxed{\frac{nT}{\sigma^2} \sim \chi^2_n}$ $= \frac{2\sigma^4}{n}$

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5)

$T = \frac{1}{n}\sum X_k^2 - \mu^2$, $ET = \sigma^2$ $RCB = \frac{2\sigma^4}{n}$

$VT = \frac{1}{n} VX^2 = \frac{2\sigma^4}{n} + \frac{4\mu^2\sigma^2}{n} > \frac{2\sigma^4}{n} = RCB$

$$VS^2 = \frac{2\sigma^4}{n-1}, \quad ES^2 = \sigma^2, \quad RCB = \frac{2\sigma^4}{n}$$ (6

$$ET = \frac{1}{mn} \cdot n\frac{m}{\lambda} = \frac{1}{\lambda}, \quad J(\lambda) = \frac{m}{\lambda^2} \qquad \tau(\lambda) = \frac{1}{\lambda}$$ (7

$$VT = \frac{1}{m^2 n^2} \cdot n\frac{m}{\lambda^2} = \frac{1}{mn\lambda^2} = RCE = \frac{\left(-\frac{1}{\lambda^2}\right)^2}{n\frac{m}{\lambda^2}} = \frac{1}{mn\lambda^2}$$

$$f(x,a) = a x^{a-1} I_{[0,1]}(x) \qquad \theta = a$$

Beta$(a,1)$

(8

RCB

$$\ln f(x,a) = \ln a + (a-1)\ln x$$

$$\frac{\partial}{\partial a}\ln f(x,a) = \frac{1}{a} + \ln x$$

$$\frac{\partial^2}{\partial a^2}\ln f(x,a) = -\frac{1}{a^2}, \qquad J(a) = -E_a\left(-\frac{1}{a^2}\right) = \frac{1}{a^2}$$

$$RCB = \frac{1}{n\frac{1}{a^2}} = \frac{a^2}{n}$$

MLE

$$L^* = n\ln a + (a-1)\sum_k \ln x_k$$

$$\frac{\partial L^*}{\partial a} = \frac{n}{a} + \sum_k \ln x_k, \quad a = \frac{-n}{\sum_k \ln x_k}$$

$$\hat{a} = \frac{n}{Z}, \quad Z = -\sum_k \ln X_k$$

$$Z_k = -\ln X_k$$

$$F_{Z_k}(t) = P\{-\ln X_k \le t\} = 1 - F_{X_k}(e^{-t}) =$$

$$= 1 - (e^{-t})^a = 1 - e^{-ta}, \quad Z_k \sim Exp(a)$$

$T = \dfrac{n-1}{\sum_k \ln X_k}$ אומד ב"מ עבור a.

$$V_a T = (n-1)^2 V_a\left(\frac{1}{Z}\right) \qquad Z \sim \Gamma(n, a)$$

$$E_a\left(\frac{1}{Z}\right)^2 = \int_0^\infty \frac{a^n}{(n-1)!} x^{n-1} e^{-ax} \frac{1}{x^2}\, dx =$$

$$= \frac{a^2}{(n-1)(n-2)} \int_0^\infty \frac{a^{n-2}}{(n-3)!} x^{n-3} e^{-ax}\, dx = \frac{a^2}{(n-1)(n-2)}$$

$$V_a\left(\frac{1}{Z}\right) = \frac{a^2}{(n-1)(n-2)} - \frac{a^2}{(n-1)^2} = \frac{a^2}{n-1}\left(\frac{1}{n-2} - \frac{1}{n-1}\right) =$$

$$= \frac{a^2}{(n-1)^2(n-2)}$$

$$V_a(T) = \frac{(n-1)^2 a^2}{(n-1)^2(n-2)} = \frac{a^2}{n-2}, \quad \underline{n>2}$$

$$\mathrm{eff}_a(T) = \frac{\frac{a^2}{n}}{\frac{a^2}{n-2}} = \frac{n-2}{n}$$

<u>יעילות של T</u>

T לא יעיל (אבל יעיל אסימפטוטית)

