Review Questions

Mark the correct answer in each part of the following questions.

1. I and II play Tic-Tac-Toe on the space $\{1, 2, \ldots, n\}^d$ for some n and d. Suppose that the rules of the game are changed, so that each player in turn makes two moves instead of a single one.

   (i) One cannot use the theorem, proved in class, regarding games with perfect information, to make conclusions about this version of the game, since that theorem relates only to games in which a player is allowed to make a single move at each turn.

   (ii) For every fixed n, if d is sufficiently large, then I has a winning strategy.

   (iii) Suppose that n is even, and that the two points each player chooses in his turn must be symmetric with respect to the center of the cube. (Namely, they are of the form $(a_1, a_2, \ldots, a_d)$ and $(n+1-a_1, n+1-a_2, \ldots, n+1-a_d)$.) Hales-Jewett Theorem cannot be used in this case to prove that I has a winning strategy.

   (iv) Suppose now that, to make the game more balanced, it is decided that I, in his first turn, selects a single point, and only from then on each player selects two points each time. For this version of the game, if n is arbitrary and d is sufficiently large, then each player has a strategy guaranteeing at least a draw.

   (v) none of the above.

2. A_1 and A_2 are both matrices of the same size $m \times n$, defining the payoffs of two two-person zero-sum games. Assume all entries of A_1 and A_2 are strictly positive. The sum $A = A_1 + A_2$ defines the payoffs of a third game. Denote by V_1, V_2, V the values of the three games.

(i) $V = V_1 + V_2$.

(ii) $V > \max\{V_1, V_2\}$, but (i) is not necessarily true.

(iii) $V \geq \max\{V_1, V_2\}$, but (i) and (ii) are not necessarily true.

(iv) $V \geq \min\{V_1, V_2\}$, but the preceding claims are not necessarily true.

(v) none of the above.

3. Consider the following two-person game. Each player chooses an integer between 1 and n, where $n \geq 2$ is a given positive integer. If both choose the same number i, then I gets i and II gets nothing. If they choose distinct numbers, then II gets 1 and I gets nothing.

   (a) The number of Nash equilibria (s_1, s_2) in pure strategies is

      (i) 0.

      (ii) 1.

      (iii) n.

      (iv) $n(n-1)$.

      (v) none of the above.

   (b) Let (x^*, y^*) be a Nash equilibrium with fully mixed strategies x^* and y^*. Then: $x_1^* =$

      (i) $\dfrac{1}{n(n+1)}$.

      (ii) $\dfrac{2}{n(n+1)}$.

      (iii) $\dfrac{1}{n}$.

      (iv) $\dfrac{2}{n+1}$.

      (v) none of the above.

   (c) As $n \to \infty$, we have $y_1^* =$

      (i) $\dfrac{1+o(1)}{\ln n}$.

      (ii) $\dfrac{1+o(1)}{n \ln n}$.

(iii) $\dfrac{1+o(1)}{\ln^2 n}$.

(iv) $\dfrac{1+o(1)}{n}$.

(v) none of the above.

4. Consider the stable matchings problem for a group of n boys and n girls.

   (a) Recall that we defined the preferences of the boys by means of an $n \times n$ matrix B. Each column of B corresponds to one of the boys and each row to one of the girls. The entries of B are the rankings the boys assign to the girls (where 1 means highest preference and n means lowest). The rankings the girls assign are similarly defined by means of a matrix G. Suppose that n is even, say $n = 2m$.

   (i) If all entries of the top left $m \times m$ block of B are between 1 and m, then, in every stable matching, the boys $b_1, b_2, \ldots, b_m$ will be matched to the girls $g_1, g_2, \ldots, g_m$ (not necessarily in the natural order).

   (ii) If the condition in (i) is satisfied by both B and G, then the conclusion of (i) is correct. However, (i) is false.

   (iii) If the condition in (i) is satisfied, then the algorithm presented in class will produce a matching satisfying the property in (i). However, not every stable matching will necessarily satisfy that property.

   (iv) If the condition in (ii) is satisfied, then the algorithm presented in class will produce a matching satisfying the property in (i). However, not every stable matching will necessarily satisfy that property.

   (v) none of the above.

   (b) Let us say that the boys *have totally distinct preferences* if no two of them have any girl at the same place on their preference lists. We define this notion similarly for girls.

(i) The matching algorithm presented in class terminates in a single step if and only if the boys have totally distinct preferences.

(ii) If the boys have totally distinct preferences, then the girls cannot possibly also have totally distinct preferences.

(iii) If the boys have totally distinct preferences, then there exists a unique stable matching.

(iv) If the boys have totally distinct preferences, then it is possible that, for each boy b, both the most preferred girl $g_{\max}(b)$ and the least preferred one $g_{\min}(b)$ on his list are attainable for him. However, this is not necessarily the case.

