Final #2

Mark the correct answer in each part of the following questions.

1. (a) Consider the following two versions of Nim:

    - Nim-B: I, at his first move, instead of removing matches from one of the heaps, adds a new heap with an arbitrary (positive) number of matches.
    - Nim-C: I, at his first move, instead of removing matches from one of the heaps, adds a (positive) number of matches to an arbitrary heap.

    Next consider the following two claims.

    Claim 1: For any initial position, I has a winning strategy in Nim-B if and only if he has one in classical Nim.

    Claim 2: For any initial position, I has a winning strategy in Nim-C if and only if he has one in classical Nim.

    (i) Both Claim 1 and Claim 2 are correct.

    (ii) Only Claim 1 is correct.

    (iii) Only Claim 2 is correct.

    (iv) None of the claims is correct. Moreover, none is correct even if we relax the "if and only if" to either "if" or "only if".

    (v) None of the above.

   (b) Consider the following additional version of Nim:

    - Nim-D: A positive integer M is given. In the beginning of the game, II splits M between an arbitrary number of heaps, as he wishes. Thereafter, the game continues as in the classical version.

    (i) For every sufficiently large M, II has a winning strategy.

(ii) For every sufficiently large even M, II has a winning strategy. Out of the odd M-s, he has a winning strategy for infinitely many, but does not for infinitely many others.

(iii) For every sufficiently large odd M, II has a winning strategy. Out of the even M-s, he has a winning strategy for infinitely many, but does not for infinitely many others.

(iv) II has a winning strategy if and only if M is even.

(v) None of the above.

2. Let
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
be the payoff matrix of a two-person zero-sum game.

(a) The organizer of the tournament offers I to change A to
$$\begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}$$
with some $a > 0$ for a suitable bribe, in such a way that II will be unaware of the change until the tournament is over.

(i) I should not pay anything for this change.

(ii) I should not pay anything if $a \leq 1$. For $a > 1$, he should pay up to a certain amount which is independent of a.

(iii) I should not pay anything if $a \leq 1$. For $a > 1$, he should pay up to a certain amount which is monotonically increasing as a function of a.

(iv) I should pay any amount less than $1/4$.

(v) None of the above.

(b) The organizer of the tournament has retracted his offer. Instead, he offers I to replace A by
$$\begin{pmatrix} -1 & 0 \\ b & 1 \end{pmatrix}$$
with some $b > 0$. However, unlike his initial proposal, II will be aware of this change in advance.

(i) I should not pay anything for this change.

(ii) I should pay up to b for the change.

(iii) I should not pay anything if $b \leq 1$. For $b > 1$, he should pay up to $\frac{1+b}{2}$.

(iv) I should pay up to b if $b \leq 1$, and up to 1 if $b > 1$.

(v) None of the above.

3. Let A be the payoff matrix of a two-person zero-sum game.

   (i) If A is a square non-invertible matrix, then there exist vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x}^T A = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. Therefore, the value of the game is 0.

   (ii) If A is of rank 1, then it has a saddle point.

   (iii) If the sum of the rows of A is the zero vector, then the value of the game is 0.

   (iv) If the sum of the columns of A is the zero vector, then the value of the game is at most 0.

   (v) None of the above.

4. There are $2n + 1$ people and $2n$ benches in a room. Of the benches, n are upholstered and the other n are bare-wood. Each person chooses a bench without knowing what the others have chosen. The utility of a person sitting alone on an upholstered bench is 6, sitting alone on a bare-wood bench – 5, sharing an upholstered bench – 4, and sharing a bare-wood bench – 3.

   (a) The number of Nash equilibria in pure strategies is:

   (i) $(2n + 1)!$.

   (ii) $\binom{2n+1}{n}$.

   (iii) $2^n \binom{2n}{n}$.

(iv) $\dfrac{n(2n+1)!}{2}$.

(v) none of the above.

(b) Now we consider Nash equilibria in fully mixed strategies, in which all people employ the same strategy. Moreover, each of them selects each of the upholstered benches with the same probability p_1, and each of the others with the same probability p_2. (Thus, $n \cdot (p_1 + p_2) = 1$.) The probabilities p_1 and p_2 satisfy the equation:

(i) $(1-p_1)^{2n} - (1-p_2)^{2n} = 1/6$.

(ii) $(1-p_1)^{2n} - (1-p_2)^{2n} = 1/3$.

(iii) $(1-p_1)^{2n} - (1-p_2)^{2n} = 1/2$.

(iv) $(1-p_1)^{2n} - (1-p_2)^{2n} = 3/5$.

(v) none of the above.

