# Final #1

Mark all correct answers in each of the following questions.

1. The first two items below deal with fixed points of a given function $g$. We start at some point $x_0$ and continue according to the iteration $x_{n+1} = g(x_n)$ for $n \geq 0$. In the last four items, we have various functions $f$, and want to find zeros of these functions. We start again from some initial point $x_0$, and continue according to Newton's method.

   (a) Let $g(x) = x \sin x$. Consider the fixed point $\pi/2$. If $x_0 \in (\pi/2, \pi/2 + \delta)$ for sufficiently small $\delta > 0$, then $x_n \underset{n\to\infty}{\longrightarrow} \pi/2$ even though $g'(\pi/2) = 1$. (Hint: You may use Taylor's expansion.)

   (b) Under the conditions of part (a), if $x_0 \in (0, \pi/2)$, then $x_n \underset{n\to\infty}{\longrightarrow} \xi$, where $\xi$ is a fixed point of $g$, but $\xi \neq \pi/2$.

   (c) For $f(x) = e^{x^2} - e$, every $x_0 \neq 0$ will yield a sequence converging to some zero of $f$.

   (d) Let $f(x) = \sin x^2$. For every zero $\xi$ of $f$, there exists a $\delta > 0$ such that, if $x_0 \in (\xi - \delta, \xi + \delta)$, then $x_n \underset{n\to\infty}{\longrightarrow} \xi$. However, the convergence is not at the same rate for all zeros of $f$. Thus, for example, if $x_0$ is near 0 then the convergence is at a rate similar to that provided by the bisection method, whereas if $x_0$ is near $\sqrt{\pi}$ then the convergence is much faster.

   (e) Let $f(x) = \cos x^2$. There exists a $\delta > 0$ such that, for every zero $\xi$ of $f$ and every $x_0 \in (\xi - \delta, \xi + \delta)$, we have $x_n \underset{n\to\infty}{\longrightarrow} \xi$.

   (f) If $f(x) = \ln(x^2 + 1/2)$, then for every $x_0 \neq 0$ we have $x_n \underset{n\to\infty}{\longrightarrow} \xi$ for some zero $\xi$ of $f$.

2. $A$ is a $3 \times 3$ invertible matrix over $\mathbf{R}$. It is given that:

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

(a) The condition number of $A$ with respect to the $\|\cdot\|_2$-norm is at least 6. However, the information we have is not enough to conclude any upper bound on this condition number.

(b) If, out of the two equalities in (1), we were given only the first, we would only be able to conclude that the condition number of $A$ with respect to the $\|\cdot\|_2$-norm is at least 3.

(c) Suppose the condition number of $A$ with respect to the $\|\cdot\|_\infty$-norm is 10. We tried to solve the system

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

and got some approximation $\hat{\mathbf{x}}$. Upon calculating $A\hat{\mathbf{x}}$, we obtained

$$A\hat{\mathbf{x}} = \begin{pmatrix} 1.01 \\ 2.01 \\ 3.01 \end{pmatrix}.$$

Denote by $\mathbf{e}$ the error. Then:

$$\frac{1}{3000} \leq \frac{\|\mathbf{e}\|_\infty}{\|\mathbf{x}\|_\infty} \leq \frac{1}{30}.$$

3. Suppose we have an approximation formula of the form

$$\int_0^1 \frac{f(x)}{\sqrt{x}} dx = w_1 f(x_1) + \ldots + w_k f(x_k) + E. \tag{2}$$

(a) If we have an approximation formula as above with $k = 1$, which is precise (namely, $E = 0$) for every polynomial $f$ of degree not exceeding 1, then $x_1 = 1/3$,

(b) If we have an approximation formula as above with $k = 2$, which is precise for every polynomial $f$ of degree not exceeding 3, then the weights $w_i$ and the points $x_i$, $1 \le i \le 2$, satisfy the equalities:

$$\begin{array}{lcrcl} w_1 & + & w_2 & = & 1, \\ w_1 x_1 & + & w_2 x_2 & = & 1/3, \\ w_1 x_1^2 & + & w_2 x_2^2 & = & 1/5, \\ w_1 x_1^3 & + & w_2 x_2^3 & = & 1/7. \end{array}$$

(c) If we have an approximation formula as above with $k = 3$, which is precise for every polynomial $f$ of degree not exceeding 5, then the error $E$ is non-negative for every continuous function $f : [0,1] \longrightarrow \mathbf{R}$.

