Final #2

Mark the correct answer in each part of the following questions.

1. Consider the collection of all subsets of even size of $\{1, 2, \ldots, n\}$.

   (a) The following two algorithms have been suggested for enumerating all sets in the collection:

   - $\mathcal{A}_1$ – Traverse the set of all subsets in lexicographic order, but take only every other set (more precisely, the first, the third, the fifth, and so forth, up to the $(2^n - 1)$-st).
   - $\mathcal{A}_2$ – Traverse the set of all subsets according to the order of the Gray code, but take only every other set.

   (i) Both algorithms are correct and work in linear time.

   (ii) Both algorithms work in linear time, but only the second is correct.

   (iii) Both algorithms are correct, but do not work in linear time.

   (iv) Both algorithms are correct, but only one of them works in linear time.

   (v) None of the above.

   (b) Consider the following three types of small changes of subsets of even size:

   - UP – addition of two elements.
   - DOWN – removal of two elements.
   - SIDEWISE – addition of one element and removal of another one.

   (i) It is possible to go over the collection with small changes, but (at least for large n) we must use all three types of changes.

(ii) It is possible to go over the collection with small changes. Moreover, we may restrict ourselves to changes of type UP and SIDEWISE. In addition, in the special case $n = 8$, it is possible to restrict ourselves to changes of type UP and DOWN.

(iii) It is possible to go over the collection with small changes. Moreover, we may restrict ourselves to changes of type UP and SIDEWISE. In the special case $n = 8$, it is impossible to restrict ourselves to changes of type UP and DOWN.

(iv) There exist infinitely many numbers n, for which it is impossible to go over the collection with small changes.

(v) None of the above.

(c) We would like to draw a uniformly random subset from the collection. The following methods have been suggested:

- Method A:
  a) Select a uniformly random subset A of $\{1, 2, \ldots, n-1\}$.
  b) If A is of an even size – take it; otherwise – take $A \cup \{n\}$.
- Method B:
  a) Select a uniformly random subset A of $\{1, 2, \ldots, n\}$.
  b) If A is of an even size – take it; otherwise – with probability $1/2$ pick a uniformly random $k \in A$ and take $A - \{k\}$, and with probability $1/2$ pick a uniformly random $k \in \{1, 2, \ldots, n\} - A$ and take $A \cup \{k\}$. (If n is odd, and the set $A = \{1, 2, \ldots, n\}$ has been selected, then with probability 1 take $A - \{k\}$.)

(i) Both methods select a subset with the required property, but only the first does so with the correct probability. Both require on the average linear time.

(ii) Both methods select a subset with the required property, but only the second does so with the correct probability. Both require on the average linear time.

(iii) Both methods select a subset as required, but only the first requires on the average linear time, whereas the second requires $\Theta(n^2)$.

(iv) Both methods select a subset as required, but only the second requires on the average linear time, whereas the first requires $\Theta(n^2)$.

(v) None of the above.

2. (a) Consider the algorithm presented in class for traversing the set of all partitions of $\{1, 2, \ldots, n\}$. Denote by a_n the number of partitions. The partition $\{1, 2, 4, 5, \ldots, n\}\{3\}$ is encountered at step:

   (i) $a_{n-1} + O(1)$.

   (ii) $a_{n-2} + O(1)$.

   (iii) $a_n - a_{n-1} + O(1)$.

   (iv) $a_n - a_{n-2} + O(1)$.

   (v) none of the above.

   (b) Denote by b_n the number of all partitions in which all sets are of size at least 2. We agree that $b_0 = 1$. The sequence (b_n) satisfies the following recurrence for all sufficiently large n:

   (i) $b_n = \sum_{k=1}^{n-1} \binom{n-1}{k} b_{n-1-k}$.

   (ii) $b_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1} b_{n-1-k}$.

   (iii) $b_n = \sum_{k=1}^{n-1} \binom{n-1}{k} b_{n-k}$.

   (iv) $b_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1} b_{n-k}$.

   (v) None of the above.

3. In this question we consider trees over n labeled vertices.

   (a) Suppose first that n is even, and we are interested in trees, in which vertices 1 and 2 are degree $n/2$ each, and all other $n-2$ vertices are leaves.

