Automata and Formal Languages

Exercises

1 Review of some Basic Notions

1. Prove by induction that for every positive integer n the number $3 \cdot 16^n + 6^n + 1$ is divisible by 5.

2. Prove by induction that for every $n \geq 1$

$$\sum_{k=1}^{n} \frac{1}{k(k+4)} = \frac{n}{4}\left(\frac{1}{n+1} + \frac{1}{2(n+2)} + \frac{1}{3(n+3)} + \frac{1}{4(n+4)}\right) .$$

3. The sequence $(a_n)_{n=0}^{\infty}$ satisfies the recurrence

$$a_n = 6a_{n-1} - 9a_{n-2}, \qquad n \geq 2,$$

and the initial conditions $a_0 = -4$, $a_1 = -9$. Prove by induction that $a_n = (n-4)3^n$ for each n.

4. Prove by induction that for every $n \geq 1$

$$\sum_{k=1}^{n} \sqrt[3]{k} > \frac{3}{4} n \sqrt[3]{n} .$$

(Can you provide a non-inductive proof as well?)

5. Prove by induction that every $n \geq 24$ can be expressed in the form $n = 5a + 7b$ for appropriately chosen non-negative integers a and b.

6. A finite number of straight lines are drawn in the plane. Prove, by induction on number of lines, that it is possible to color the resulting map by two colors in such a way that any two neighboring states will have different colors.

7. There are n lines in the plane in general position (i.e., no two of them are parallel and no three meet at the same point). Prove that the number of states in the resulting map is $\frac{n(n+1)}{2} + 1$.

8. There are n pairs of socks of different colors in a drawer. How many need to be drawn to make sure that they will contain at least one pair?

9. Let $a_1, a_2, \ldots, a_n$ be any integers. Show that there exist $1 \le i \le j \le n$ such that the sum $a_i + a_{i+1} + \ldots + a_j$ is divisible by n.

10. Prove that, for any finite set of (at least two) people, there are at least two people in the set who know exactly the same number of people from the set. (Assume that if A knows B then B knows A).

11. Prove that, in any group of six people or more, either there are three each of whom knows both of the two others or there are three each of whom knows none of the two others.

12. For every subset of {rreflexive, symmetric, transitive}, construct a relation with the properties in this partial set, but not with the other properties.

13. For each of the following relations on a set A, determine whether it is reflexive, symmetric, transitive:

(a) A – the set of integers, aRb if $b - a$ is divisible by m (where m is a fixed positive integer)

(b) A – the set of integers, aRb if $|b - a| \ge 5$.

(c) A – the set of reals, aRb if $|b - a| \le 0.001$.

(d) A – the set of vectors in $\mathbf{R}^n$, aRb if every component of $b - a$ is integer.

(e) A – the set of vectors in $\mathbf{R}^3$, aRb if a is longer than b.

(f) A – the set of integers, aRb if a divides b.

(g) A – the set of integers, aRb if a and b are relatively prime (i.e., have no common divisor greater than 1).

(h) A – the set of integers, aRb if a and b have the same prime factors.

(i) A – the set of integers, aRb if $|a - b|$ is not a prime.

(j) A – the set of $n \times n$ matrices with rational entries, aRb if there exists a matrix $c \in A$ with $\det(c) = 1$ such that $b = ca$.

(k) A – the set of $n \times n$ matrices with rational entries, aRb if there exists an invertible matrix $c \in A$ with $b = c^{-1}ac$.

(l) A – the set of Hebrew words, aRb if the number of letters of b is not less than that of a.

(m) A – the set of Hebrew words, aRb if the words a and b contain at least one common letter.

(n) A – the set of Hebrew words, aRb if the words a and b contain at least one common letter at the same position.

14. Prove that for any partial ordering of a finite set there exists a minimal element.

15. Describe the transitive closure for the following relations:

(a) A – the set of integers, aRb if $a + b$ is divisible by m ($m \geq 2$ a fixed integer).

(b) the relation from Question 13.b.

(c) A – the set of all points in the plane, aRb if the distance between a and b is at most 1.

(d) the relation from Question 13.g.

(e) A ={1,2,3,4,5}, R = { (1,5), (2,3), (4,1), (5,3)}.

(f) A – the set of vertices of a finite undirected graph, aRb if $a = b$ or there is an edge from a to b.

(g) A – the set of all permutations of $\{1, 2, \ldots, n\}$, aRb if it is possible to pass from a to b by exchanging two adjacent entries.

(h) A – the set of integers, aRb if $|a - b| = m$, where m is an arbitrary fixed positive integer.

(i) A – the set of integers, aRb if the greatest common divisor of a and b is greater than 1.

16. There are n functions $f_1, f_2, \ldots, f_n$ in a Java program. Define a graph on the set of these functions as follows: there is an edge from function f_i to function f_j if f_i calls f_j. For example, a function f is directly recursive if there is an edge from f to itself, and recursive if there is a path from f to f. Let M be the adjacency matrix of the graph ($M_{ij} = 1$ if there is an edge from f_i to f_j and $M_{ij} = 0$ otherwise).

(a) Show that f_i is recursive iff $(M^n)_{ii} > 0$ for some positive integer n.

(b) Show that the set of functions that have to be put in every program that uses f_i is: $\{f_j : \text{there exists } n \text{ such that } M_{ij}^n > 0\}$.

17. A path in a directed graph is *Hamiltonian* if it passes exactly once through each vertex of the graph. (It is not necessary that there exists an edge from last vertex of the path to the first). Prove by induction that for any finite graph the following is true: If between every pair of vertices there is an edge in at least one direction, then the graph contains a Hamiltonian path.

2 Decidable and Undecidable Problems

18.

(a) Present a machine which, given a sequence of digits representing a non-negative integer in base 10, starting with the least significant digit, determines whether this integer is even or odd.

(b) Same as the preceding part if the digits are given starting with the most significant.

(c) Same as part (a) for the problem of divisibility by 125.

(d) Same as part (a) for the problem of divisibility by 3.

19. The ring of *Gaussian integers* consists of the subset

$$\mathbf{Z}[i] = \{a + bi : a, b \in \mathbf{Z}\}$$

of the complex plane, with the operations of addition and multiplication. An element $z \in \mathbf{Z}[i]$, distinct from $0, \pm 1, \pm i$, may be either prime or composite; it is *composite* if it may be written as a product of two such numbers, and *prime* otherwise.

(a) Present an algorithm to decide whether a given $z \in \mathbf{Z}[i]$ is prime or not.

(b) Show that every element $z \in \mathbf{Z}[i]$, distinct from $0, \pm 1, \pm i$, is a product of finitely many primes. Present an algorithm for finding such a factorization.

20. Show that the 1-dimensional tiling problem, namely that of tiling the line by finitely many types of unit intervals, is decidable.

21. Show that if a certain tiling system enables tiling some $m \times n$ rectangle, with $m, n \geq 2$, in such a way that the rightmost column is tiled the same as the leftmost column and the top row is tiled the same as the bottom row, then there exists a legal tiling of the whole plane.

22. Consider the diophantine equation $x^2 + y^2 = a$.

(a) Prove that it has no solutions if a is divisible by one of the primes $p = 3, 7, 11$, but not by its square.

(b) Prove that the set of integers a for which the equation is solvable is closed under multiplication.

23. Prove that there exist infinitely many positive integers a for which the diophantine equation $x^2 + y^2 + z^2 = a$ has no solutions. (Hint: Consider the equation modulo 8.)

24. Prove that for "most" integers a the diophantine equation $x^4 + y^4 + z^4 = a$ has no solutions.

3 Alphabets and Languages

25. Characterize those pairs of words $x, y \in \Sigma^*$ such that:

(a) $xy = yx$.

(b) $xyx = yxy$.

(c) $xyx = yxy^2$.

(d) $xy^2x = yx^2y$.

26. Define words a_n and b_n in $\{0,1\}^*$ for each $n \geq 0$ as follows:

$$\begin{array}{lll} a_0 = 0, & b_0 = 1 & \\ a_n = a_{n-1}b_{n-1}, & b_n = b_{n-1}a_{n-1}, & n \geq 1\ . \end{array}$$

(a) Find the length of a_n and b_n for each n.

(b) Show that, for each n, the word b_n is obtained from a_n by replacing each 0 by a 1 and vice versa.

(c) Show that both words a_n and b_n are palindromes for each even n. What happens for odd n?

(d) Show that, for each n, the word a_n does not contain the strings 000 and 111.

(e) Show that a_n is cube-free for each n (namely, a_n contains no string of the form www, where $w \neq \epsilon$).

27. Let L_1 and L_2 be languages.

(a) Show that if $L_1 \subseteq L_2$ then $L_1^* \subseteq L_2^*$.

(b) Conclude that for any L_1 and L_2 we have $L_1^* \cup L_2^* \subseteq (L_1 \cup L_2)^*$.

(c) Show that it is possible to have $L_1^* \cup L_2^* \neq (L_1 \cup L_2)^*$.

(d) Find languages L_1 and L_2, neither of which contains the other, such that $L_1^* \cup L_2^* = (L_1 \cup L_2)^*$.

28. Let L_1, L_2, L_3 be languages.

(a) Show that

$$(L_1 \cup L_2)\, L_3 = L_1 L_3 \cup L_2 L_3 \ .$$

(b) Show that the following is not necessarily true:

$$(L_1 \cap L_2)\, L_3 = L_1 L_3 \cap L_2 L_3 \ .$$

29. Give an example of infinite languages L_1, L_2 satisfying $L_1 \cap L_2 = \emptyset$ and $L_1 L_2 = L_2 L_1$.

30. Denote by L^C the complement language of $L \subseteq \Sigma^*$. Is it possible that $\left(L^C\right)^* \neq (L^*)^C$? that $\left(L^C\right)^* = (L^*)^C$?

31. Prove by induction that:

(a) $\left(w^R\right)^R = w$ for every word w.

(b) $\left(w^R\right)^n = (w^n)^R$ for every word w and $n \geq 0$.

32. Prove that:

(a) if $L \subseteq \Sigma^*$ then $(L^*)^* = L^*$.

(b) if $L_1, L_2 \subseteq \Sigma^*$ then $(L_1 \cup L_2)^* = (L_1^* L_2^*)^*$. Show that it is impossible in general to replace the right hand side by $(L_1 L_2)^*$. Under what conditions can we do it?