(v) none of the above.

5. The matrix

$$A' = \begin{pmatrix} -4 & 3 & 12 \\ -7 & 0 & 15 \\ -8 & -1 & 20 \end{pmatrix}$$

consists of the first three rows of the payoff matrix A of a certain two-person zero-sum game, in which I has m pure strategies and II has three. Denote I's pure strategies by $s_1, s_2, \ldots, s_m$ and II's by t_1, t_2, t_3.

(a)

(i) The data above enables us to provide a non-trivial lower bound on the value of the game, but not an upper bound.

(ii) The data above enables us to provide a non-trivial upper bound on the value of the game, but not a lower bound.

(iii) The data above enables us to provide both a non-trivial lower bound and a non-trivial upper bound on the value of the game.

(iv) If y^* is an optimal strategy for II, then $y_3^* = 0$.

(v) none of the above.

(b) II would like to reduce I's options. Thus, II offers I some amount to agree to eliminate s_2. We are interested in the amount c for which it is worthwhile for I to accept the offer.

(i) Player I should accept the offer for any $c > 0$.

(ii) For $c > 4/3$ it is certainly worthwhile for I to accept the offer. For $0 < c \leq 4/3$, it may be worthwhile or not, depending on the other rows of A.

(iii) For $c > 5/3$ it is certainly worthwhile for I to accept the offer. For $0 < c \leq 5/3$, it may be worthwhile or not, depending on the other rows of A.

(iv) For $c > 8/3$ it is certainly worthwhile for I to accept the offer. For $0 < c \leq 8/3$, it may be worthwhile or not, depending on the other rows of A.

(v) none of the above.

6. I and II play Nim, with n heaps of sizes $3, 3^2, 3^3, \ldots, 3^n$.

(i) For every sufficiently large n, II has a winning strategy.

(ii) For infinitely many values of n, I has a winning strategy, and for infinitely many II does.

(iii) For every n, I has a winning strategy.

(iv) For every sufficiently large n, I may make any move in the initial position and still force a win.

(v) none of the above.

Solutions

1. The theorem regarding games with perfect information does apply to this game. We have simply changed the set of allowed moves in the game, but the theorem holds just as well. Hence, (i) is false.

   In regular Tic-Tac-Toe, Hales-Jewett Theorem was used to prove that, given n, for a sufficiently large dimension, the game cannot possibly end in a draw. It serves to prove this point here as well, as it does not matter in what way points on the board are captured by the players. It only matters that a completely filled board must contain a monochromatic combinatorial line. To prove that I has a winning strategy, we use the idea of strategy stealing just as in the original game. It follows that (ii) is true, while (iii) is false.

   For the version in (iv), strategy stealing cannot be used any more to prove that I has a winning strategy. In fact, by making an arbitrary first move and ignoring it later, I does not "become II"; he will have the other player occupy two squares before he is allowed to respond. Still, by Hales-Jewett Theorem, draw is impossible, so either I or II does have a winning strategy. Thus, (iv) is false.

2. For the matrices
$$A_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
   we clearly have $V_1 = V_2 = 3$, yet
$$A = A_1 + A_2 = \begin{pmatrix} 5 \\ 5 \end{pmatrix},$$
   so that $V = 5 \neq 3 + 3 = V_1 + V_2$. Thus, (i) is false.

   Suppose, say, that $V_1 \geq V_2$, so that $\max\{V_1, V_2\} = V_1$. Let $x^* \in \Delta_m$ and $y^* \in \Delta_n$ be optimal strategies for I and II, respectively, in the game defined by A_1. Pick some $c > 0$ such that all entries of A_2 are at least c. Denoting by J the all-ones $m \times n$ matrix, we obtain

$$\begin{aligned}
V &= \max_{x\in\Delta_m} \min_{y\in\Delta_n} x^T A y \\
&\geq \min_{y\in\Delta_n} x^{*T} A y \\
&\geq \min_{y\in\Delta_n} x^{*T} (A_1 + cJ) y \\
&= \min_{y\in\Delta_n} \left(x^{*T} A_1 y + c\right) \\
&= \min_{y\in\Delta_n} x^{*T} A_1 y + c \\
&= x^{*T} A_1 y^* + c \\
&= V_1 + c,
\end{aligned}$$

and therefore (ii) is true.

3. (a) Suppose that there exists a Nash equilibrium (i, j). If $i \neq j$, then I would be better off if he played j instead of i. If $i = j$, then II would be better off if he played anything but i. Hence there is no Nash equilibrium in pure strategies.

   Thus, (i) is true.

