Numerical Analysis

Exercises

1 MATLAB

1. Given: $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Use MATLAB to check the veracity of the following propositions:

(a) The vector v is an eigenvector of A.

(b) The vector v is an eigenvector of the inverse to A.

(c) The vector v is an eigenvector of the transpose of A.

(d) The vector v is an eigenvector of A multiplied by itself.

(e) The vector v is an eigenvector of B , where $B_{ij} = A_{ij}^2$.

Present your code (one line for each proposition should be enough). Note: In order to find C, where $C_{ij} = A_{ij}/B_{ij}$, you may use the MATLAB statement: $C = A./B$.

2. Implement your own MATLAB function for matrix multiplication.

3.

(a) Write a MATLAB function which, given a vector containing the digits in the decimal expansion of a non-negative integer L, provides the binary expansion of L.

(b) Write a MATLAB function which, given a non-negative integer, provides its binary expansion.

4. Consider the following polynomial:

$$f(x) = 2x^3 + 5x^2 + 3x + 4. \quad (1)$$

(a) Use only paper and pencil (no calculator!) to find out the extreme values of the function.

(b) Use MATLAB symbolic tools to find the extreme values of the function.

(c) Use MATLAB 'plot' and 'subs' commands to plot $f(x)$ and $f'(x)$ on the interval $[-2, 0]$, on the same figure. Which extreme value is a minimum and which a maximum?
Present your code and plots.
(See plot options at:
www.mathworks.com/access/helpdesk/help/techdoc/ref/plot.html)

2 Introduction

5. Let $\mathbf{p} = (1/4, 1/4, 1/4, 1/4)$ and

$$T = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

(a) Convince yourself using MATLAB that the sequence of vectors $(\mathbf{p}T^n)$ converges to some vector $\mathbf{p}_0$ as $n \to \infty$.

(b) Use MATLAB to find the eigenvalues and eigenvectors of T. Verify that $\mathbf{p}_0$ is a row eigenvector of T, corresponding to the eigenvalue 1.

6. A person uses the internet to play two games, A and B. After a game of A he chooses between another game of A and a game of B with equal probabilities, whereas after a game of B he continues with another game of B with probability 2/3 and reverts to A with probability 1/3 only.

(a) Describe the process in terms of a transition matrix T.

(b) Find matrices R and Λ, with Λ diagonal, such that $T = R^{-1}\Lambda R$.

(c) Show that $T^n \underset{n\to\infty}{\longrightarrow} T_0$ for an appropriate matrix T_0.

(d) Find the asymptotic frequencies with which the player plays each of the games.

7. Let $T = (t_{ij})_{i,j=1}^d$ be a square matrix.

(a) Prove that, if T has the same sum of entries in each row, say $\sum_{j=1}^d t_{ij} = C$ for each $1 \le i \le d$, then C is an eigenvalue of T. In particular, if T is a *stochastic matrix*, i.e., a matrix with non-negative entries, such that the sum of entries in each row is 1, then 1 is an eigenvalue of T.

(b) Prove that the product of stochastic matrices is stochastic as well. In particular, positive integer powers of stochastic matrices are such.

(c) Prove that all eigenvalues of a stochastic matrix are of absolute value not exceeding 1.

(d) Suppose T is a stochastic $d \times d$ matrix, all of whose eigenvalues, except for the eigenvalue 1, are of absolute value strictly less than 1. Suppose also that all eigenvalues are simple. Show that, for every probability vector (namely, a vector with non-negative entries, summing up to 1) $\mathbf{p}$, there exists a probability vector $\mathbf{p}'$ such that $\mathbf{p}T^n \underset{n\to\infty}{\longrightarrow} \mathbf{p}'$.

(e) Show that the vector $\mathbf{p}'$ from the preceding part is an eigenvector of T, corresponding to the eigenvalue 1. Conclude that the vector $\mathbf{p}'$ is the same for all probability vectors $\mathbf{p}$.

(f) For any $d \ge 2$, give an example of a $d \times d$ stochastic matrix T and a probability d-vector $\mathbf{p}$ such that the sequence $(\mathbf{p}T^n)_{n=0}^\infty$ does not converge.

8. Let T be the transformation of the interval $[0, 1)$, defined in class.

(a) Which points $x \in [0, 1)$ do have the property that $T^n(x) = 0$ for all sufficiently large n?

(b) Prove that, if x is not such a point, then we cannot even have $T^n(x) \underset{n\to\infty}{\longrightarrow} 0$.