(c) if $L_1, L_2 \subseteq \Sigma^*$ and $\varepsilon \in L_1$ then $(L_1 \cup L_1 L_2)^* = L_1 \left(L_1 \cup L_2 L_1\right)^*$.

33. How many subwords does a word of length n have?

4 Regular Expressions

34. Replace the following regular expressions by simpler regular expressions representing the same languages:

(a) $(0 \cup ((1(1^*))0))$.

(b) $(((0^*)(1^*))^* \cup ((0 \cup 1)^*))$.

(c) $((((0^*)1)^*) \cup (((01^*10)0)^*))$.

(d) $(a \cup \phi)^* \cup (b \cup \phi)^* \cup (ab \cup \phi)^* \cup (a \cup b \cup \phi)^*$.

(e) $a \cup bb^*a$.

(f) $(a^*b)^* \cup (ba^*)^*$.

(g) $(a \cup b)^*a(a \cup b)^*$.

35. Let $\Sigma = \{0, 1\}$. Show that the following languages over Σ are regular:

(a) The collection of all words containing an even number of 0's.

(b) The collection of all words containing an odd number of 0's.

(c) The collection of all words in which the number of 0's is divisible by 3.

(d) The collection of all words containing at least three 0's.

(e) The collection of all words in which the block 01 appears exactly once.

(f) The collection of all words in which the block 00 appears exactly once.

(g) The collection of all words in which the block 000 appears exactly once.

(h) The collection of all words ending with 10.

(i) The collection of all words not ending with 10.

36. How many words of length n do the languages, corresponding to the following regular expressions, contain? (In each case Σ is the set of letters appearing in the expression.)

(a) $(01^*)^*$.

(b) 0^*1^*.

(c) $0^*1^*2^*$.

(d) $0^*1^*0^*$.

(e) $0^*1^*2^*3^*4^*5^*6^*$.

(f) $(0 \cup 10)^*$.

(g) $(0 \cup 11 \cup 22)^*$.

(h) $(01 \cup 02 \cup 12 \cup 000 \cup 111)^*$.

(i) $0^* \cup 1^* \cup (01)^*$.

(j) $(1^*011^*)^*$.

(k) $(01 \cup 10)^*$.

37. Show that, if L is a regular language, then so are the following languages:

(a) L^R.

(b) The language consisting of all suffixes of words in L.

(c) The language consisting of all words in L containing the letter σ (where σ is an arbitrary fixed letter in Σ).

(d) The language consisting of all words in L not containing the letter σ (where σ is an arbitrary fixed letter in Σ).

(e) The language consisting of all words of even length in L.

(f) The language obtained from L by taking only the words starting with the letter σ and deleting it:

$$L = \{w \in \Sigma^* : \sigma w \in L\} .$$

(g) The language obtained from L by deleting the last letter in every non-empty word:

$$L' = \{w : w\sigma \in L\} .$$

(h) The language obtained from L by doubling the last letter in every non-empty word:

$$L' = \{w\sigma^2 : w\sigma \in L\} .$$

(i) The language obtained from L by taking, for each word in L, all words obtained from it by adding a single σ to it:

$$L' = \{w_1 \sigma w_2 : w_1 w_2 \in L\} .$$

(j) The language obtained from L by taking, for each word in L, all words obtained from it by adding any number of times the letter σ:

$$L' = \bigcup_{\sigma_1\sigma_2\ldots\sigma_k \in L} \{\sigma\}^*\{\sigma_1\}\{\sigma\}^*\{\sigma_2\}\{\sigma\}^* \ldots \{\sigma\}^*\{\sigma_k\}\{\sigma\}^* .$$

38. The $*$-*height* of a regular expression is inductively defined as follows:

$h(\phi) = 0.$

$h(a) = 0, \qquad a \in \Sigma.$

$h((\alpha \cup \beta)) = h((\alpha\beta)) = \max(h(\alpha), h(\beta)).$

$h(\alpha^*) = h(\alpha) + 1.$

For example, $h((0^*1 \cup 101)^*01 \cup 0^*1^*) = 2$.

For each of the following regular expressions, find a regular expression of smaller $*$-height representing the same language:

(a) $(0^* \cup 1^* \cup 01)^*$.

(b) $(01^21^*0)^*$.

(c) $(01^*)^*$.

(d) $(00^*1^* \cup 1^*0^* \cup 10^21)^*$.

(e) $((012)^*01)^*$.

(f) $(0(1^*2)^*)^*$.

(g) $((01)^* \cup (10)^*)^*$.

(h) $(\alpha \cup \alpha^*)^*$ (α – an arbitrary regular expression).

(i) $(\alpha\beta^* \cup \beta\alpha^*)^*$ (α, β – arbitrary regular expressions).

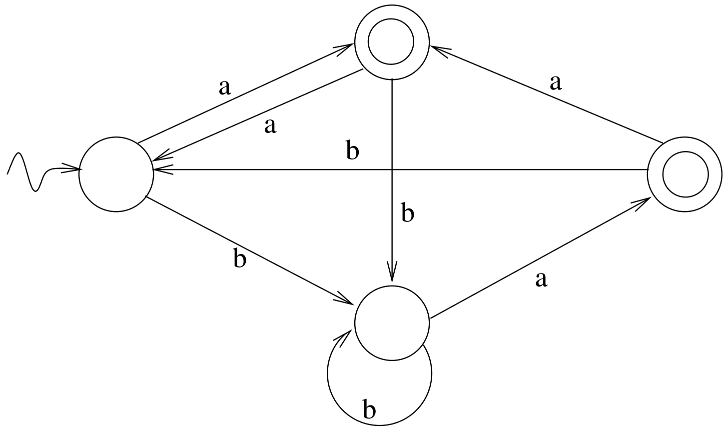
39. Let α, β be regular expressions. Assume that the language corresponding to α does not contain the word ε. Consider the following "equation":

$$\alpha x \cup \beta = x .$$

(Namely, a regular expression x for which both sides represent the same language is sought). Find the solution x and show it is unique.

5 Deterministic Finite Automata

40. Consider the automaton M given in the figure, and:



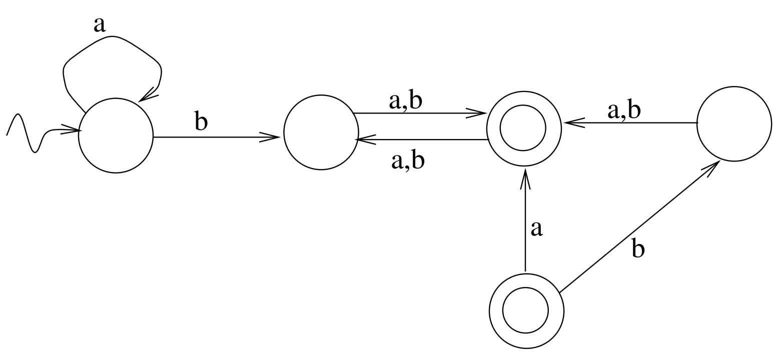
(a) Find three words accepted by M.

(b) Write the configuration sequences corresponding to these words.

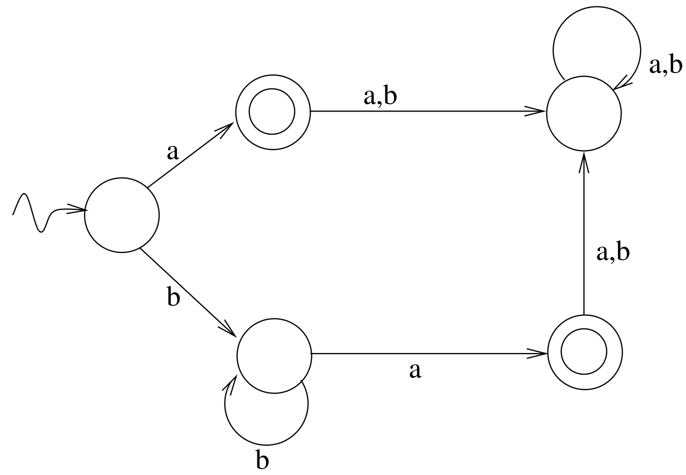
(c) Find three words not accepted by M.

41. Describe (in words or by a regular expression) the languages accepted by the following deterministic finite automata:

(a)



(b)



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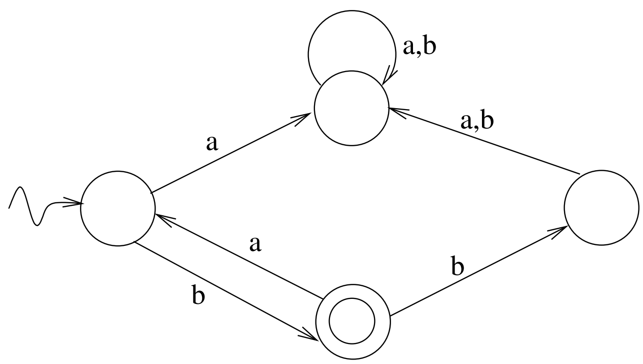

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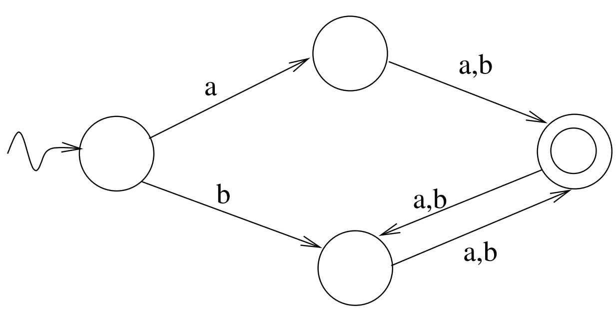