(c) Now suppose that one of the people is indifferent to the bench type (i.e., his utilities are 6 if sitting alone and 4 if sharing a bench), and that he has chosen a certain bare-wood bench and sat on it before the others have chosen. Suppose the others are trying to reach an equilibrium similar to that in the preceding part, taking into account the choice made by that person.

(i) A player may never gain by revealing his strategy. In our case, the person who sat first will lose due to it.

(ii) A player may never gain by revealing his strategy. In our case, the person who sat first will neither gain nor lose due to it.

(iii) A player may gain by revealing his strategy. For example, in the game of chicken, if one of the drivers makes an action that proves that he will not deviate from the path (by breaking the wheel, say), he will gain by it. However, in our case the person who sat first will not gain due to it.

(iv) A player may gain by revealing his strategy. For example, in the game of chicken, if one of the drivers makes an action that proves that he will not deviate from the path (by breaking GTE wheel, say), he will gain by it. In our case also, the person who sat first will gain due to it.

(v) None of the above.

5. Consider the stable matchings problem for a group of n boys and n girls.

   (a) Recall that the data concerning an instance of the problem is given by means of a matrix B, defining the preferences of the boys, and a matrix G, defining the preferences of the girls. Here we would like to find out what we can conclude, if at all, if we know only one of the matrices.

      (i) If we know only B, we cannot conclude anything about the matching that will be obtained by the algorithm taught in class. Namely, for every B, and for each of the $n!$ possible matchings, there exists a G for which the algorithm will produce this matching.

      (ii) B does not determine in general which matching will be produced by the algorithm, but there exist matrices B that do determine uniquely this matching.

      (iii) B by itself can never tell us exactly how many steps the algorithm will take.

      (iv) G does not determine in general which matching will be produced by the algorithm, but there exist matrices G that do determine uniquely this matching.

      (v) None of the above.

   (b) Now suppose that the preferences are given by a matrix $U = (u_{ij})_{i,j=1}^{n}$, measuring compatibility of possible pairs. Due to a security breach, one of the girls is able to make certain changes in U. We are interested in the possibilities of this girl to use it to her advantage.

      (i) If she can replace but one entry of U, whichever she chooses, by any number she wants, she can make sure to be matched, under the stable matching corresponding to the matrix, to the boy at the top of her preference list.

      (ii) If she can replace one column of U, whichever she chooses, by any numbers she wants, she can make sure to be matched, under the stable matching corresponding to the matrix, to the boy at the top of her preference list. However, the preceding claims are incorrect.

(iii) If she can replace both a row and a column of U, whichever she chooses, by any numbers she wants, she can make sure to be matched, under the stable matching corresponding to the matrix, to the boy at the top of her preference list. However, the preceding claim is incorrect.

(iv) A change of a single entry of U can never change the stable matching corresponding to the matrix.

(v) None of the above.

Solutions

1. (a) Consider Nim-B first. Suppose a position is winning for I in classical Nim. Then the Nim-sum of the heap sizes is non-zero. If I adds a heap whose size is the Nim-sum of all heap sizes, the new Nim-sum will be 0, so that II will be in a losing position. If, however, the position is a losing position for I, then the Nim-sum of the heap sizes is 0. Then, no matter what size the heap I adds is, the Nim-sum will become non-zero, and II will have a winning strategy. Hence, in Nim-B, a position is winning for I if and only if it is winning for him in classical Nim, so that Claim 1 is correct.

   In Nim-C, it is still the case that a position, in which I is to lose in classical Nim, is losing. In fact, the Nim-sum of the heap sizes is 0, and any change in one of the heaps, be it a subtraction (as in classical Nim) or addition (as in Nim-C), makes the Nim-sum non-zero, and therefore allows II to win. However, not every position which is winning in classical Nim is such in Nim-C. For example, any position with a single heap is a winning position for I in classical Nim, but it clearly losing in Nim-C. Hence, Claim 2 is incorrect.

   Thus, (ii) is true.

(b) If M is even, then II wins by splitting the matches between two heaps of size $M/2$ each. Indeed, the Nim-sum of two equal numbers is certainly 0.

On the other hand, if M is odd, then any splitting of the matches has the property that there is an odd number of heaps of odd size. Hence, the Nim-sum of the sizes must have a 1 at the lowest digit, and in particular be non-zero. Hence, whatever II does, I will have a winning strategy.

Thus, (iv) is true.

2. (a) Since II is unaware of the change, he will play as if the payoff matrix is A. Since strategy 1 of II dominates strategy 2, he will continue playing it. Hence, even after the change, I will have to play his strategy 2, and the change is irrelevant for him.

Thus, (i) is true.