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9)

$$RCB = \frac{1}{n\lambda^2} \qquad I(\lambda) = \frac{1}{\lambda^2}, \quad \tau(\lambda) = \frac{1}{\lambda}$$

$$Y_1 = \min_k X_k \sim Exp(n\lambda) \qquad ET = nEY_1 = \frac{n}{n\lambda} = \frac{1}{\lambda}$$

$$VT = n^2 \cdot VY_1 = \frac{n^2}{n^2\lambda^2} = \frac{1}{\lambda^2} > \frac{1}{n\lambda^2} = RCB.$$

$$\underline{n>1}$$

[illegible]

9) $Exp(\lambda)$ $\qquad \theta = \frac{1}{\lambda}$ $\qquad \tau(\lambda) = \frac{1}{\lambda}$

$$f(x,\lambda) = \lambda e^{-\lambda x}$$

$$\ln f(x,\lambda) = \ln\lambda - \lambda x$$

$$\frac{\partial}{\partial\lambda}\ln f(x,\lambda) = \frac{1}{\lambda} - x$$

$$J(\lambda) = E_\lambda\left(\frac{1}{\lambda} - X\right)^2 = V_\lambda X = \frac{1}{\lambda^2}$$

$$RCB = \frac{\left(\left(\frac{1}{\lambda}\right)'\right)^2}{n\frac{1}{\lambda^2}} = \frac{1}{n\lambda^2}$$

$\underset{\sim}{X} = \min_k X_k$, $\qquad T = n\min_k X_k$

$F_{\underset{\sim}{X}}(t) = 1 - e^{-n\lambda x}$, $\qquad \underset{\sim}{X} \sim Exp(n\lambda)$

$$E(n\underset{\sim}{X}) = nE(\underset{\sim}{X}) = \frac{n}{n\lambda} = \frac{1}{\lambda} = \theta$$

$$V(n\underset{\sim}{X}) = n^2 V(\underset{\sim}{X}) = \frac{n^2}{n^2\lambda^2} = \frac{1}{\lambda^2}$$

$$F_T(t) = P\{n\underset{\sim}{X} \le t\} = F_{\underset{\sim}{X}}\left(\frac{t}{n}\right) = 1 - e^{-\lambda t}$$

$T \sim Exp(\lambda)$

$$eff_T(\lambda) = \frac{\frac{1}{n\lambda^2}}{\frac{1}{\lambda^2}} = \frac{1}{n} \to 0 \not\to 1$$

$$P\left\{\left|T - \frac{1}{\lambda}\right| \le \varepsilon\right\} = \left(1 - e^{-\lambda\left(\frac{1}{\lambda}+\varepsilon\right)}\right) - \left(1 - e^{-\lambda\left(\frac{1}{\lambda}-\varepsilon\right)}\right) =$$

$$= e^{\varepsilon\lambda - 1} - e^{-\lambda\varepsilon - 1} \underset{n\to\infty}{\not\to} 1$$

T אינו עקבי

$\tau(\lambda) = e^{-\lambda}$ $\qquad\qquad$ $\mathcal{P}(\lambda)$ (10

$$f(x,\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \ldots$$

$$\frac{\partial}{\partial \lambda} \ln f = \frac{x}{\lambda} - 1, \quad \frac{\partial^2}{\partial \lambda^2} \ln f = -\frac{x}{\lambda^2}$$

$$I(\lambda) = -E\left(-\frac{X}{\lambda^2}\right) = \frac{1}{\lambda^2} EX = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$e^{-\lambda} = P\{X = 0\} = E(I_0(X))$$

$$ET = E(I_0(X_k)) = e^{-\lambda}$$

$$VT = \frac{1}{n} p(1-p) = \frac{e^{-\lambda}(1 - e^{-\lambda})}{n}$$

$I_0(X) \sim B(1,p)$
$p = e^{-\lambda}$

$$RCB = \frac{e^{-2\lambda}}{n \cdot \frac{1}{\lambda}} = \frac{1}{n} \lambda e^{-2\lambda}$$

$$eff_T(\lambda) = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} = \frac{\lambda}{e^{\lambda} - 1} < 1$$

$\lambda > 0$