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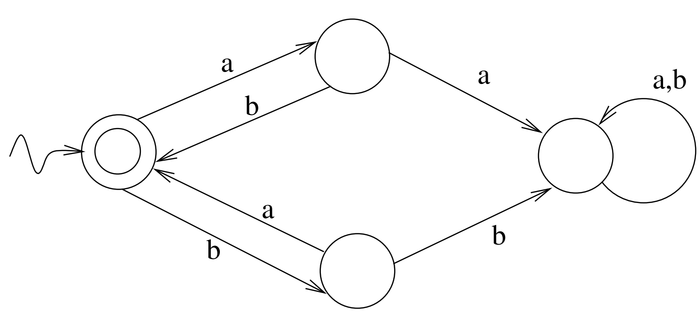
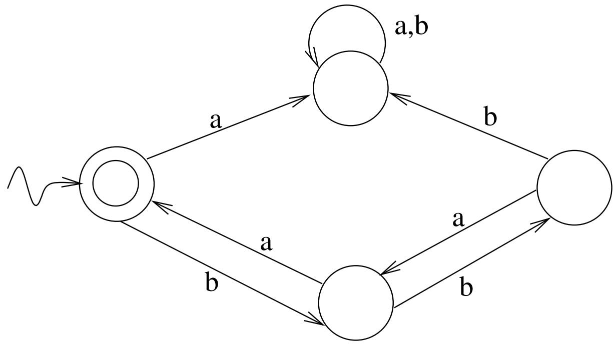
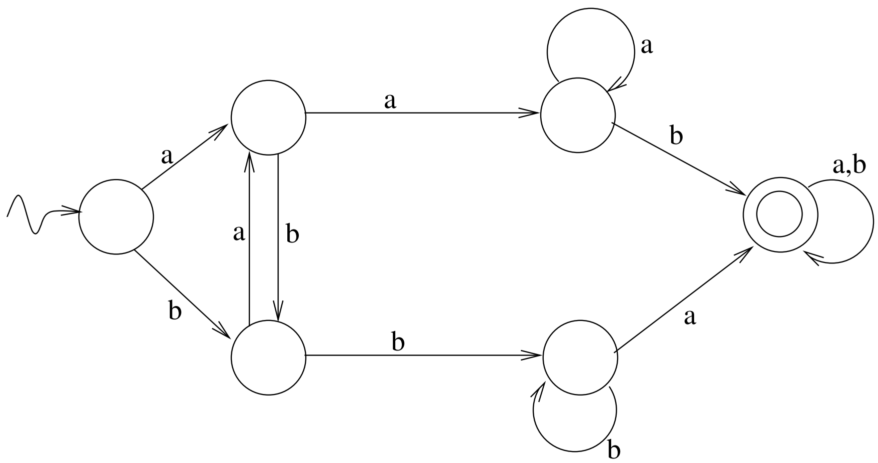


(f)



(g)



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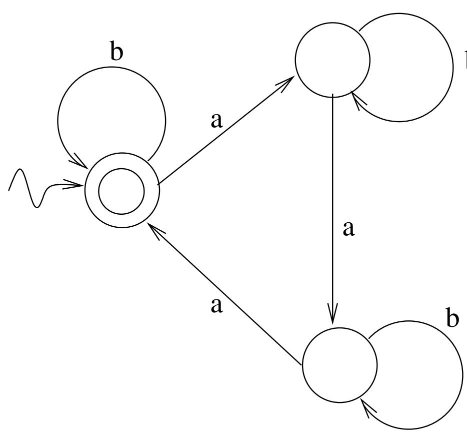
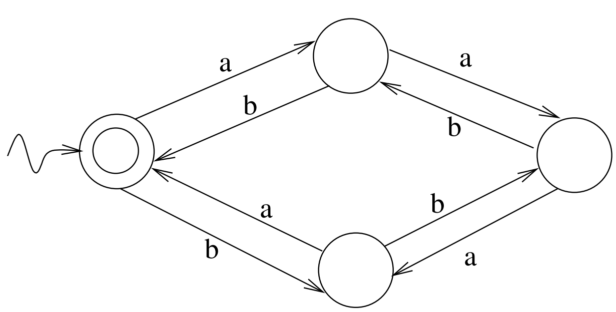


(i)



42. Let $\Sigma = \{a, b\}$. Construct DFA's accepting the following languages:

(a) All words of even length.

(b) All words of odd length.

(c) All words of length 2 modulo 5.

(d) All words in which the number of a's is 2 modulo 5.

(e) All words in which the number of a's is 2 modulo 3 and the number of b's is 2 or 3 modulo 4.

(f) Any given finite language.

(g) Any given language whose complement is finite.

(h) All words not containing ba as a subword.

(i) All words not ending with aab.

(j) All words ending with neither ab nor $aabb$.

(k) All words containing $abab$ as a subword.

(l) All words containing both ab and ba as subwords.

(m) All words containing neither a^2 nor b^2 as subwords.

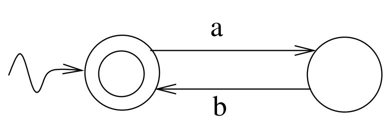
(n) All words in which the third letter from the right is b.

(o) All words with at most one occurrence of a^2 and one occurrence of b^2.

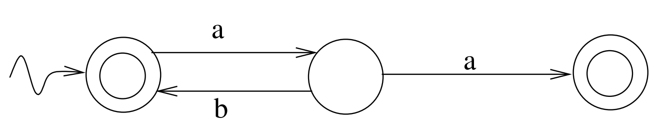
6 Non-Deterministic Finite Automata

43. Describe (in words or by a regular expression) the languages accepted by the following automata:

(a)



(b)



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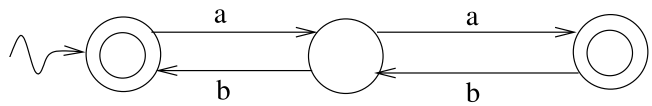

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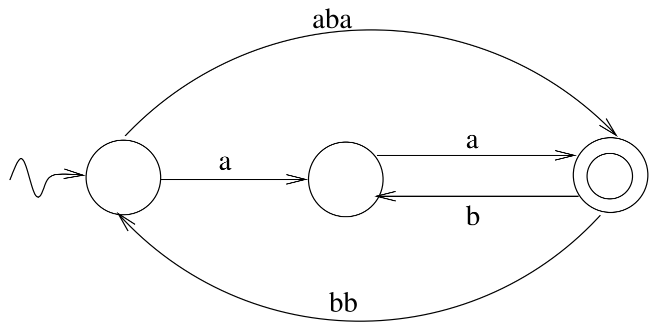

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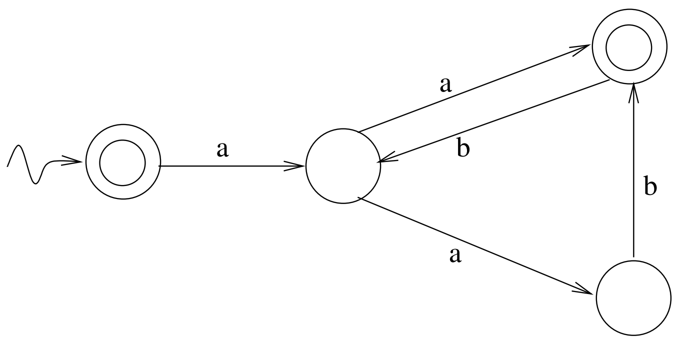

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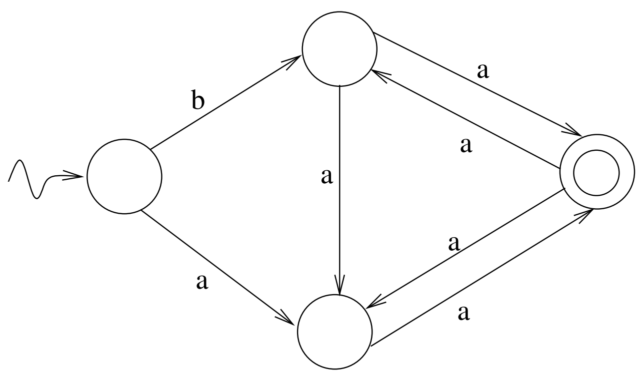

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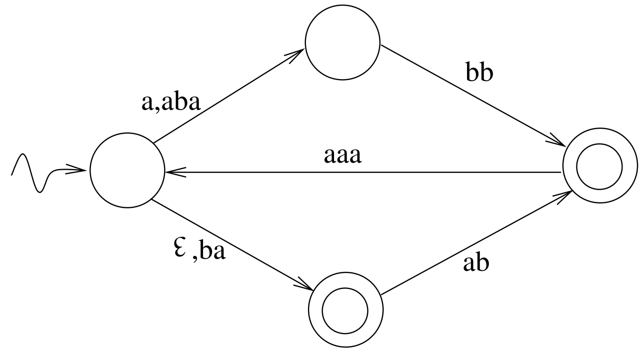
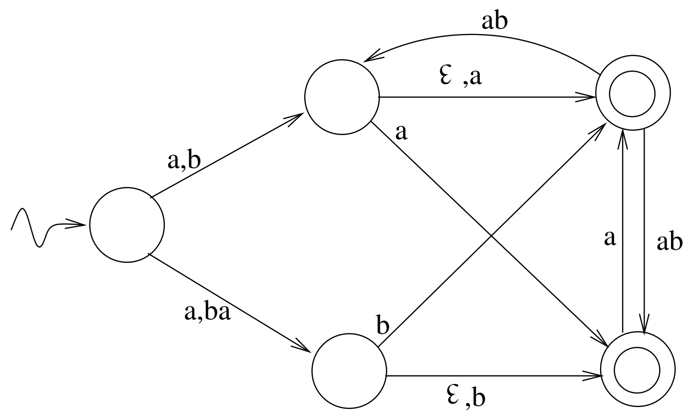


(h)



44.

(a1) Construct an NFA accepting the language consisting of all words over $\{a, b\}$ ending with $aaba$.

(a2) Construct a DFA accepting the same language.

(b) Do the same for the language $(ab \cup a^3 \cup aba)^*$.

45. Construct NFA's accepting the following languages:

(a) $(101 \cup 010)^*$.

(b) $01(10 \cup 001)^*101$.

(c) $(10 \cup 101)1^*$.

(d) $(00 \cup 1)^* \cup (01 \cup 0)^*$.

(e) $((0^*1^*0^*)^*1)^*$.

(f) $(01 \cup 110)^*(1101 \cup 010)^*$.

46. The concept of an NFA may be generalized in such a way that, instead of having a single initial state s, we shall have some set

$S \subseteq Q$, each of whose elements may be the initial state. Show that the family of languages accepted by all automata using this extended definition is identical with that obtained by using only "conventional" NFA's.

47.

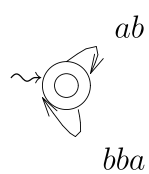
(a) Show that, if we restrict our attention to automata satisfying $A = \{s\}$, then the family of accepted languages is different than that obtained by using all automata.

(b) What family of languages is obtained by automata in which A consists of a single state?

(c) What family of languages is obtained by automata in which $s \notin A$?