(b) For $b > 0$, strategy 2 of I dominates his strategy 1, so that he will still play it. Hence, for $b \leq 1$, player 2 will play his strategy 1, and I will get b instead of 0. For $b > 1$, player 2 will play his strategy 1, and I will get 1 instead of 0. Thus, I should pay for the change up to b if $b \leq 1$, and up to 1 if $b > 1$.

Thus, (iv) is true.

3. Vectors $\mathbf{x}$ and $\mathbf{y}$ as in (i) indeed exist, but they may well not be probability vectors (or vectors with all entries of the same sign, so they can be normalized to probability vectors). Indeed, if, say,

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \tag{1}$$

then the value of the game is clearly 1, even though A is non-invertible. Consequently, (i) is false.

If

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

then A is of rank 1, yet has no saddle point, so that (ii) is false.

The row sum of the matrix in (1) is $\mathbf{0}$, yet the value of the game is 1. (The condition does imply, though, that the value is at least 0, analogously to the reasoning in the next paragraph.)

If the column sum of A is $\mathbf{0}$, then, employing all his pure strategies with the same probability, II guarantees himself 0 on the average. (Note that he may well have a better strategy that will guarantee more.)

Thus, (iv) is true.

4. (a) First note that, in a Nash equilibrium in pure strategies, all benches are taken. Otherwise, any person who shares a bench could have made better by switching to a vacant bench. Thus, all benches are occupied by a single person, except for one, shared by two people. Also, these two clearly share an upholstered bench. Otherwise, it would be better for one of them to move to such a bench. On the other hand, any way of seating two people on some upholstered bench, and all others on separate benches, is clearly an equilibrium. Hence the number of equilibria is

$$\binom{2n+1}{2} \cdot n \cdot (2n-1)! = \frac{n(2n+1)!}{2}.$$

Thus, (iv) is true.

(b) If at some Nash equilibrium, some player's strategy is fully mixed, then all pure strategies are equally good for him, assuming all other players stick to their strategies. In our case, if a person selects an upholstered bench, he gets 6 if nobody else selects the same bench, which happens with probability $(1-p_1)^n$, and 4 if someone else does select it, which happens with probability $1-(1-p_1)^n$. Hence, selecting an upholstered bench, he gets on the average

$$6 \cdot (1-p_1)^n + 4 \cdot (1-(1-p_1)^n).$$

Similarly, if he selects a bare-wood bench, he gets on the average

$$5 \cdot (1-p_2)^n + 3 \cdot (1-(1-p_2)^n).$$

The reasoning above yields:

$$6\cdot(1-p_1)^n+4\cdot(1-(1-p_1)^n)=5\cdot(1-p_2)^n+3\cdot(1-(1-p_2)^n).$$

Simplifying the equations, we obtain

$$2\cdot(1-p_1)^n+4=2\cdot(1-p_2)^n+3,$$

so that

$$(1-p_1)^n-(1-p_2)^n=-1/2.$$

Thus, (v) is true.

(c) In the chicken game, if one of the drivers is certain that the other will not deviate from the path, then he should deviate, as this strategy gives him a much smaller loss than continuing on the path. In our case, if a certain person sits on a bare-wood bench, it makes no sense for other players to select that bench with a positive probability. In fact, if anyone sits there, he will certainly get 3, whereas if, say, he selects an upholstered bench, he guarantees himself at least 4. Hence, the person who sat first actually guarantees that he gets 6.

Thus, (iv) is true.

5. (a) In general, B does not determine the matching produced by the algorithm. For example, suppose that all rows of B are equal, which is tantamount to saying that all boys have the exact same preferences. By symmetry, it is clear that in this case all $n!$ possible matchings are possible outcomes of the algorithm. (To be more specific, if some matching gives each girl the boy at the top of her preference list, then the algorithm will produce this matching.)

However, for some matrices B, the matrix G is irrelevant to the algorithm. This is the case if each boy has a different girl at the top of his preference list. In fact, in this case, at the first stage, each girl will get just one proposal, which she will therefore accept, and the result will not take G into account. Notice also that the algorithm will make just one step in this case. Moreover, this situation shows also that G by itself never determines the result of the algorithm.

Thus, (ii) is true.

(b) Suppose that the girl in question changes the entry relating to the compatibility of herself and the boy at the top of her list, so that it will be larger than all other entries of both the row and the column of this entry. We claim that they will be matched. In fact, we know that, when the preferences are determined by a matrix U of compatibility measures, there exists a unique stable matching. Thus, we may consider the result of the algorithm taught in class. According to the algorithm, that boy will propose to her at the first stage, and, even if she gets more proposals at any point, she will reject them and remain matched to him.

Thus, (i) is true.