Final #2

Mark all correct answers in each of the following questions.

1. The first item below deals with the performance of the bisection method under certain conditions. The following three parts deal with fixed points of various functions g. We start at some point x_0 and continue according to the iteration $x_{n+1} = g(x_n)$ for $n \geq 0$. In the last two parts, we have two functions f, and want to find zeros of these functions. We start again from some initial point x_0, and continue according to Newton's method.

   (a) Let $f : [a, b] \longrightarrow \mathbf{R}$ be increasing and continuously twice differentiable, with $f(a) < 0$ and $f(b) > 0$. Suppose that at the zero ξ of f in $[a, b]$ we have also $f'(\xi) = f''(\xi) = 0$. Then the bisection method works faster than is usually the case with this method. More precisely, denoting by c_n the center of the interval, known to contain ξ, which remains after n iterations of the bisection method, we have $|\xi - c_n| = O(\alpha^{2^n})$ for some $\alpha < 1$.

   (b) Let
   $$g(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$
   Then there exists an $\varepsilon > 0$ such that for every $x_0 \in \left(\frac{2}{\pi}, \frac{2}{\pi} + \varepsilon\right)$ we have $x_n \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi}$.

   (c) Let g be as in the preceding part. Then there exists no $\varepsilon > 0$ such that, for every $x_0 \in (0, \varepsilon)$, we have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

   (d) Let
   $$g(x) = \begin{cases} |x|^{3/2} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then there exists no $\varepsilon > 0$ such that, for every $x_0 \in (0, \varepsilon)$, we have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

(e) Let $f(x) = e^{\sin x} - 1$. Then f has zeros near which Newton's method converges quadratically (namely, $|\xi - x_{n+1}| = O(|\xi - x_n|^2)$), but it also has roots at which the rate of convergence is only linear (namely, $|\xi - x_{n+1}| = O(|\xi - x_n|)$), and not quadratic.

(f) Let $f(x) = \sin^2 x e^{\sin x}$. Then there exists an $\varepsilon > 0$ such that, for every initial point $x_0 \in (-\varepsilon, \varepsilon)$, we have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

2. (a) The function $f : [0, 2] \longrightarrow \mathbf{R}$ is twice differentiable, increasing and convex. We approximate the two integrals $\int_0^1 f(x)dx$ and $\int_1^2 f(x)dx$ by dividing each of the intervals $[0, 1]$ and $[1, 2]$ into n sub-intervals of equal length, and approximating the integral on each sub-interval using the rectangle rule. Then the error in the approximation of $\int_1^2 f(x)dx$ is larger in absolute value than the error in the approximation of $\int_0^1 f(x)dx$.

(b) Let I_{3n} be the approximation obtained for $\int_{-\pi/2}^{\pi} \sin x dx$, when we divide the interval $[-\pi/2, \pi]$ into $3n$ sub-intervals of length $\pi/2n$ each, and approximate the integral on each sub-interval by the midpoint rule. Then the error $E = \int_{-\pi/2}^{\pi} \sin x dx - I_{3n}$ is positive.

(c) Let E be the error when approximating $\int_{10}^{11} x^3 e^x dx$ by dividing the interval $[10, 11]$ into n sub-intervals, not necessarily of equal length, by means of division points $10 = x_0 < x_1 < \ldots < x_n = 11$, and estimating the integral on each sub-interval using the trapezoid rule. Then for every $\delta > 0$ there exists an $\varepsilon > 0$ such that, if $\max_{0\le i\le n-1}(x_{i+1} - x_i) > \delta$, then $|E| > \varepsilon$.

(d) Let E_1 and E_2 the errors when approximating $\int_3^4 e^{3x} dx$ and $\int_4^5 e^{3x} dx$, respectively, by dividing each interval into n sub-intervals of equal length and employing Simpson's rule on each. Then $|E_1| > |E_2|$.