48. Construct an equivalent DFA for each of the following NFA's:

(a)



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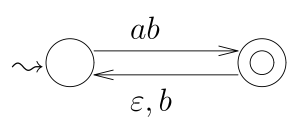

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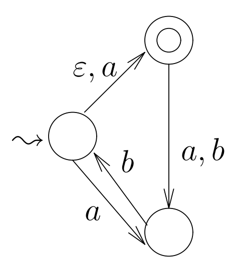

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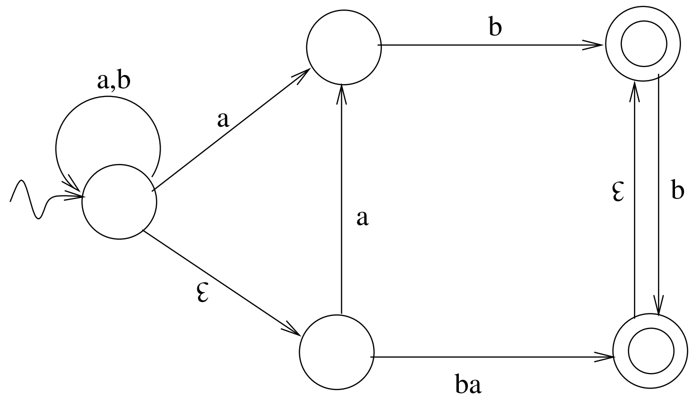


49. Let L be a language accepted by a finite automaton. Show that the following languages are also accepted by finite automata:

(a) L^R.

(b) The language consisting of all words in L whose length is l modulo k for some $0 \le l < k$.

(c) The language consisting of all words in L containing the letter a at least twice and at most 4 times.

(d) The language consisting of all prefixes of words in L.

(e) The language consisting of all suffixes of words in L.

(f) The language consisting of all subwords of words in L.

(g) The set of subsequences of all words in L. (Here a word w_2 is a *subsequence* of a word w_1 if $w_1 = \sigma_1\sigma_2\ldots\sigma_k$ for some $\sigma_1, \sigma_2, \ldots, \sigma_k \in \Sigma$ and $w_2 = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_l}$, where $1 \le i_1 < i_2 < \ldots < i_l \le k$.)

(h) All words in L which are not strict prefixes of words in L, that is $\{w \in L \,:\, L \cap (\{w\}(\Sigma^* - \{\varepsilon\})) = \emptyset\}$.

(i) The right quotient L/L_1 of L over some language L_1, where: $L/L_1 = \{w \in \Sigma^* \,:\, \exists w_1 \in L_1,\ ww_1 \in L\}$.

(j) $L' = \{w_1\sigma^2 \,:\, w_1\sigma \in L,\ w_1 \in \Sigma^*,\ \sigma \in \Sigma\}$.

50. Given two automata $M_1,\ M_2$, with state sets $Q_1,\ Q_2$, respectively, construct an automaton M_3 whose state set is $Q_1 \times Q_2$ such that $L(M_3) = L(M_1) \cap L(M_2)$.

51. A language L is *absolute* if there exists a positive integer l such that, for any word $w \in \Sigma^*$, belonging to L depends only on the l last letters of w.

(a) Prove that every absolute language is accepted by a finite automaton.

(b) Show that the family of absolute languages over Σ is closed under finite unions, finite intersections and complementation, but not under concatenation and Kleene-$*$.

52. Let Σ_1, Σ_2 be some alphabets. Let $h : \Sigma_1 \to \Sigma_2^*$ be any function. We can extend h to a function defined on all Σ_1^* by the following inductive definition:

$$h(\varepsilon) = \varepsilon,$$

$$h(w\sigma) = h(w)h(\sigma), \qquad w \in \Sigma_1^*,\ \sigma \in \Sigma_1,$$

(where the extension of h is denoted by h as well). Notice that h is well defined and

$$h(w_1 w_2) = h(w_1)h(w_2), \qquad w_1, w_2 \in \Sigma_1^*.$$

An h satisfying these properties is a *homomorphism* from Σ_1^* to Σ_2^*.

(a) Show that, if a language $L \subseteq \Sigma_1^*$ is accepted by a finite automaton, then the language $h(L) = \{h(w) :\ w \in L\} \subseteq \Sigma_2^*$ is also accepted by a finite automaton.

(b) Show that, if a language $L \subseteq \Sigma_2^*$ is accepted by an FA, then so is the language $h^{-1}(L) = \{w \in \Sigma_1^* :\ h(w) \in L)\} \subseteq \Sigma_1^*$. (Hint: start from an FA accepting L and construct one with the same state set. Given a letter $\sigma \in \Sigma_1$, act as in the original automaton for the word $h(\sigma)$.)

(c) In section a (b, respectively), if it is given that L is not accepted by any FA, is it necessarily the case that $h(L)$ ($h^{-1}(L)$, respectively) is also not accepted by any FA?

53. Given a language $L \subseteq \{a\}^*$, denote $P(L) = \{n :\ a^n \in L\}$. Prove that L is accepted by an FA if and only if the sequence $P(L)$ is eventually periodic, that is, it satisfies the following condition: there exist a non-negative integer n_0, a positive integer d and integers $0 \le d_1 < d_2 < \cdots < d_l < d$ such that for $n \ge n_0$ we have $n \in P(L)$ if and only if $n \bmod d = d_i$ for some $1 \le i \le l$.

54. Given a language $L \subseteq \Sigma^*$, denote $P(L) = \{|w| :\ w \in L\}$.

(a) If L is known to be accepted by an FA, what can you say about $P(L)$ in view of exercises 52.a and 53?

(b) Show that the condition you found in the preceding part on $P(L)$ is not sufficient for L to be accepted by an FA.

55. Write a program which determines whether some words are accepted by a given NFA. The data regarding the NFA is given in a text file named *automaton.txt*, as follows. The first line of the file contains a positive integer, which is the number of states of the NFA. If the number is n, then $Q = \{q_1, q_2, \ldots, q_n\}$. The alphabet Σ is (a subset of) $\{a, b, \ldots, z\}$. The initial state is $s = q_1$. The second line of the input file contains a few positive integers. If these integers are $i_1, i_2, \ldots, i_k$, then $A = \{q_{i_1}, q_{i_2}, \ldots, q_{i_k}\}$. The set of rules Δ is given by all other lines of the file, from the third on, each rule in a separate line. Each line consists of an integer i, a single space, a word w over Σ, another space and then another integer j. The data above signifies that $(q_i, w, q_j) \in \Delta$. If $w = \varepsilon$ then the integer i is succeeded by two spaces and then the integer j. The program asks the user to supply words (from the keyboard), and for each such word decides whether it is accepted by the NFA or not (and sends a message accordingly). If it is accepted, then the program further provides a sequence of configurations showing how to get from s to a final state. Each configuration is given in a separate line composed of an integer i (meaning that the automaton is in state q_i) and a word w (still to be read) preceded by a single space.

56.

(a) Present a simple algorithm which, given a DFA, determines whether the language it accepts is finite or infinite.

(b) Implement the algorithm in a computer program.

57. Write a program which, given an NFA, constructs an equivalent DFA.

58. The well-known Towers of Hanoi problem gives rise to several languages, as follows. Let n, the number of disks, be arbitrary and fixed. Consider the 6-letter alphabet

$$\Sigma = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}.$$

Any "letter" $(i, j) \in \Sigma$ should be interpreted as the instruction "move the disk on top of peg i to the top of peg j". A word in Σ^* is thus a sequence of instructions. For example, if $n = 2$, the word $(1,2)(2,3)(1,2)(3,2)$ represents a sequence of instructions for moving both disks from the initial position, in which both are placed on peg 1, to the final position, in which both are placed on peg 2 (although this is done non-optimally). The word $(1,2)(1,3)(2,3)$ is another word which represents a sequence of legal moves, but does not lead at the end to the required final position. The word $(1,2)(1,2)(2,3)$ represents an illegal sequence of moves, as at the second step disk 2 is placed above disk 1.

(a) Let $L_{\text{legal}} \subseteq \Sigma^*$ denote the language consisting of all words representing sequences of instructions which, when starting from the initial position (all disks on peg 1), correspond to legal sequences of disk moves. (Thus, for $n = 2$, we have $(1,2)(1,3)(2,3) \in L_{\text{legal}}$, whereas $(1,2)(1,2)(2,3) \notin L_{\text{legal}}$.) Explain why L_{legal} is accepted by a finite automaton. Write a computer program which reads the number of disks n (where n may be assumed to be bounded by some large constant) and gives an automaton accepting L_{legal}. The format of the automaton is as described above.

(b) The same for the language $L_{\text{good}} \subseteq L_{\text{legal}}$, consisting of those words representing sequences of instructions leading in a legal way from the initial position to the final position. (Thus, for $n = 2$, we have $(1,2)(2,3)(1,2)(3,2) \in L_{\text{good}}$, whereas $(1,2)(1,3)(2,3) \notin L_{\text{good}}$.)

59. Use the idea of the proof that every regular language is accepted by an FA to construct FA's accepting the languages corresponding to the following expressions:

(a) $a^*ba^*ab^*$.

(b) $(a^* \cup b^*)c^*$.

(c) $a^*(b \cup a^3b^2)b^*$.

(d) $a^*bb^*(a \cup a^2ba)ba^*$.

(e) $ba \cup (a \cup bb)a^*b$

(f) $ab(((ab)^* \cup b^3)^* \cup a)^*$.

(g) $((a \cup b)^2)^* \cup ((a \cup b)^3)^*$.

(h) $a((a^2b)^* \cup (ac)^* \cup (b \cup c)^*)b^2$.

(i) $b((b^2ab^* \cup ab)^*)a^*$.

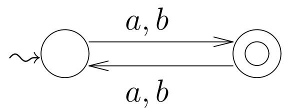
(j) $((ab)^* \cup (bc)^*)ab$.

(k) $((a \cup b)^*(\emptyset^* \cup c)^*)^*$.

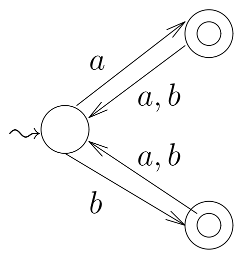
(l) $a^*(ab \cup ba \cup \emptyset^*)b^*$.