   We would like to go over this set of set with changes as small as possible. More formally, let us define:

   - A *minimal change* in a tree consists of removing one edge and adding another (in such a way that we still have a tree).

- A *small change* in a tree consists of removing two edges and adding two others (in such a way that we still have a tree).

(i) It is possible to go over the class of trees in question with minimal changes.

(ii) It is possible to go over the class of trees in question with small changes, and so that from the last tree in the sequence it will be possible to pass to the first in the sequence. However, it is impossible to go over the class with minimal changes.

(iii) It is possible to go over the class of trees in question with small changes, but not so that from the last tree in the sequence it will be possible to pass to the first in the sequence. Also, it is impossible to go over the class with minimal changes.

(iv) It is impossible to go over the class of trees in question with small changes.

(v) None of the above.

(b) The number of trees, in which all vertices are of degree either 1 or 2, is

(i) $\frac{(n-1)!}{2}$.

(ii) $(n-1)!$.

(iii) $\frac{n!}{2}$.

(iv) $n!$.

(v) none of the above.

4. Consider Young tableaux with 3 rows, so that the first row consists of $2m$ squares, and the second and the third of m each. Denote by $n = 4m$ the total number of squares. When selecting a random Young table of this shape in the way we have discussed in class, we first select the square to hold n. Let p_n be the probability that n should be placed at the top right corner. Then $p_n \underset{n\to\infty}{\longrightarrow}$

(i) $\frac{2}{7}$.

(ii) $\frac{1}{3}$.

(iii) $\frac{3}{7}$.

(iv) $\frac{1}{2}$.

(v) none of the above.

Solutions

1. (a) When moving with minimal changes, each time we either add an element or remove one. Hence the size of the set changes each time from even to odd or vice versa. Hence, starting with the empty set, and taking only every other set, we get exactly all sets of even size.

    In the lexicographic order, half the changes are of adding a single element, and thus change the size of the set from even to odd or vice versa. However, a fourth of the changes are of removing an element and adding another one, and thus do not change the parity of the size of the set. Hence, $\mathcal{A}_1$ yields both subsets of even size (and not all of them) and subsets of odd size.

    Both algorithms are, as we know, linear in 2^n. Since the collection of subsets of even size includes $2^{n-1} = \Theta(2^n)$ sets, the algorithms are linear also as functions of the size of the collection we deal with here.

    Thus, (ii) is true.

   (b) Recall that we can traverse the set of all subsets of any given size by SIDEWISE changes. Moreover, it is clear by symmetry that we can start with any subset of the given size. Hence, we can traverse the collection in question by first going over subsets of size 0 (which consists only of $\emptyset$, but this is immaterial), then add any two elements and go over all subsets of size 2, and so forth.

    For $n = 8$, the number of subsets of size $0, 2, 4, 6, 8$ is $1, 28, 70, 28, 1$, respectively. Now UP and DOWN changes take us from subsets of size 4 to subsets of size either 2 or 6. Since $70 > 28 + 28 + 1$, it is impossible to have after every subset of size 4 a different subset of size 2 or 6.

    Thus, (iii) is true.

   (c) We need to select each set of even size with probability $1/2^{n-1}$. Let A be such a set. Consider the first method. If $n \notin A$, then we clearly select A with the required probability. If $n \in A$, then we select A if and only if at the first step we select the subset $A - \{n\}$ of $\{1, 2, \ldots, n-1\}$, which again happens with probability $1/2^{n-1}$.

The second method selects only subsets of even size, but not with the required probability. Take, for example, $A = \emptyset$. It is selected if and only if we either select it to begin with, or select first a singleton, which happens with probability $n/2^n$, and the second drawing dictates to remove the single element from that singleton. The total probability is therefore

$$\frac{1}{2^n} + \frac{n}{2^n} \cdot \frac{1}{2} = \frac{1 + n/2}{2^n} \neq \frac{1}{2^{n-1}}.$$

In both methods, most of the work is to select a uniformly random subset of a set of size $n-1$ or n, which takes $\Theta(n)$ time.

Thus, (i) is true.