9. (requires Probability Theory) Use the Borel-Cantelli Lemma to prove that, for $l = 0, 1, \dots, 2^k - 1$, if x is chosen randomly (uniformly) in the interval $[0, 1)$, then there is probability 1 that the interval $[l/2^k, (l + 1)/2^k)$ will contain infinitely many terms of the sequence $(T^n(x))_{n=0}^\infty$. Conclude that, moreover, if x is chosen randomly in $[0, 1)$, then there is probability 1 that every interval $[a, b) \subseteq [0, 1)$ of positive length will contain infinitely many terms of that sequence.

10.

(a) Use the infinite binary expansion of real numbers to find a point x for which the sequence $T^n(x)$ is asymptotically 30% of the time in the subinterval $[0, 1/2)$ (and 70% of the time in the subinterval $[1/2, 1)$).

(b) Does there exist a point x such that $T^n(x)$ lies in the subinterval $[0, 1/4)$ asymptotically 40% of the time, in $[1/4, 1/2)$ – 30% of the time, in $[1/2, 3/4)$ – 20% of the time and in $[3/4, 1)$ – 10% of the time?

11. Consider the family of transformations T_a, $a = 2, 3, \ldots$, of the interval $[0, 1)$, defined by $T_a(x) = \{ax\}$, where $\{t\}$ denotes the fractional part of a real number t.

(a) Given a transformation S of an arbitrary space X (namely, S is a function from X to itself), a point $x \in X$ is *eventually periodic* under S if there exist positive integers $m < n$ such that $S^n(x) = S^m(x)$ (or, equivalently, if the orbit of x under S is finite). Which points of $[0, 1)$ are eventually periodic under T_a?

(b) In the setup of the preceding part, a point $x \in X$ is *periodic* under S if $S^n(x) = x$ for some $n \in \mathbf{N}$. Perform a few experiments, and conjecture which points of $[0, 1)$ are periodic under T_a.

12. Let T_a be as in the preceding question.

(a) Test how T_a-orbits look like on the computer for odd numbers a. Do you think the computer results are closer to the truth than they are for $a = 2$?

(b) Now consider even values of a. You can easily verify that, for some of them, the orbits received on the computer contain much less non-zero terms than in the case of $a = 2$, whereas for others these orbits contain just a few less such terms. Explain this dependence on a.

13. Given any transformation T of an interval (or a more general space), one may consider orbits $\{T^n(x) : n = 0, 1, 2, \ldots\}$ of points x. In the case considered in class, we saw that, if the transformation is defined as multiplication by 2 modulo 1, then even a very small error in the initial determination of x causes the computed values of $T^n(x)$ to be quite far from their true values already for medium-sized n. Can you suggest a sufficient condition on T under which the initial error causes only small errors in the computation of $T^n(x)$ even for large values of n? (Hint: Formulate your condition in terms of the relation between the size of $|T(x_2) - T(x_1)|$ and that of $|x_2 - x_1|$ for various pairs of points (x_1, x_2).)

3 Nonlinear Equations

14.

(a) Find a zero of the function $e^x - 2\cos x$ in the interval $[0, 1]$ with an error not exceeding 0.01 by means of the Bisection Method, using only a calculator. Does the function have additional zeros?

(b) Same for the function $e^x \sin x - 1$.

15. It is required to find all the zeros of the function f defined by

$$f(x) = xe^{2x} - 10xe^x - 5e^x + 50 \ .$$

(a) Show that f has a single zero in the interval $[1, 2]$, another single zero in the interval $[2, 3]$, and no additional zeros.

(b) How many flops are required to find each of the zeros with an error not exceeding 10^{-6} if we use the Bisection Method in MATLAB?

(c) Simplify the form of f to obtain a better result (still using the Bisection Method when required).

16. Same as the preceding question for the function

$$f(x) = xe^{2x} \sin x - 2xe^x - 5e^x \sin 2x + 20 \cos x \ ,$$

where the intervals $[1, 2]$ and $[2, 3]$ are replaced by $[0, 1]$ and $[1, 2]$, respectively.

17.

(a) Suppose instead of the Bisection Method we use "trisection". Namely, at each step we divide the current interval into three equal subintervals, and check which of them contains a zero of the given function. Assuming that the function contains a single zero in the interval $[a, b]$, how fast would you expect the method to converge "on the average"? Compare the result with that of the Bisection Method. (The analysis should be rather heuristic. Assume that, at each step, the zero of the function may belong to each of the three subintervals with equal likelyhoods.)