60. Use the idea of the proof that every language accepted by an FA is regular to find regular expressions corresponding to the languages accepted by the following automata:

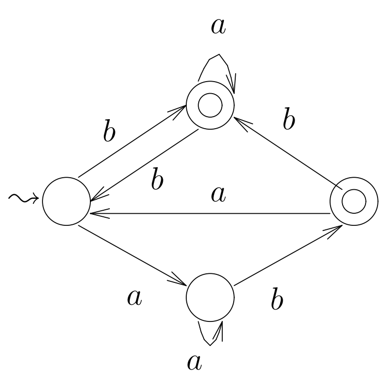
(a)



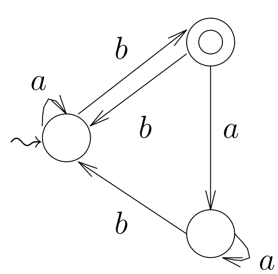
(b)



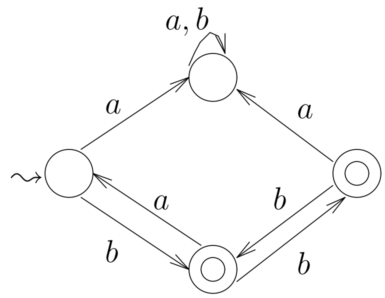
(c)



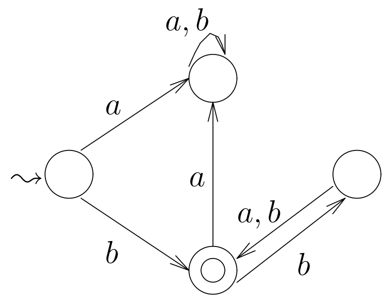
(d)



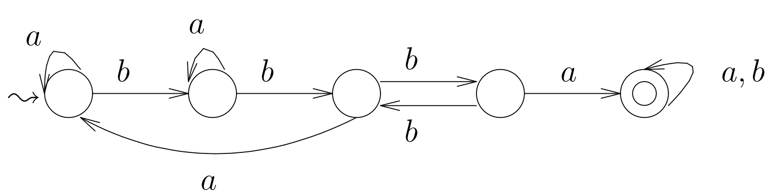
(e)



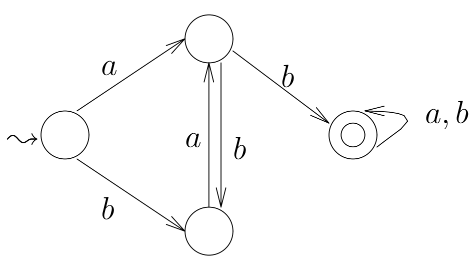
(f)



(g)



(h)



61. Show directly (without using automata) that, if a language L is regular, then so is the language $L^R = \{w^R : \ w \in L\}$.

62. Show that, if L is regular, then so is $\{w : \ w \in L, \ \ w^R \in L\}$.

63. Given a language L define: $L_1 = \{vw : \; v \in L, \; w \notin L\}$.

(a) Show that if L is regular then so is L_1.

(b) Show that the converse is not necessarily true.

64. Suppose we have an eventually periodic infinite sequence of letters from Σ:

$$\sigma_1, \; \sigma_2, \ldots, \sigma_{n_0}, \; \sigma_{n_0+1}, \ldots, \sigma_{n_0+p-1}, \; \sigma_{n_0}, \; \sigma_{n_0+1}, \ldots, \sigma_{n_0+p-1}, \ldots .$$

Put: $L = \{w \in \Sigma^* : \; w = \sigma_m \sigma_{m+1} \ldots \sigma_n, \;\; m, n \geq 1\}$ (where $w = \varepsilon$ for $m > n$). Show that L is regular.

65. Show that, if a regular language L is infinite, then there exist regular infinite languages $L_1, \; L_2$ such that $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$.

7 Pumping Lemma

66. Show that the following languages are not regular:

(a) $\{a^n b^{n+1} : \; n \geq 0\}$.

(b) $\{a^n b^{2n} : \; n \geq 0\}$.

(c) $\{a^m b a^n b a^{m+n} : \; m, n \geq 0\}$.

(d) $\{a^{n_1} b^{n_2} c^{n_3} d^{n_1+n_2+n_3} : \; n_1, n_2, n_3 \geq 0\}$.

(e) All words in $\{a, b\}^*$ in which the number of a's is distinct from the number of b's.

(f) $\{a^m b^n c^n : \; m, n \geq 0\}$.

67. Prove the following strengthened version of the pumping lemma: Let M be a finite automaton and let x be any word in $L(M)$ with $|x| \geq |Q|$ (where $|Q|$ denotes the number of elements of Q). Then there exist words $u, \; v, \; w \in \Sigma^*$ such that $uvw = x, \;\; |uv| \leq |Q|, \;\; v \neq \varepsilon$ and $uv^n w \in L$ for every $n \geq 0$.

68. Use the preceding exercise to show that the following languages are not regular:

(a) $\{w^2 : \;\; w \in \Sigma^*\}, \qquad (|\Sigma| \geq 2)$.

(b) $\{w^k : \;\; w \in \Sigma^*\}, \qquad (|\Sigma| \geq 2, \;\; k \geq 2)$.

(c) $\{ww^R : \ w \in \Sigma^*\}, \qquad (|\Sigma| \geq 2)$.

(d) $\{w\overline{w} : \ w \in \{a,b\}^*\}$, where $\overline{w}$ is the word obtained from w by replacing each a with b and vice versa.

69. Prove the following strengthened version of the pumping lemma: Given a regular language L there exists a constant s such that, if z_1, z_2, z_3 are any words with $z_1z_2z_3 \in L$ and $|z_2| = s$, then there exist words u, v, w with $z_2 = uvw$ and $v \neq \varepsilon$ such that $z_1uv^nwz_3 \in L$ for every $n \geq 0$.

70. Use the preceding exercise to show that the following languages are not regular:

(a) $\{a^mb^n : \ m,n \geq 0, \ \gcd(m,n) > 1\}$. (Recall that $\gcd(m,n)$ is the greatest common divisor of m and n.)

(b) $\{a^mb^n : \ m,n \geq 0, \ \gcd(m,n) = 1\}$.

71. Which of the following languages are regular and which are not:

(a) $\{a^{n^3+2n^2} : \ n \geq 0\}$.

(b) $\{\sigma_1\sigma_2^n\sigma_3 : \ \sigma_1,\sigma_2,\sigma_3 \in \Sigma, \ \sigma_1 \neq \sigma_2, \ \sigma_1 \neq \sigma_3, \ \sigma_2 \neq \sigma_3, \ n \geq 2\}$.

(c) $\{ww^Rw : \ w \in \{a,b\}^*\}$.

(d) All words in $\{a,b\}^*$ in every proper prefix of which the difference between the number of a's and the number of b's does not exceed 100 in its absolute value. (Thus, for the word itself the difference in question may be 101.)

(e) $\{a^mb^n : \ m,n \geq 0, \ m \leq n \leq 2m\}$.

(f) All words over the Hebrew alphabet starting with a final letter and ending with a non-final letter.

(g) All words over the Hebrew alphabet in which no letter occurs more than three times.

72. Given a set of non-negative integers, the set of their expansions in base 10 forms a partial language of $\{0,1,\ldots,9\}^*$. For each of the following sets show that the corresponding language is regular or not (as indicated):

(a) All even numbers (regular).

(b) All numbers divisible by 10 (regular).

(c) All numbers congruent to 333 modulo 625 (regular).

(d) All numbers divisible by 3 (regular. Hint: You may use the fact that an integer is divisible by 3 if and only if the sum of its digits is such.)

(e) All numbers divisible by 11 (regular. Hint: A number is divisible by 11 if and only if the alternating sum of its digits is such. For example: 379181737 is divisible by 11 since $3-7+9-1+8-1+7-3+7=2\cdot 11$.)

(f) All numbers congruent to r modulo d, where $0 \le r < d$ (regular).

(g) All powers of 10 (regular).

(h) All powers of 2 (not regular. Hint: Use the pumping lemma and the fact that no power of 10 but 10^0 is a power of 2.)

(i) All perfect squares (not regular. Hint: Start with special perfect squares.)

(j) $\{a_n : \ n \ge 0\}$ where $a_0 = 3$ and $a_n = a_{n-1}^2$ for $n \ge 1$ (not regular).

(k) $\{2^m 5^n : \ m, n \ge 0, \ |m-n| \le 10\}$ (regular).

73. With definitions as in the preceding exercise, determine whether the following sets are regular:

(a) $\{10^m + 10^n + 1 : \ m > n \ge 1\}$.

(b) $\{10^{2n} + 10^n + 1 : \ n \ge 1\}$.

(c) $\{27 \cdot 10^m + 19 \cdot 10^n + 53 : \ m > n+3, \ n > 2\}$.

(d) $\{27 \cdot 10^m - 19 \cdot 10^n - 53 : \ m \ge \max(2, n)\}$.

(e) $\{3^{n^2} : \ n \ge 0\}$.

(f) $\{n! : \ n \ge 0\}$.

(g) The set $\{a_n : \ n \ge 0\}$, where

$$a_n = \begin{cases} 0, & n = 0, \\ a_{n-1} + 1, & n \ge 1, \ a_{n-1} \text{ has an odd number of digits,} \\ a_{n-1} + 2, & n \ge 1, \ a_{n-1} \text{ has an even number of digits.} \end{cases}$$

(g) The set of all Fibonacci numbers $\{F_n : \ n \ge 0\}$, where $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. (You may use the fact that no power of $\frac{1+\sqrt{5}}{2}$ is an integer.)

(h) All perfect cubes.

74. Given a word $w \in \{a,b\}^*$, denote $R_{a,b}(w) = \#_a(w)/\#_b(w)$, where $\#_\sigma(w)$ is the number of occurrences of σ in w. (Thus, $R_{a,b}(\varepsilon)$ is undefined, and we may take $R_{a,b}(w)$ to be infinite for $w \in \{b\}^+$.) A non-negative real number r (or ∞) is *a limit ratio* for a language $L \subseteq \{a,b\}^*$ if there exists a sequence $(w_n)_{n=0}^{\infty}$ of distinct words in L such that $R_{a,b}(w_n) \underset{n\to\infty}{\longrightarrow} r$. Denote by $R_{a,b}(L)$ the set of all limit ratios for L. For example, $R_{a,b}(\{a,b\}^*) = [0,\infty]$ and $R_{a,b}(L) = [\alpha,\beta]$ if $L = \{w \in \{a,b\}^* : \alpha \le R_{a,b}(w) \le \beta\}$ (for any $0 \le \alpha \le \beta \le \infty$).