2. (a) The partitions preceding $\{1, 2, 4, 5, \ldots, n\}\{3\}$ are exactly those partitions in which the 3 elements $1, 2, 3$ reside in the same set. This set of partitions is in 1-1 correspondence with the set of all partitions of a set of size $n-2$. In fact, the 3 elements may be considered as a single element, and we need to decide where to place all the other $n-3$ elements.

   Thus, (ii) is true.

   (b) Let us split the set of all partitions in question to classes, depending on the number k of elements belonging to the same set as does the element 1. This k needs to be at least 1 due to the condition that all sets are of size at least 2. For each k, we have $\binom{n-1}{k}$ possibilities of choosing the elements in the same set as 1, and for each of these we have b_{n-k-1} possibilities of partitioning the remaining elements to sets of size at least 2. (Note that, in fact, k cannot be $n-2$, as then the set containing 1 will contain $n-1$ elements altogether, so the remaining element will make it impossible to complete the partition as required. However, we do not need to worry about it, as the fact that $b_1 = 0$ will take care of it.)

   Thus, (i) is true.

3. (a) A tree in the collection in question is determined by the set of $n/2-1$ leaves connected to vertex 1. Hence it is in 1-1 correspondence with the collection of all subsets of $\{3, 4, \dots, n\}$ of size $n/2-1$. Recall the algorithm for going over all subsets of size k of a set of size n, and apply it with k and n replaced by $n/2-1$ and $n-2$, respectively. Go over all such subsets, and consider for each such subset V' that tree in our collection, in which vertex 1 is connected to all vertices in V', and vertex 2 is connected to all vertices in $\{3, 4, \dots, n\} - V'$. To pass from V' to the next subset, we remove one element and add another. The corresponding action in the tree consists of disconnecting a certain leaf from 1 and connecting it to 2, and disconnecting some other leaf from 2 and connecting it to 1. Moreover, since the algorithm for going over all subsets of some fixed size ends with a subset which is adjacent to the initial subset, in our case it will be possible to pass from the last tree in the sequence to the first by a small change.

   It is impossible to go over the collection by minimal changes, as no (non-trivial) minimal change will leave us in the family. In fact, if we first discconect, say, 1 from one of the leaves connected to it, then for this leaf to stay a leaf we need to connect it to 2, so 1 and 2 will be connected to distinct numbers of leaves. If we disconnect 1 and 2, then they cannot have degree $n/2$ after any addition of an edge (different from the one we have removed).

   Thus, (ii) is true.

   (b) A tree with the required properties must be a path. To choose a path, we need to choose first its two endpoints i and j, which can be done in $\binom{n}{2}$ ways. Then we need to choose the unique neighbor of i, which can done in $n-2$ ways, the second neighbor of that vertex – which can be done in $n-3$ ways – and so forth. Altogether, the number of possibilities is

$$\binom{n}{2} \cdot (n-2)(n-3) \cdot \ldots \cdot 1 = \frac{n!}{2}.$$

   Thus, (iii) is true.

4. According to the formula developed in class, The probability for n to be at a corner c is
$$P(c) = \frac{1}{n} \cdot \prod \frac{h(s)}{h(s)-1},$$
where the product ranges over all squares at the same row or column of c (not including c), and $h(s)$ is the hook length of s. Now the hook lengths of the first m squares in the first row are $2m+2, 2m+1, 2m, \ldots, m+3$, and those of the next $m-1$ squares are $m, m-1, \ldots, 2$. Consequently:
$$\begin{aligned} p_n &= \frac{1}{n} \cdot \frac{2m+2}{2m+1} \cdot \frac{2m+1}{2m} \cdot \ldots \cdot \frac{m+3}{m+2} \cdot \frac{m}{m-1} \cdot \frac{m-1}{m-2} \cdot \ldots \cdot \frac{2}{1} \\ &= \frac{1}{4m} \cdot \frac{2m+2}{m+2} \cdot m = \frac{m+1}{2(m+2)} = \frac{n+4}{2n+16}. \end{aligned}$$
It follows that
$$p_n \xrightarrow[n\to\infty]{} \frac{1}{2}.$$
Thus, (iv) is true.