(e) The integrals $\int_a^b \sin^2 x dx$ and $\int_a^b \cos^2 x dx$ are approximated by the integrals $\int_a^b P_1(x)dx$ and $\int_a^b P_2(x)dx$, where P_1 and P_2 are the interpolation polynomials of the functions $f_1(x) = \sin^2 x$ and

$f_2(x) = \cos^2 x$, respectively, coinciding with these functions at the points $x_0, x_1, \ldots, x_n \in [a, b]$. Let E_1 and E_2 be the resulting errors. Then $|E_1| = |E_2|$.

3. (a) Let $f : [-a, a] \longrightarrow \mathbf{R}$ be an odd function (namely, a function satisfying $f(-x) = -f(x)$ for each x). Then for every even $n \geq 2$ there exist points $x_0, x_1, \ldots, x_n \in [-a, a]$ such that the interpolation polynomial of degree at most n, coinciding with f at $x_0, x_1, \ldots, x_n$, is actually of degree at most $n - 1$.

   (b) Let $f : [-a, a] \longrightarrow \mathbf{R}$ be an even function (namely, a function satisfying $f(-x) = f(x)$ for each x). Then, for every n and every points $x_0, x_1, \ldots, x_n \in [-a, a]$, the interpolation polynomial of degree at most n, coinciding with f at $x_0, x_1, \ldots, x_n$, is also an even function.

   (c) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function for which $f^{(n+1)}(x)$ does not exist at any point x. For each n we are given $n+1$ points $x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}$, satisfying $x_{i+1}^{(n)} - x_i^{(n)} > n$ for every $0 \leq i \leq n - 1$. Let P_n be the interpolation polynomial of degree at most n, coinciding with f at the points $x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}$. Then, for every sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive numbers and every sufficiently large n, there exists a point $x \in [x_0^{(n)}, x_1^{(n)}]$ such that $|f(x) - P_n(x)| > \varepsilon_n$.

   (d) For each n, let P_n be the interpolation polynomial of degree at most n, coinciding with the function $f(x) = e^x$ at the $n+1$ points $n, n + 1, \ldots, 2n$. Then, for every number M, there exists some $n_0 = n_0(M)$ such that for every $n > n_0$ there exists a point $x \in [n, n + 1]$ for which $|e^x - P_n(x)| > M$.

   (e) For each n, let P_{2n} be the interpolation polynomial of degree at most $2n$, coinciding with the function $f(x) = e^x$ at the $2n + 1$ points $-n, -n + 1, \ldots, 0, \ldots, n - 1, n$. Denote

$$\delta_n = \max_{0 \leq x \leq 1} |e^x - P_{2n}(x)| .$$

   Then $\delta_n \underset{n\to\infty}{\longrightarrow} 0$. (Hint: You may use the inequalities $\frac{2^{2n}}{2n+1} \leq \binom{2n}{n} \leq 2^{2n}$ for any non-negative integer n.)

(f) Let P_2 be the interpolation polynomial of degree at most 2, coinciding with the function $f(x) = \mathrm{tg}x$ at the points $\pi/6, \pi/4, \pi/3$. Then $P_2(x) \neq \mathrm{tg}x$ for every $x \in (-\pi/2, \pi/2)$ with $x \neq \pi/6, \pi/4, \pi/3$.

4. Consider the initial value problem:

$$y' = \sqrt{y}, \qquad y(1) = \frac{1}{4}.$$

Note that the function $y(t) = t^2/4$ forms a solution of the problem.

(a) Using the equation to calculate higher-order derivatives of $y(t)$, and expanding the function around the point $t_0 = 1$, we obtain the above solution of the problem.

(b) Using Euler's method with step size $h = 5/8$, the approximation we obtain for $y(9/4)$ is $33/32$.

(c) Using Euler's method with any positive step size, the resulting sequence $(y_n)_{n=1}^{\infty}$ is increasing.

Solutions

1. (a) The length of the interval we have in the bisection method after n iterations is $(b-a)/2^n$. As c_n is the center of this interval, and c_{n+1} is the center of either the right half or the left half of this interval, we have $|c_{n+1} - c_n| = \frac{b-a}{4 \cdot 2^n}$. It follows that either $|\xi - c_n| \geq \frac{b-a}{8 \cdot 2^n}$ or $|\xi - c_{n+1}| \geq \frac{b-a}{8 \cdot 2^n}$. In particular, the errors cannot decrease faster than exponentially. (Namely, some of them may be much smaller, but one out of any two consecutive errors is at least a constant divided by 2^n.)