(a) Show that, if F is any finite set of (non-negative) rationals, then there exists a regular language L with $R_{a,b}(L) = F$.

(b) The same for any interval with rational endpoints.

(c) Prove that, if L is a regular language such that $R_{a,b}(L)$ contains an irrational, then $R_{a,b}(L)$ contains a sequence of rationals converging to it. (Hint: You need to pump large parts, perhaps not consecutive, of given words.)

8 Context-Free Grammars

75. Find $L(G)$ for each of the following grammars:

(a) $S \to \varepsilon \mid abSba$.

(b) $S \to SS \mid a \mid b$.

(c) $S \to aS \mid b$.

(d) $S \to ba^2b^3 \mid a^2bSb$.

(e) $S \to SS \mid aS \mid ab$.

(f) $S \to \varepsilon \mid aaSb \mid bbSa$.

(g) $S \to bA \mid abA$,

$A \to \varepsilon \mid S$.

(h) $S \to aAb$,

$A \to \varepsilon \mid S \mid Sb$.

(i) $S \to aAb \mid bAa$,

$A \to \varepsilon \mid S \mid aSa$.

(j) $S \to aA \mid bS \mid b$,

$A \to aB \mid bA$,

$B \to aS \mid bB \mid a$.

(k) $S \to aA \mid bB$,

$A \to bB$,

$B \to \varepsilon \mid S$.

(l) $S \to AB$,

$A \to aS \mid bB \mid \varepsilon$,

$B \to bS \mid aA \mid \varepsilon$.

(m) For any alphabet $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$:

$S \to \varepsilon$,

$S \to \sigma_i S \sigma_j, \quad 1 \le i, j \le m.$

(Thus, R contains $m^2 + 1$ elements.)

76. Show that the following grammars accept the indicated languages:

(a) $S \to aS \mid aSbS \mid \varepsilon$.

$L(G)$ – all words in which the number of a's in any prefix is at least as large as the number of b's in that prefix.

(b) $S \to \varepsilon \mid aSb \mid bSa \mid SS$.

$L(G)$ – all words in which the number of a's is equal to the number of b's.

(c) $S \to aB \mid bA$,

$A \to a \mid aS \mid bAA$,

$B \to b \mid bS \mid aBB$.

$L(G)$ – all words of positive length with the same number of a's and b's.

(d) $S \to SS \mid aSb \mid c$

$L(G)$ – all non-empty words in $\{a, b, c\}^*$ satisfying:

i) The number of a's is equal to the number of b's.

ii) In each prefix, the number of a's is no less than the number of b's.

iii) ab is not a subword.

(e) $S \to SS \mid aSb \mid aSbc \mid \varepsilon$.

$L(G)$ – all words in $\{a, b, c\}^*$ satisfying:

i) The number of a's is equal to the number of b's.

ii) In each prefix, the number of a's is no less than the number of b's.

iii) Every c is preceded by a b.

77. Recall that the grammar
$S \to S \to \epsilon \mid aSb \mid SS$
accepts the language consisting of all words in $\{a,b\}^*$ satisfying:

i) The number of a's is equal to the number of b's.

ii) In each prefix, the number of a's is no less than the number of b's.

Write a computer program which, given a word w in $\{a,b\}^*$ determines whether it satisfies these two properties, and if so – provides a sequence of productions leading from S to w.

78. Construct context-free grammars accepting the following languages:

(a) $\{a^{m+3}b^{2m+1} \;:\; m \geq 0\} \cup \{a^{3m+1}b^{2m} \;:\; m \geq 0\}$.

(b) $\{a^m b^n c^n d^m e^3 \;:\; m, n \geq 0\}$.

(c) $\{a^m b^n \;:\; 0 \leq m \leq n\}$.

(d) $\{a^i b^j c^k d^l \;:\; i, j, k, l \geq 0, \;\; i \neq l, \;\; j > k\}$.

(e) $\{a^m b^{m+n} a^n \;:\; m, n \geq 0\}$.

(f) $\{a^i b^j c^k d^l \;:\; i + j = k + l\}$.

(g) $\{a^m b^n \;:\; m \leq n \leq 3m\}$.

(h) $\{a^m b^n \;:\; m = 3n \;\vee\; n = 3m\}$.

(i) $\{ww^R \;:\; w \in \Sigma^*\}$ (Σ – an arbitrary alphabet).

(j) $\{w \in \Sigma^* \;:\; w^R = w\}$ (Σ – an arbitrary alphabet).

(k) $\cup_{m=2}^{\infty} L\left((a^*b)^m(b^*a)^m\right)$.

(l) $\cup_{m=0}^{\infty} L\left((a^*b)^m c(b^*a)^m\right)$.

(m) $\cup_{m=0}^{\infty} L\left(a^*b^{2m}a(a^*c)^m\right)$.

(n) $\{a^l b^m c^n \;:\; l, m \geq 1, \; n = |l - m|\}$.

(o) All words in $\{a,b\}^*$ starting with an a and containing an even number of a's.

(p) All words of length 2 modulo 5 over any alphabet Σ.

(q) All words in $\{a,b,c\}^*$ in which a and b appear exactly once each.

(r) All words in $\{a, b\}^*$ not containing the word aaa as a subword.

(s) $\{w \in \Sigma^* \ : \ w^R \neq w\}$ (Σ – an arbitrary alphabet).

(t) All words in $\{a, b, c, d\}^*$ starting and ending with an a such that between any two consecutive a's there is an even number of b's.

(u) $\{avbw \ : \ v, w \in \{a, b\}^*, \ |v| = |w|\}$.

(v) $\{wcw^R \ : \ w \in \{a, b\}^*\}$.

(w) All words in $\{a, b, c, d\}^*$ satisfying the following conditions:

i) All a's appear before the b's and the c's.

ii) All d's appear after the b's and the c's.

iii) The number of a's is equal to the number of d's and is positive.

iv) The number of c's is even and positive.

79. Consider the following grammars:

1. $S \to a \mid aS \mid bSS \mid SbS \mid SSb$.
2. $S \to bSa \mid aSb \mid SS \mid ba \mid ab \mid \varepsilon$.
3. $S \to b \mid bS \mid aSS \mid SaS \mid SSa$.

(a) Prove that the language accepted by the first grammar consists of all words over $\{a, b\}$ containing more a's than b's, by the second – all words containing the same number of a's and b's, and by the third – all words containing less a's than b's.

(b) Write a program which reads words over $\{a, b\}$, determines to which of the three languages each word belongs, and then shows step by step a sequence of productions which leads from S to the required word.

9 Regular Grammars

80. Construct DFA's accepting the same languages as the following grammars:

(a) $S \to aA \mid b^2B \mid ca^2b$,

$A \to bA \mid ab \mid aB \mid cac$,

$B \to ba \mid A \mid ba^2B$.

(b) $S \to \varepsilon \mid A \mid ba^2A \mid ab^2B,$

$A \to bS \mid ba,$

$B \to a^2S.$

(c) $S \to aS \mid bS \mid bA,$

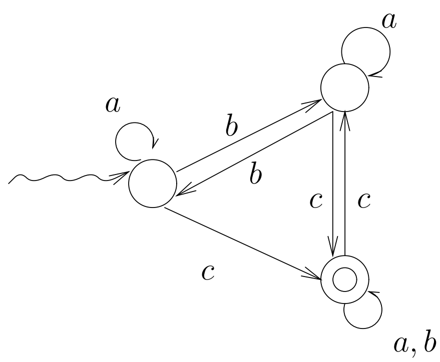
$A \to bB,$

$B \to bC,$

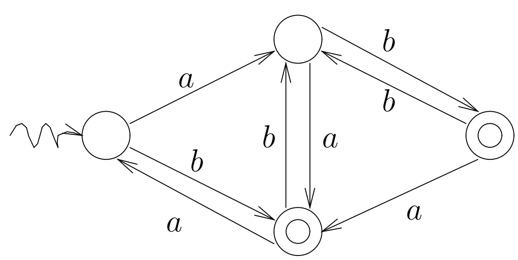
$C \to aC \mid bC \mid \varepsilon.$

81. Construct regular grammars accepting the same languages as the following DFA's:

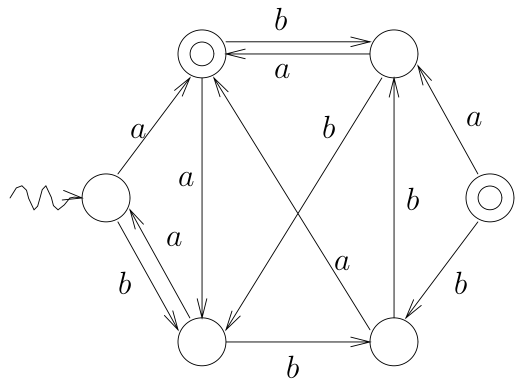
(a)



(b)



(c)



82. A context-free grammar is *left-regular* if $R \subset N \times (N \cup \{\varepsilon\})\Sigma^*$. Prove that a language is regular if and only if it is accepted by a left-regular context-free grammar.

83. Show that, if L is a context-free language, then so is L^R.

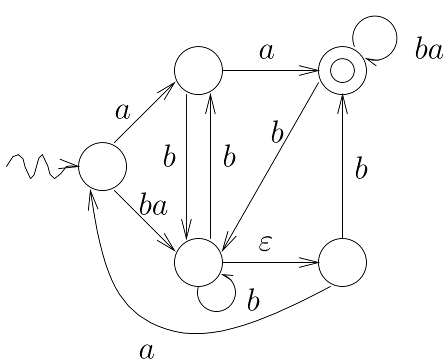
84. Consider the following condition for a context-free grammar:

$$R \subset N \times (\Sigma^*(N \cup \{\varepsilon\}) \cup N\Sigma^*) .$$

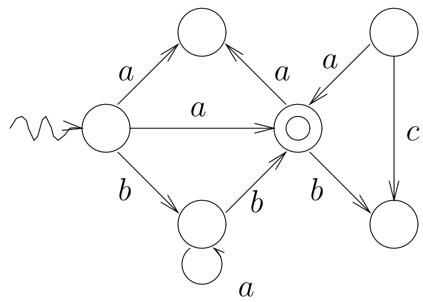
Is a language accepted by a grammar satisfying this condition necessarily regular?