(d) Suppose (2) holds precisely for all polynomials $f$ of degree not exceeding $2k-1$. Define an inner product $\langle \cdot, \cdot \rangle$ on the space of all polynomials over $\mathbf{R}$ by:

$$\langle Q_1, Q_2 \rangle = \int_0^1 \frac{Q_1(x) Q_2(x)}{\sqrt{x}} dx.$$

Put $P(x) = (x - x_1) \cdot \ldots \cdot (x - x_k)$. Then $\langle P, Q \rangle = 0$ for every polynomial $Q$ of degree not exceeding $2k - 1$.

(e) If we have an approximation formula as above with $k = 2$, which is precise for every polynomial $f$ of degree not exceeding 3, then the points $x_i$ are rational.

4. Let $[a, b]$ be an interval on the real line, $x_0, x_1, \ldots, x_n$ distinct points in $[a, b]$, and $f, f_1, f_2$ functions from $[a, b]$ to $\mathbf{R}$.

(a) Let $P$ be the interpolation polynomial of degree at most $n$, coinciding with $f$ at $x_0, x_1, \ldots, x_n$. There exists a constant $\varepsilon > 0$ such that, if $|f(x) - P(x)| < \varepsilon$ for every $x \in [a, b]$, then $f$ is $n+1$ times differentiable in $[a, b]$. Moreover, $f^{(n+1)}$ is bounded in the interval.

(b) Let $P_1, P_2$ be the interpolation polynomials of degrees at most $n$, coinciding with $f_1, f_2$, respectively, at the points $x_0, x_1, \ldots, x_n$. If $P_1 = P_2$, then there exist infinitely many points $x$ in $[a, b]$ for which $f_1(x) = f_2(x)$.

(c) Let $P_1, P_2$ be the interpolation polynomials of degrees at most $n$, coinciding with $f_1, f_2$, respectively, at the points $x_0, x_1, \ldots, x_n$. Then the interpolation polynomial of degree at most $n$, coinciding with the function $f_1 + f_2$ at the points $x_0, x_1, \ldots, x_n$, is $P_1 + P_2$.

(d) Let $P_1, P_2$ be the interpolation polynomials of degrees at most $n$, coinciding with $f_1, f_2$, respectively, at the points $x_0, x_1, \ldots, x_n$. Then there exist points $x_{n+1}, \ldots, x_{2n}$ such that $P_1P_2$ is the interpolation polynomial of degree at most $2n$, coinciding with $f_1f_2$ at the points $x_0, x_1, \ldots, x_{2n}$.

(e) Let $P$ be the interpolation polynomial of degree at most $n$, coinciding with $f$ at $x_0, x_1, \ldots, x_n$. Let $g : [a/2, b/2] \longrightarrow \mathbf{R}$ be the function defined by

$$g(x) = f(2x), \qquad x \in [a/2, b/2].$$

Then there exist points $x'_0, x'_1, \ldots, x'_n$ in the interval $[a/2, b/2]$ such that $P(2x)$ is the interpolation polynomial of degree at most $n$, coinciding with $g$ at $x'_0, x'_1, \ldots, x'_n$.

(f) Let $P$ be the interpolation polynomial of degree at most $n$, coinciding with $f$ at $x_0, x_1, \ldots, x_n$. If $f$ is continuously differentiable throughout $[a, b]$, then there exist points $x'_0, x'_1, \ldots, x'_{n-1}$ in the interval $[a, b]$ such that $P'$ is the interpolation polynomial of degree at most $n - 1$, coinciding with $f'$ at $x'_0, x'_1, \ldots, x'_{n-1}$.

## Solutions

1. (a) As

$$g'(x) = x\cos x + \sin x, \qquad g'(\pi/2) = 1,$$

and

$$g''(x) = -x\sin x + 2\cos x,$$

we have

$$g(x) = \pi/2 + 1 \cdot (x - \pi/2) + \frac{-\eta\sin\eta + 2\cos\eta}{2!}(x - \pi/2)^2,$$

where $\eta \in (\pi/2, x)$. Thus, if $x \in (\pi/2, \pi/2+\delta)$, then $g(x) < x$ and, moreover, if $\delta > 0$ is sufficiently small, then $g(x) > \pi/2$. Hence the conditions imply that the sequence $(x_n)$ is strictly decreasing and $x_n > \pi/2$ for each $n$. It follows that $x_n \underset{n\to\infty}{\longrightarrow} \xi$ for some point $\xi \in [\pi/2, \pi/2+\delta)$. The point $\xi$ must be a fixed point of $g$, and therefore $\xi = \pi/2$.