(b) Same as the first part for the following variant of the Bisection Method: At each step the current interval $[a', b']$ is bisected into two subintervals $[a', (1-\alpha)a' + \alpha b']$ and $[(1-\alpha)a' + \alpha b', b']$, where $0 < \alpha < 1$ is fixed. How does your answer vary with α?

18. Suggest a few ways of rewriting Problem 14(a) as one of locating fixed points of an appropriately chosen function.

19.

(a) Let f be differentiable in the interval $[a, b]$. Assume f has at least one zero at the interval. Show that Newton's Method leads within one step to a zero of f for every starting point $x_0 \in [a, b]$ if and only if f is linear.

(b) Construct a function, differentiable on the whole real line, having an infinite number of zeros $(\xi_n)_{n=1}^{\infty}$, so that each ξ_n has the following property: There exists an infinite set A_n such that Newton's Method, if started at any point of A_n, leads within one step to ξ_n.

20.

(a) Find a family of cubic polynomials for which Newton's Method, if started at an appropriate point, yields a periodic orbit.

(b) (requires Differential Equations) Find a function f defined over the whole real line with the following properties: 1) f has a single zero. 2) f is differentiable everywhere, except perhaps at its zero. 3) Newton's Method, with any initial point except for the zero of f, yields a periodic orbit.

21. Consider the following functions from $\mathbf{R}$ to itself. Characterize the fixed points of each, and determine which points are attracted to each of these under successive applications of the function. If possible, try to find the rate of convergence of those points attracted to the fixed points.

(a) $g(x) = 2x/3 + 7$.

(b) $g(x) = 2x + 5$.

(c) $g(x) = x + x^3$.

(d) $g(x) = x - x^3$.

(e) $g(x) = x + x^2$.

(f) $g(x) = \sin x$.

(g) $g(x) = \cos x$.

(h)

$$g(x) = \begin{cases} x + x^3 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

22. The same as the preceding question for the following functions from $\mathbf{R}^2$ to itself:

(a) $g(x, y) = (x/2, y/3)$.

(b) $g(x, y) = (x/2 + 1, y/3 + 2)$.

(c) $g(x, y) = (2x, 3y)$.

(d) $g(x, y) = (x/2 + 1, 3y - 2)$.

(e) $g(x, y) = (x/2 + y, y/2)$.

(f) $g(x, y) = (x + y, -2x + 4y)$.

(g) $g(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha), \qquad (\alpha \in \mathbf{R})$.

(h) $g(x, y) = (x^3 y^4, x^{-4} y^3)$ (here take $\mathbf{R}^2_+$ as your domain.)

23. For the following functions all zeros can be explicitly found. Find the biggest interval you can around each zero such that, starting Newton's Method at a point belonging to this interval, we obtain a sequence converging to that zero. Determine the rate of convergence in each case.

(a) $f(x) = e^x - a$.

(b) $f(x) = x^2 - 8x + 15$.

(c) $f(x) = (x - a)(x - b)$.

(d) $f(x) = (x - 1)(x - 2)(x - 3)$.

(e) $f(x) = (x - 1)^k$.

(f) $f(x) = \sin x$.

(g) $f(x) = e^x - (1 + x + x^2/2)$.

(h) $f(x) = e^x - 1 - \sin x \ , \quad x \in [-\pi/2, \pi/2]$.

(i) $f(x) = \operatorname{tg} x$.

24. Show that, if ξ is a root of order k of f, then, defining g by

$$g(x) = x - kf(x)/f'(x) \ ,$$

we have $g'(\xi) = 0$.

25.

(a) Suppose Newton's Method is applied to the function $f(x) = x^k$. Show that, the larger is k, the slower is the convergence.

(b) Consider the function defined by:

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0\ . \end{cases}$$

Find the rate of convergence of Newton's Method for this function. Explain.

26. How does Newton's Method work for the function f defined by:

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0. \end{cases}$$

Explain how this corresponds to the theory.

27. Consider the following two iterative formulas, designed to solve the equation $x^2 = a$:

$$\begin{aligned} x_{n+1} &= \tfrac{x_n + a/x_n}{2}, & n = 0, 1, 2, \ldots, \\ x_{n+1} &= \tfrac{x_n + a/x_n}{2} - \tfrac{(x_n - a/x_n)^2}{8x_n}, & n = 0, 1, 2, \ldots\ . \end{aligned}$$

(a) Show that the first method converges quadratically.

(b) Show that the second method is a third-order method.

(c) Write a MATLAB function implementing the two methods, and compare their performance.