(b) For $x \neq 0$ we have

$$g'(x) = -\frac{1}{x} \cos \frac{1}{x} + \sin \frac{1}{x},$$

and in particular $g'(2/\pi) = 1$. It follows that, if $x \in \left(\frac{2}{\pi}, \frac{2}{\pi} + \varepsilon\right)$, where $\varepsilon > 0$ is sufficiently small, then $g(x) > 2/\pi$. Also, since $0 <$

$\sin\frac{1}{x} < 1$ for such x, we have $g(x) < x$. Thus, if $x_0 \in \left(\frac{2}{\pi}, \frac{2}{\pi} + \varepsilon\right)$, then the sequence (x_n) is decreasing, and is bounded below by $2/\pi$. Hence it converges to a fixed point of g in the interval $\left[\frac{2}{\pi}, \frac{2}{\pi} + \varepsilon\right)$, which must be the point $2/\pi$.

(c) Every point of the form $\frac{1}{(2n+1/2)\pi}$, where n is a positive integer, forms a fixed point of g. As the sequence formed of these points converges to 0, this means that there are points arbitrarily close to 0 such that, if we start the iteration from those points, we do not get convergence to 0.

(d) Since for $x \in (0,1)$ we have

$$0 \leq |x|^{3/2} \sin\frac{1}{x} \leq x^{3/2} < x,$$

if we start with $x_0 \in (0,1)$, we obtain a non-increasing sequence in the interval $[0,1)$. This sequence converges to a fixed point of g in $[0,1)$. Now 0 is clearly the only fixed point of g in this interval, and therefore we do have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

(e) The function f vanishes exactly at those points where the sine function does, namely at all integer multiples of π. Since $f'(x) = \cos xe^{\sin x}$, when $f(x) = 0$ we have $f'(x) = \pm 1$. Hence Newton's method converges quadratically near all zeros of f.

(f) We have:

$$\begin{aligned} g(x) =& x - \frac{\sin^2 xe^{\sin x}}{2\sin x\cos xe^{\sin x} + \sin^2 x\cos xe^{\sin x}} \\ =& x - \frac{\sin x}{2\cos + \sin x\cos x} \\ =& x - \frac{1}{2\cos + \sin x\cos x} \cdot \frac{\sin x}{x} \cdot x \approx \frac{1}{2}x. \end{aligned}$$

It follows that, if we start from a point near 0, then we get convergence, although only at a linear rate.

Thus, (b), (c) and (f) are true.

2. (a) According to the formula for the error in the rectangle rule, for the first integral the error is

$$E_1 = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{f'(\eta_i)}{2}$$

for some points $\eta_i \in [(i-1)/n, i/n], 1 \le i \le n$. Similarly, the error in the second integral is

$$E_2 = \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{f'(\eta_i')}{2}$$

for appropriate points $\eta_i' \in [1 + (i-1)/n, 1 + i/n]$. Since f is convex, the function f' is increasing, and therefore the sum on the right-hand side of the formula for E_1 is term-by-term smaller than the corresponding sum for E_2. Since f is increasing, both sums consist of non-negative terms, and consequently $E_1 < E_2$.

(b) Due to the fact that the integrand here is an odd function, the errors on the sub-intervals $[-\pi/2, 0]$ and $[0, \pi/2]$ cancel each other, and therefore the total error is actually the same as the error on the interval $[\pi/2, \pi]$. In view of the general estimate for the error when using the mid-point rule, the error on each sub-interval of $[\pi/2, \pi]$ is of the same sign as $\sin'' = -\sin$, and therefore negative.