85. Construct regular grammars accepting the same languages as the following automata:

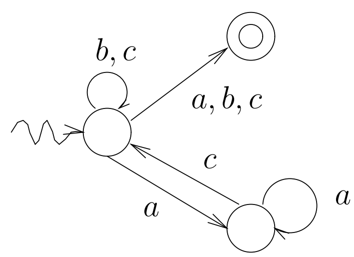
(a)



(b)



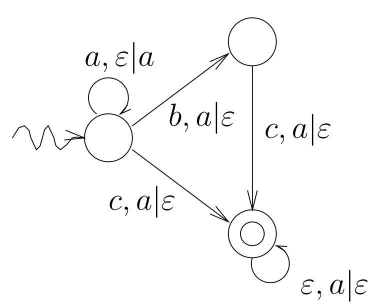
(c)



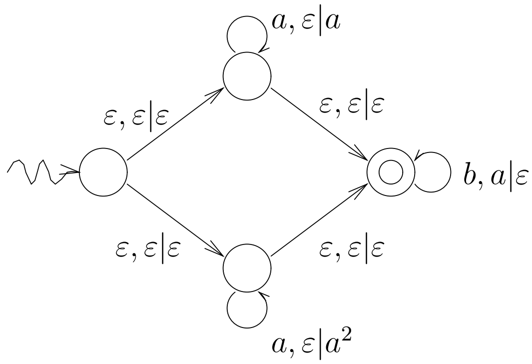
10 Pushdown Automata

86. Describe the languages accepted by the following pushdown automata:

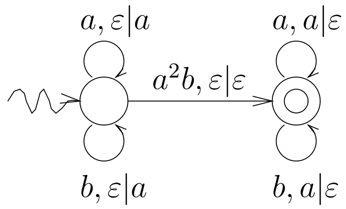
(a)



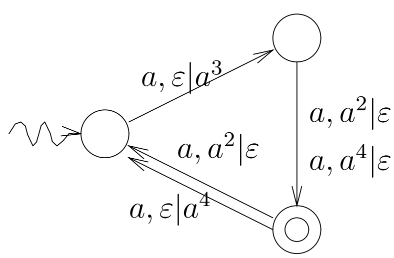
(b)


$a, \varepsilon|a$
$\varepsilon, \varepsilon|\varepsilon$
$\varepsilon, \varepsilon|\varepsilon$
$b, a|\varepsilon$
$\varepsilon, \varepsilon|\varepsilon$
$\varepsilon, \varepsilon|\varepsilon$
$a, \varepsilon|a^2$

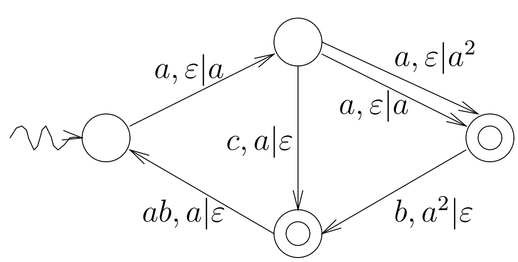
(c)


$a, \varepsilon|a$
$a, a|\varepsilon$
$a^2b, \varepsilon|\varepsilon$
$b, \varepsilon|a$
$b, a|\varepsilon$

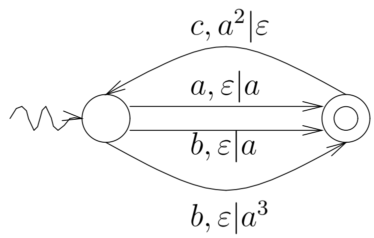
(d)


$a, \varepsilon|a^3$
$a, a^2|\varepsilon$
$a, a^2|\varepsilon$
$a, a^4|\varepsilon$
$a, \varepsilon|a^4$

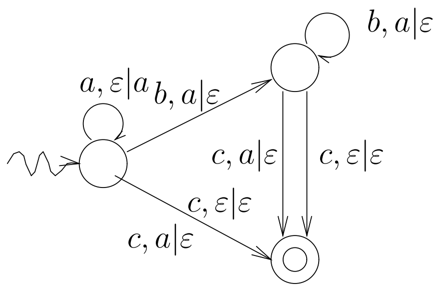
(e)


$a, \varepsilon|a$
$a, \varepsilon|a^2$
$a, \varepsilon|a$
$c, a|\varepsilon$
$ab, a|\varepsilon$
$b, a^2|\varepsilon$

(f)



(g)



87. Construct pushdown automata accepting the same languages as the following grammars:

(a) $S \to \varepsilon \mid SS \mid AS \mid SB$,

$A \to aB \mid a^2$,

$B \to b^2S \mid baA \mid b^2$.

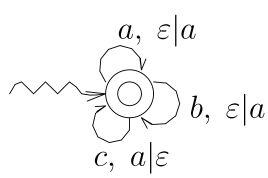
(b) $S \to aS \mid ASB \mid ScS$,

$A \to cAc \mid bBb \mid ba$,

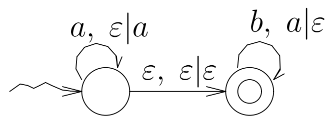
$B \to Sb \mid AcA \mid BB$.

88. Construct context-free grammars accepting the same languages as the following pushdown automata:

(a)



(b)



89. Write a program receiving as its input a pushdown automaton and producing a grammar accepting the same language. The automaton is defined in a text file named *PDA* as follows: In the first line – a positive integer giving the number of states of the automaton. The states are $1,\ 2,\dots$. In the second line – a sequence of arbitrary length of characters; this is the alphabet Σ. In the third line – Γ, given similarly. Fourth line – the initial state number. Fifth line – a sequence of numbers; these are the accepting states. Sixth line on – the elements of Δ. These are given one per line, in the order p, w, α, q, β, where p, q are positive integers (states), w is a word over Σ, and $\alpha,\ \beta$ are words over Γ. These five components are separated by single spaces from each other. The empty word is denoted by 0 (assume that Σ and Γ do not include the characters 0 and "space"). The output of the program consists first of a file called *simple*, containing a simple automaton equivalent to the original one. This file has in principle the same format as the file *PDA*. The program also generates a file named *grammar*, where the grammar is described: In the first line appears the title *non-terminals*, and thereafter in each line one non-terminal character of the grammar is written. Each non-terminal appears in the form $< p, A, q >$, where p, q are numbers (signifying states) and A is either a letter or 0 (the empty word). In the next line we have the heading *rules*, and all the following lines are devoted to stating the rules, each rule being on a separate line.

90. Write a program receiving as its input a context-free grammar and producing a PDA accepting the same language. Use representations similar to those of the preceding exercise.

11 Closure Properties; Non-Context-Freeness Proofs

91. Use the property of closure under union to show that the following languages are context free:

(a) $\{a^m b^n : \ m \neq n\}$.

(b) $\{a, b\}^* - \{a^n b^n : \ n \geq 1\}$.

(c) $\{a^i b^j c^k d^l : \ \ j = l \ \vee \ i \le k \ \vee \ i + j = k + l\}$.

(d) $\{w \in \{a, b\}^* : \ \ w^R = w\}$.

92. Prove that the following languages are not context-free:

(a) $\{a^p : \ \ p \ - \ $ a prime $\}$.

(b) $\{a^{2^n} : \ n \ge 0\}$.

(c) $\{w \in \{a, b, c\}^* : \ \ |w|_a = |w|_b = |w|_c\}$ (where $|w|_\sigma$ denotes the number of occurrences in w of the letter σ).

(d) $\{a^n b^{n^2} : \ \ n \ge 0\}$.

(e) $\{www : \ \ w \in \Sigma^*\}$, $\qquad (|\Sigma| \ge 2)$.

(f) $\{a^m b^m c^n : \ \ 0 \le m \le n \le 2m\}$.

(g) $\{a^n b a^n b a^n : \ \ n \ge 0\}$.

93. Are the following languages context-free?

(a) $\{a^m b^m a^n b^n : \ \ m, n \ge 0\}$.

(b) $\{w w^R w : \ \ w \in \{a, b\}^*\}$.

(c) $\{A_n H c_1 c_2 \ldots c_n \ : \ \ n \in \mathbf{N}, c_i \in \{a, b\}, \ 1 \le i \le n\} \subseteq \{0, 1, H, a, b\}^*$, where A_n is the binary expansion of n.

(d) $\{x_1 y_1 x_2 y_2 \ldots x_m y_m : \ x_i \in \{a, b\}, y_i \in \{\bar{a}, \bar{b}\}, \ m \ge 1, y_i = \bar{x_i}, 1 \le i \le m\}$.

94. Let $h : \Sigma_1^* \to \Sigma_2^*$ be a homomorphism (see exercise 52). Prove that:

(a) If $L \subseteq \Sigma_1^*$ is context-free then so is $h(L) \subseteq \Sigma_2^*$.

(b) If $L \subseteq \Sigma_2^*$ is context-free then so is $h^{-1}(L) \subseteq \Sigma_1^*$.

95. Prove that the pumping lemma for context-free languages can be strengthened as follows: Given a context-free grammar G, there exist constants $K = K(G)$, $\ k = k(G)$, such that if $w \in L(G)$, $\ |w| \ge K$, then there exist words $u, v, x, y, z \in \Sigma^*$, with $|vxy| \le k$, $\ vy \ne \varepsilon$ and $w = uvxyz$, such that $uv^n xy^n z \in L(G)$ for every $n \ge 0$.

96. Let L be a context-free language and R a regular language. Is the language L–R necessarily context-free? What about R–L?

97. Are the following languages context-free? (You may use exercises 94 – 95.)

(a) $\{ww : \ w \in \{a,b\}^*\}$.

(b) $\{baba^2ba^3 \dots ba^{n-1}ba^nb : \ n \geq 1\}$.

(c) $\{a^{2^l 3^m 5^n} : \ l, m, n \geq 0\}$.

(d) $\{a^{pq} : \ p, q \text{ primes}\}$.

(e) $\{a^l b^m c^n : \ l \neq m \ \vee \ l \neq n \ \vee \ m \neq n\}$.

(f) $\{a^l b^m c^n : \ l = m \ \vee \ l = n \ \vee \ m = n\}$.

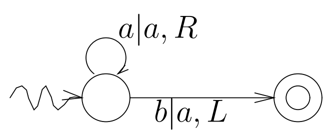
(g) $\{1a10, 10a11, 11a100, 100a101, \dots, A_n a A_{n+1}, \cdots : \ n \geq 1\}$, where A_n is the binary expansion of n.