(b) For any $x \in (0, \pi/2)$ we have $0 < g(x) < x$. Hence the sequence $(x_n)$ is strictly decreasing, and therefore converges to a point in $\xi \in [0, \pi/2)$. Now $\xi$ is a fixed point of $g$. Obviously, the only fixed point of $g$ in $[0, \pi/2)$ is 0, and hence $\xi = 0$.

(c) As $f$ is an even function, it suffices, by symmetry, to deal with the case $x_0 > 0$. The only zero of $f$ in the positive half-line is $\xi = 1$. Take an interval $[a, b]$ containing both $x_0$ and 1. Since $f$ is increasing and convex in $(0, \infty)$, replacing $b$ by a sufficiently large number if necessary, we obtain an interval on which the sufficient condition for Newton's method to converge holds.

(d) Since $f'(x) = 2x\cos x^2$, no zero of $f$ is also a zero of $f'$, except for 0. Thus, if we start from a point near a zero $\xi \neq 0$ of $f$, the convergence to $\xi$ is quadratic. Now:

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\sin x^2}{2x\cos x^2} = x - \frac{1}{2\cos x^2} \cdot \frac{\sin x^2}{x^2} x.$$

Since $\frac{\sin x^2}{x^2} \underset{x\to 0}{\longrightarrow} 0$, for points $x$ near 0 we have $g(x) \approx x/2$, so that Newton's method converges only linearly in the case $\xi = 0$.

(e) The zeros of $f$ are all numbers of the form $\sqrt{(2n+1)\pi/2}$ for non-negative integers $n$. None of these zeros is a zero of $f'$, so that Newton's method converges quadratically when we start it from a sufficiently small neighborhood of each. However, as the distances between consecutive zeros of $f$ go to 0 as $n$ grows, these small neighborhoods cannot possibly be independent of $n$.

(f) The zeros of $f$ are $\pm\sqrt{2}/2$. Now:

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} = x - \frac{\ln(x^2+1/2)}{(1/(x^2+1/2)) \cdot 2x} \\ &= x - \frac{\ln(x^2+1/2)}{2} \frac{x^2+1/2}{x^2} x. \end{aligned}$$

One easily sees that, for large $x$, this implies that $|g(x)| > |x|$. Hence Newton's method does not converge if $|x_0|$ is large enough.

Thus, (a), (b), (c) and (d) are true.

2. (a) Out of the two vectors whose images under the action of $A$ are known, one goes to a vector whose $\|\cdot\|_2$-norm is three times as large as that of the given vector and the other goes to a vector whose $\|\cdot\|_2$-norm is half that of the given vector. Hence $\|A\|_2 \geq 3$ and $\|A^{-1}\|_2 \geq 2$. Thus, the condition number of $A$ with respect to the $\|\cdot\|_2$-norm is at least 6.

   (b) In this case it is impossible to conclude anything non-trivial. For example, it is possible that each vector is taken by $A$ to a vector whose $\|\cdot\|_2$-norm is three times as large as that of the given vector. (This is the case, for example, if the transformation defined by $A$ first multiplies each vector by 3, and then applies a suitable rotation to the resulting vector. In this case, the condition number of $A$ is 1.

   (c) According to one of the formulas developed in class:

$$\frac{1}{\|A\| \cdot \|A^{-1}\|} \cdot \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|e\|}{\|x\|} \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}\,.$$

   Plugging in the values $\mathrm{cond}(A) = 10, \|\mathbf{b}\| = 3, \|\mathbf{r}\| = 0.01$, we obtain the required inequalities.

Thus, (a) and (c) are true.

3. (a) We have:

$$\int_0^1 \frac{x^l}{\sqrt{x}} dx = \left[\frac{x^{l+1/2}}{l+1/2}\right]_{x=0}^1 = \frac{2}{2l+1}\,, \qquad l = 0, 1, 2 \ldots.$$

   Hence, if the given approximation formula is precise for every polynomial $f$ of degree not exceeding $2k-1$ for some $k$, then:

$$\begin{array}{ccccccccc} w_1 & + & w_2 & + \ldots & + & w_k & = & 2, \\ w_1 x_1 & + & w_2 x_2 & + \ldots & + & w_k x_k & = & 2/3, \\ \vdots & & \vdots & & & \vdots & & \vdots \\ w_1 x_1^{2k-1} & + & w_2 x_2^{2k-1} & + \ldots & + & w_k x_k^{2k-1} & = & 2/(4k-1). \end{array}$$

In the special case $k = 1$, we get the system

$$\begin{array}{lcl} w_1 & = & 2, \\ w_1 x_1 & = & 2/3, \end{array}$$

Whose solution is $w_1 = 2, x_1 = 1/3$.