   (b) Since (x^*, y^*) is a Nash equilibrium with fully mixed strategies, if any player deviates from his strategy in any way (while the other player does not), he still gets the same. Now if II plays 1 then he gets on the average $1 - x_1^*$, if II plays 2 then he gets on the average $1 - x_2^*$, and so forth. It follows that

$$1 - x_1^* = 1 - x_2^* = \ldots = 1 - x_n^*,$$

   which implies

$$x_1^* = x_2^* = \ldots = x_n^*,$$

   and therefore I plays each i with the same probability $1/n$.

   Thus, (iii) is true.

   (c) Similarly to the previous part, if I plays 1 then he gets on the average $1 \cdot y_1^*$, if I plays 2 then he gets on the average $2 \cdot y_2^*$, and so forth. Consequently:

$$y_1^* = 2y_2^* = \ldots = ny_n^*.$$

Denote by c the common value of all expressions in the preceding system of equations. Then $y_i^* = c/i$ for $1 \le i \le n$. Since $y^* \in \Delta_n$, we obtain

$$y_i^* = \frac{1}{iH_n}, \qquad 1 \le i \le n,$$

where

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

Recall that, as $n \to \infty$, we have $H_n = \ln n + O(1)$. It follows that

$$y_i^* = \frac{1}{i(\ln n + O(1))} = \frac{1}{(1 + o(1)) \cdot i \ln n} = \frac{1 + o(1)}{i \ln n}.$$

In particular:

$$y_1^* = \frac{1 + o(1)}{\ln n}.$$

Thus, (i) is true.

4. (a) The information given by the condition in (i) is that $b_1, b_2, \ldots, b_m$ all have $g_1, g_2, \ldots, g_m$ at the top half of their preference lists. It may well be the case that $b_{m+1}, b_{m+2}, \ldots, b_n$ also have $g_1, g_2, \ldots, g_m$ at the top half of their lists. If (i) (or (iii)) was correct, then it would imply that, in each stable matching (or under the algorithm presented in class), $b_{m+1}, b_{m+2}, \ldots, b_n$ are also matched to $g_1, g_2, \ldots, g_m$, which is impossible. Hence, (i) and (iii) false.

   Under the condition in (ii), however, the conclusion in (i) does hold. In fact, suppose we have a matching in which $b_1, b_2, \ldots, b_m$ are not matched (exactly) with $g_1, g_2, \ldots, g_m$. Then there exists a boy $b_i, 1 \le i \le m$, matched with some $g_{i'}, m + 1 \le i' \le n$, and a girl $g_j, 1 \le j \le m$, matched with some $b_{j'}, m+1 \le j' \le n$. Now b_i and g_j certainly prefer each other to their current match, so that the matching is unstable.

   Thus, (ii) is true.

   (b) For the algorithm to terminate in a single step, it is necessary that no two boys have the same girl at the top of their preference list. This condition is strictly weaker than having totally distinct preferences. Hence, (i) is false.

The stable matchings problem does not assume any relation between the boys' preferences and the girls'. Hence, (ii) is false.

Suppose that the boys have totally distinct preferences, and the girls' preferences are such that each girl has at the top of her preference list that very boy who has her last on his list. Then the algorithm will match each boy with his top choice, while the algorithm, with opposite sexes, will match each boy with his least preferred girl. In particular, both the most and the least preferred girls on each boy's list are attainable for him. Of course, this is not necessarily the case. For example, if each boy's most preferred girl has him as her most preferred boy, then there exists a unique stable matching. It follows that (iv) is true, while (iii) is false.

Thus, (iv) is true.

5. (a) Playing s_1, the first player ensures that he gets at least -4, and hence $V \geq -4$. On the other hand, suppose, for example, that all entries in the fourth row of A are some number c. Then $V \geq c$. Since this c may be arbitrarily large, the information does not provide any upper bound on V.

   Since the first column of A' dominates the second, which in turn dominates the third, if A were just A', the second player would have to play t_1. However, if, for example, $m = 4$, and the fourth row of A is $(40, 40, 30)$, then the game has a saddle point; I will play s_4 and II will play t_3. Hence, (iv) is false.

   Thus, (i) is true.

   (b) The set of strategies $\{s_1, s_3\}$ dominates s_2. Playing each of s_1 and s_3 with a probability of $1/2$ yields a better result for I than does playing s_2, whatever II may play. Hence, I never has to use s_2, so that he should agree to sell his right to use it for any positive price.

   Thus, (i) is true.

6. The largest heap in this game is more than double in size that each of the others. It follows that, when writing the sizes in base 2, the

representation of the size of the largest heap is strictly longer than that of each of the other heaps. In particular, the nim-sum of all heap sizes is non-zero, so that I has a winning strategy.

If I plays in any heap but the largest, the nim-sum of all heap size after his move will still be non-zero, so that II will be able to force a win. (In fact, notice that, even when playing in the largest heap, one has a unique move that will ensure that he can win.)

Thus, (iii) is true.