(c) For suitable points $\eta_i \in (x_i, x_{i+1})$ we have

$$\begin{aligned} E &= -\frac{1}{12} \sum_{i=1}^{n} \frac{d^2(x^3 e^x)}{dx^2}(\eta_i)(x_{i+1} - x_i)^3 \\ &= -\frac{1}{12} \sum_{i=1}^{n} (6\eta_i + 6\eta_i^2 + \eta_i^3) e^{\eta_i} (x_{i+1} - x_i)^3, \end{aligned}$$

Hence, if $\max_{0 \le i \le n-1}(x_{i+1} - x_i) > \delta$, then

$$|E| > \frac{1}{12}(6 \cdot 10 + 6 \cdot 10^2 + 10^3) e^{10} \delta^3 .$$

It follows that $\varepsilon = 100 e^{10} \delta^3$ satisfies the requirements.

(d) According to the formula for the error in Simpson's rule, we have

$$E_1 = -\frac{1}{90}\sum_{i=1}^{n} 3^4 e^{3\eta_i} \cdot \left(\frac{1}{2n}\right)^5$$

and

$$E_2 = -\frac{1}{90}\sum_{i=1}^{n} 3^4 e^{3\xi_i} \cdot \left(\frac{1}{2n}\right)^5,$$

where $3+(i-1)/n \le \eta_i \le 3+i/n$ and $4+(i-1)/n \le \xi_i \le 4+i/n$ for $1 \le i \le n$. Comparing the sums obtained for E_1 and E_2 term by term, we readily see that $|E_1| < |E_2|$.

(e) The identity $f_2 = 1-f_1$ implies that $P_2 = 1-P_1$, and consequently

$$\begin{aligned} E_2 &= \int_a^b f_2(x)dx - \int_a^b P_2(x)dx \\ &= \int_a^b (1-P_2(x))dx - \int_a^b (1-f_2(x))dx \\ &= \int_a^b P_1(x)dx - \int_a^b f_1(x)dx = -E_1, \end{aligned}$$

which yields $|E_1| = |E_2|$.

Thus, (a), (c) and (e) are true.

3. (a) Suppose the points $x_0, x_1, \ldots, x_n$ are taken as follows: $x_0 = 0$, $x_1, x_2, \ldots, x_{n/2}$ are any distinct points in $(0, a]$ and $x_{n/2+1} = -x_1$, $x_{n/2+2} = -x_2, \ldots, x_n = -x_{n/2}$. It is easy to verify that the interpolation polynomial of degree at most n, coinciding with f at the points $x_0, x_1, \ldots, x_n$, is odd. Moreover, the same holds if the point x_0 is omitted. Hence the interpolation polynomial of degree at most n, coinciding with f at the points $x_0, x_1, \ldots, x_n$, is the same as the interpolation polynomial of degree at most $n-1$, coinciding with f only at the points $x_1, \ldots, x_n$. It follows that the interpolation polynomial of degree at most n, coinciding with f at the points $x_0, x_1, \ldots, x_n$, is in fact of degree at

most $n-1$. (We mention in passing that the required interpolation polynomial P can be effectively found by setting $P(x) = a_1x + a_3x^3 + a_5x^5 + ... + a_{n-1}x^{n-1}$ and solving the linear system

$$\begin{array}{ccc} P(x_1) & = & f(x_1) \\ P(x_2) & = & f(x_2) \\ \vdots & \vdots & \vdots \\ P(x_{n/2}) & = & f(x_{n/2}) \end{array}$$

whose matrix of coefficients is basically a Vandermonde matrix.)

(b) If the points $x_0, x_1, \ldots, x_n$ are all chosen from $[0, a]$, then the fact that f is even plays no role when the interpolation polynomial is constructed, so that this polynomial may be any polynomial. (In fact, to be specific, let P be any non-even polynomial, and define f by:

$$f(x) = \begin{cases} P(x), & x \geq 0, \\ P(-x), & x < 0. \end{cases}$$

Then f is even, but the interpolation polynomial, constructed as above, will be the polynomial P.)

(c) Let $f(x) = 1$ if x is rational and $f(x) = 0$ otherwise. Clearly, f is non-continuous at any point. Take $\varepsilon_n = 1$ for each n. Choosing all points $x_i^{(n)}$ to be rational, we see that each P_n is identically 1, so that $|f(x) - P_n(x)| \leq 1$ at every point x.