98. Prove that the family of context-free languages is closed under union, concatenation and Kleene-$*$ by starting in each case from PDA's accepting the given languages and constructing a PDA accepting the language in question.

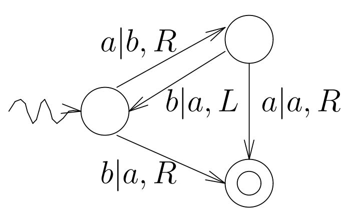
12 Turing Machines

99. For each of the following Turing machines, determine for which inputs they eventually halt, for which they hang and for which they never halt. In all cases $\Gamma = \Sigma \cup \{B\}$.

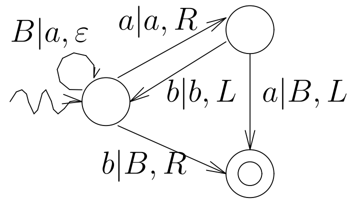
(a) $\Sigma = \{a, b\}$.



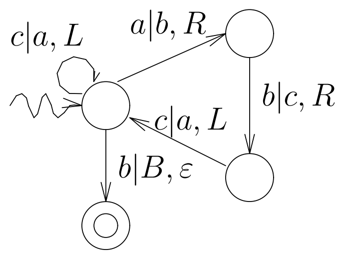
(b) $\Sigma = \{a, b\}$.



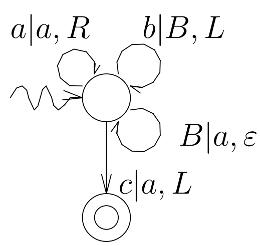
(c) $\Sigma = \{a, b\}$.



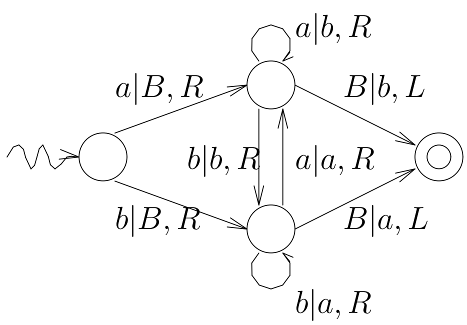
(d) $\Sigma = \{a, b, c\}$.



(e) $\Sigma = \{a, b, c\}$.

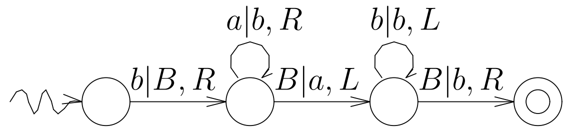


(f) $\Sigma = \{a, b\}$.

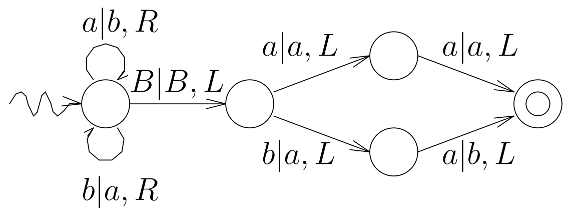


100. Find the languages accepted by the following Turing machines:

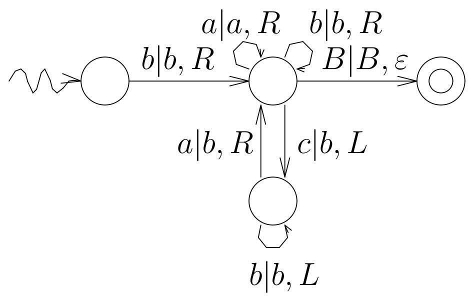
(a)


a|b, R
b|b, L
b|B, R
B|a, L
B|b, R

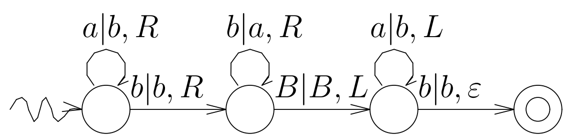
(b)


a|b, R
B|B, L
a|a, L
a|a, L
b|a, L
a|b, L
b|a, R

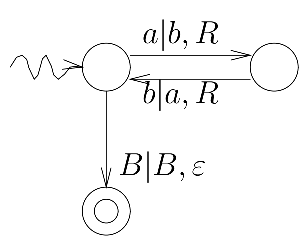
(c)


a|a, R
b|b, R
b|b, R
B|B, ε
a|b, R
c|b, L
b|b, L

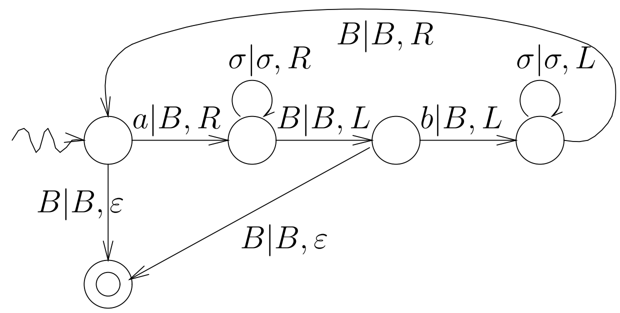
(d)


a|b, R
b|a, R
a|b, L
b|b, R
B|B, L
b|b, ε

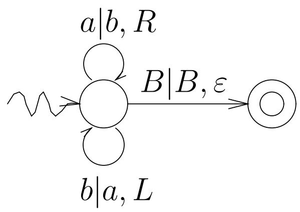
(e)


a|b, R
b|a, R
B|B, ε

(f)



(g)



101. Construct Turing machines computing the following functions (in each case $\Sigma = \{a, b\}$):

(a) $f(w) = \begin{cases} w, & |w| \le 2, \\ w\sigma_3, & |w| \ge 3, \quad w = \sigma_1\sigma_2\ldots\sigma_n \,. \end{cases}$

(b) $f(w) = \sigma_1\sigma_2\sigma_4\sigma_5\sigma_7\sigma_8\ldots, \qquad w = \sigma_1\sigma_2\ldots\sigma_n\,.$

(c) $f(w) = w^R$.

(d) f replaces each occurrence of ab by a c (for example, $f(baabaabb) = bacacb$).

(e) $f(w) = \begin{cases} wc, & w \text{ contains the block } abba, \\ wd, & \text{otherwise.} \end{cases}$

(f) f leaves the letters in the odd places but reverses the order of letters in the even places (for example, $f(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7) = \sigma_1\sigma_6\sigma_3\sigma_4\sigma_5\sigma_2\sigma_7$).

(g) $f(w) = \begin{cases} wgood, & w \in L(a^*b^*), \\ wbad, & w \notin L(a^*b^*)\,. \end{cases}$

(h) $f(w) = \begin{cases} wgreater, & |w|_a > |w|_b, \\ wless, & |w|_a < |w|_b, \\ wequal, & |w|_a = |w|_b\,. \end{cases}$

102. Explain how we can, given any homomorphism (see exercise 52) from Σ^* to Γ^*, construct a Turing machine computing the homomorphism. Exemplify for the homomorphism defined by $h(a) = cacb,\ \ h(b) = bab$.

103. Construct Turing machines accepting the following languages:

(a) a^*b^2a.

(b) $a^*b^*a^*b^*$.

(c) $\{a^nb^n : \ n \geq 0\}$.

(d) $\{a^nb^nc^n : \ n \geq 0\}$.

(e) $\{a^nb^{2n}c^{n+1} : \ n \geq 0\}$.

(f) $\{a^mb^ma^nb^n : \ m, n \geq 0, \ m \neq n\}$.

(g) $\{a^lb^ma^n : \ m = l + n\}$.

(h) $\{w \in \{a, b\}^* : \ |w|_a \geq |w|_b + 1\}$.

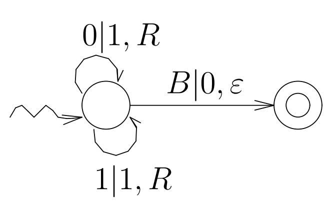
(i) The language consisting of all words over $\{a, b\}^*$ in which the number of a's is equal to the number of b's and the number of a's in each prefix is not smaller than the number of b's in the same prefix.

(j) $\{a^p : \ p \text{ is prime}\}$.

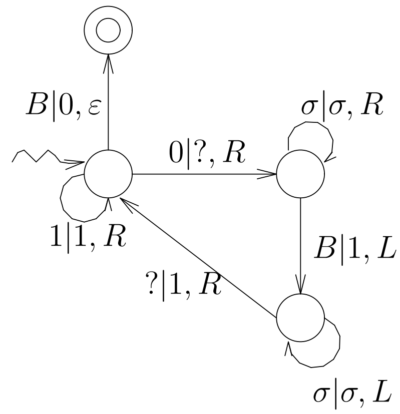
(k) $\{a^{2^n} : \ n \geq 0\}$.

104. Determine, for every positive integer k, which functions of k integer variables are computed by the following Turing machines:

(a)



(b)



105. Construct Turing machines computing the following functions:

(a) $f(n)=\min\{n,2\}, \qquad n\in\mathbf{Z}_+$.

(b) $f(n)=n^2, \qquad n\in\mathbf{Z}_+$.

(c) $f(n)=[\log_2(n+1)], \qquad n\in\mathbf{Z}_+$.

(d) $f(m,n)=mn, \qquad m,n\in\mathbf{Z}_+$.

(e) $f(m,n)=(m+1)^n, \qquad m,n\in\mathbf{Z}_+$.

106. Consider the following grammar:

$S\to AA,$

$A\to AAA\mid Aa\mid aA\mid b$.

(a) Show that the language accepted by this grammar is composed of all words in $\{a,b\}^*$ with an even number of b's.

(b) Show that this grammar is ambiguous.

(c) Construct an equivalent unambiguous grammar.

107. Construct unambiguous grammars for the following languages:

(a) All words in $\{a,b,c\}^*$ in which the letters a and b occur exactly once.

(b) All words in $\{a,b\}^*$ in which the string ab^2 occurs at least three times.

(c) All words in $\{a,b,c\}^*$ satisfying the following conditions:

i) Any two consecutive a's are separated by at least one b.

ii) Any two consecutive b's are separated by at least one a.

108. Same as the preceding question for the languages:

(a) $\{w \in \Sigma^* : \ w^R = w\}$.

(b) $\{w \in \Sigma^* : \ w^R \neq w\}$.