(b) The special case $k = 2$ of the system of equations, developed in the preceding part, is:

$$\begin{array}{lclcl} w_1 & + & w_2 & = & 2, \\ w_1 x_1 & + & w_2 x_2 & = & 2/3, \\ w_1 x_1^2 & + & w_2 x_2^2 & = & 2/5, \\ w_1 x_1^3 & + & w_2 x_2^3 & = & 2/7. \end{array}$$

(c) Let $f(x) = -(x - x_1)^2 (x - x_2)^2 (x - x_3)^2$ . As the integrand is non-positive throughout the interval, and is 0 only at finitely many points, $\int_0^1 \frac{f(x)}{\sqrt{x}} dx < 0$. Since $f$ vanishes at the points $x_1, x_2, x_3$, the approximation formula gives 0 as the approximation. Hence the error is negative.

(d) Similarly to the classical case, discussed in class, we must have $\langle P, Q \rangle = 0$ for every polynomial $Q$ of degree not exceeding $k - 1$. However, this is generally not the case for polynomials $Q$ of higher degrees. For example, we clearly have $\langle P, P \rangle > 0$.

(e) Let $P(x) = (x - x_1)(x - x_2) = x^2 + a_1 x + a_2$. Since $P$ is orthogonal to each polynomial of degree up to 1, we have:

$$\begin{array}{lclcl} \langle P, 1 \rangle & = & 0 & \Longrightarrow & \frac{2}{5} + \frac{2}{3} a_1 + 2 a_2 = 0. \\ \langle P, x \rangle & = & 0 & \Longrightarrow & \frac{2}{7} + \frac{2}{5} a_1 + \frac{2}{3} a_2 = 0. \end{array}$$

Solving the system, we obtain $a_1 = -6/7, a_2 = 3/35$. The points $x_1, x_2$ are the zeros of the quadratic $x^2 - \frac{6}{7}x + \frac{3}{35}$, namely $(4 \pm \sqrt{13})/35$.

Thus, only (a) is true.

4. (a) The fact that $|f(x) - P(x)| < \varepsilon$ throughout the interval does not imply any differentiability properties of $f$. For example, let

$$f(x) = \begin{cases} \varepsilon/2, & x \text{ rational}, \\ 0, & x \text{ irrational}, \end{cases}$$

If all points $x_0, x_1, \ldots, x_n$ are irrational, then $P$ is the 0 polynomial, and $|f(x) - P(x)| < \varepsilon$ for all $x$. However, $f$ is not continuous at any point.

(b) Let $f_1(x) = 0$ and $f_2(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$ for all $x$. Then both $P_1, P_2$ are identically 0, but $f_1$ and $f_2$ coincide only at the points $x_0, x_1, \ldots, x_n$.

(c) The polynomial $P_1 + P_2$ clearly coincides with $f_1 + f_2$ at the points $x_0, x_1, \ldots, x_n$, and it is of degree at most $n$. Hence it is the required interpolation polynomial.

(d) Let $f_1(x) = f_2(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$. Then $P_1 = P_2$ is the 0 polynomial. Now, $f_1 f_2$ does not vanish at any point besides the points $x_i$, and therefore $P_1 P_2$ cannot be an interpolation polynomial coinciding with $f_1 f_2$ at $2n + 1$ points.

(e) Taking $x'_i = x_i/2$ for $0 \leq i \leq n$, we obtain:

$$g(x'_i) = f(2x'_i) = f(x_i) = P(x_i) = P(2x'_i), \qquad 0 \leq i \leq n.$$

The polynomial $P(2x)$ thus coincides with $g(x)$ at the points $x'_0, x'_1, \ldots, x'_n$, and it is of degree at most $n$. Hence it is the required interpolation polynomial.

(f) The function $f(x) - P(x)$ vanishes at all points $x_0, x_1, \ldots, x_n$. Between any two zeros of this function, the derivative $f'(x) - P'(x)$ must vanish at least once. Hence there exist points $x'_0, x'_1, \ldots, x'_{n-1}$ in $[a, b]$ such that $f'(x'_i) - P'(x'_i) = 0$ for $0 \leq i \leq n - 1$. Since $P'$ is of degree at most $n - 1$, it is the interpolation polynomial coinciding with $f'$ at $x'_0, x'_1, \ldots, x'_{n-1}$.

Thus, (c), (e) and (f) are true.