(d) According to the formula for the error of the interpolation polynomial, for every $x \in [n, 2n]$ we have

$$e^x - P_n(x) = \frac{(x-n)(x-n-1)\ldots(x-2n)}{(n+1)!}(e^x)^{(n+1)}(\eta)$$

for some $\eta = \eta(x) \in [n, 2n]$. For $x = n + 1/2$, this yields

$$\begin{aligned} |e^{n+1/2} - P_n(n+1/2)| &> \frac{1/2 \cdot 1/2 \cdot 3/2 \cdot 5/2 \cdot \ldots \cdot (n-1/2)}{(n+1)!}e^n \\ &> \frac{(n-1)!}{4(n+1)!}e^n = \frac{e^n}{4n(n+1)}. \end{aligned}$$

The right-hand side grows exponentially as a function of n, and in particular the absolute error eventually exceeds any real number.

(e) The error at any point $x \in [-n, n]$ is

$$e^x - P_{2n}(x) = \frac{(x+n)(x+n-1)\dots x \dots (x-n)}{(2n+1)!}(e^x)^{(2n+1)}(\eta)$$

for some $\eta = \eta(x) \in [-n, n]$. In particular, if $x \in [0, 1]$, then:

$$|e^x - P_{2n}(x)| \le \frac{(n+1)n(n-1)\dots \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot \dots (n-1)n}{(2n+1)!} e^n.$$

Therefore:

$$|e^x - P_{2n}(x)| \le \frac{(n+1)!n!}{(2n+1)!} e^n \le \frac{e^n}{\binom{2n}{n}} \le \frac{(2n+1)e^n}{4^n} \xrightarrow[n\to\infty]{} 0.$$

(f) The error at any point $x \neq \pi/6, \pi/4, \pi/3$ satisfies

$$\mathrm{tg}x - P_2(x) = \frac{(x-\pi/6)(x-\pi/4)(x-\pi/3)}{3!}\mathrm{tg}^{(3)}(\eta)$$

for some $\eta \in (-\pi/2, \pi/2)$. Now

$$\mathrm{tg}^{(3)}(\eta) = \frac{6\sin^2\eta}{\cos^4\eta} + \frac{2}{\cos^2\eta},$$

which does not vanish for any η. Since the error does not vanish, the function does not coincide with the interpolation polynomial at any point besides the interpolation points.

Thus, (a), (d), (e) and (f) are true.

4. (a) We have

$$y'' = \frac{1}{2\sqrt{y}} \cdot y' = \frac{1}{2},$$

and therefore

$$y''' = y^{(4)} = \dots = 0.$$

Hence:

$$\begin{aligned} y(t) &= y(1) + y'(1)(t-1) + \frac{y''(1)}{2!}\cdot(t-1)^2 \\ &= \frac{1}{4} + \frac{1}{2}(t-1) + \frac{1}{4}\cdot(t-1)^2 = \frac{t^2}{4}. \end{aligned}$$

(b) Euler's method yields

$$
\begin{aligned}
y_0 &= y(1) = \frac{1}{4}\,,\\
y_1 &= y_0 + \sqrt{y_0}\cdot h = \frac{1}{4} + \sqrt{\frac{1}{4}}h = \frac{9}{16}\,,\\
y_2 &= y_1 + \sqrt{y_1}\cdot h = \frac{9}{16} + \frac{3}{4}\cdot\frac{5}{8} = \frac{33}{32}\,.
\end{aligned}
$$

Since y_0 is the value of the function at the point $t = 1$ and the step size is $5/8$, the values of y_1 and y_2 are the approximations for the value of the function at the points $t = 13/8$ and $t = 9/4$, respectively.

(c) We claim that, for every non-negative integer n, we have $y_{n+1} > y_n > 0$. In fact, for $n = 0$ we have $y_1 = y_0 + \sqrt{y_0}\cdot h > y_0$, and by induction we similarly see that this holds for each n.

Thus, all three claims are true.