28. Consider the function defined by:

$$f(x) = \begin{cases} x^4 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0\ . \end{cases}$$

Does there exist an interval around 0 such that, starting Newton's Method at any point x_0 of this interval with $f(x_0) \neq 0$ and $f'(x_0) \neq 0$, we obtain an orbit converging to 0?

29.

(a) Find the general solution of the recursion $d_{n+1} = 2d_n + C$, where C is a constant.

(b) What does the first part hint regarding the error in Newton's Method?

30.

(a) Suppose a new method for solving non-linear equations has been proposed. Given any three points, the first of these is replaced by a new one, determined by the values of the given function at the initial points. More precisely, given three consecutive approximations x_{n-2}, x_{n-1}, x_n to the root, we calculate the next approximation x_{n+1} according to the given procedure, and continue from the triple x_{n-1}, x_n, x_{n+1}. Suppose the error in this method satisfies $e_{n+3} \approx K e_{n+2} e_{n+1} e_n^2$ for some constant K. Compare the new method with the others we studied.

(b) Same for a method based on the last four points at each stage, and satisfying $e_{n+4} \approx K e_{n+3}^2 e_{n+2} e_{n+1}^4 e_n^6$.

4 Systems of Linear Equations

31. Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ satisfy all properties of a norm.

32.

(a) Prove that the sum of a finite number of norms is a norm.

(b) Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$. For which $n \times n$ matrices A does the function $\|\cdot\|'$, defined by

$$\|v\|' = \|Av\|, \qquad v \in \mathbf{R}^n,$$

constitute a norm on $\mathbf{R}^n$?

33. Show that for any norm $\|\cdot\|$ on $\mathbf{R}^n$ we have:

$$\|u - v\| \geq |\|u\| - \|v\||, \qquad u, v \in \mathbf{R}^n.$$

34. Let $\|\cdot\|$ be any norm on $\mathbf{R}^n$. Show that the function $d : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$, defined by

$$d(u, v) = \|u - v\|, \qquad u, v \in \mathbf{R}^n,$$

is a metric on $\mathbf{R}^n$. Namely, d satisfies the following properties:

(i) $d(u, v) \geq 0$ for every $u, v \in \mathbf{R}^n$, with equality if and only if $u = v$.

(ii) $d(u, v) = d(v, u)$ for every $u, v \in \mathbf{R}^n$.

(iii) $d(u, w) \leq d(u, v) + d(v, w)$ for every $u, v, w \in \mathbf{R}^n$.

35. Prove the following properties of matrix norms:

(a) $\|A\| \geq 0$ for every $n \times n$ matrix A, with equality if and only if $A = 0$.

(b) $\|\alpha A\| = |\alpha| \cdot \|A\|$ for every $\alpha \in \mathbf{R}$ and $n \times n$ matrix A.

(c) $\|A + B\| \leq \|A\| + \|B\|$ for every two $n \times n$ matrices A and B.

(d) $\|AB\| \leq \|A\| \cdot \|B\|$ for every two $n \times n$ matrices A and B.

36. Show that, if $A = (a_{ij})_{i,j=1}^n$, then $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$.

37. Calculate $\|A\|_1$ for a general $n \times n$ matrix A.

38.

(a) Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $\mathbf{R}^n$. Suppose

$$c_1\|v\| \leq \|v\|' \leq c_2\|v\|, \qquad v \in \mathbf{R}^n,$$

for some constants $c_2 \geq c_1 > 0$. What relations can you infer between the corresponding matrix norms?

(b) Provide non-trivial upper and lower bounds for $\|A\|_2$ in terms of the entries of A.

39. Calculate the 2-norm of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} .$$

40. Let A be an $n \times n$ matrix over $\mathbf{R}$. Suppose all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are real and known.

(a) Find a lower bound for $\|A\|$ in terms of the eigenvalues of A, where $\|\cdot\|$ is any norm.

(b) Show that it is impossible to give an upper bound for $\|A\|$ in terms of the eigenvalues of A alone. (Here you may take any norm you prefer).

41. Check experimentally using MATLAB that, in general, as the condition number of the matrix of coefficients increases, so does the error in the solution, as follows:

(a) Select randomly matrices of coefficients A. Next select corresponding vectors b for which the solution of the system $Ax = b$ is simple and known in advance (say, all the entries of x are 1). Solve the system and compare the size of the (known) error vector (measured by the norm of your liking) with the condition number. (Instead of calculating the condition number, you may use MATLAB's **cond** and **rcond**.)

(b) Find the size of the errors when A is the Hilbert matrix of order n (generated in MATLAB by **hilb**(n)), again with the solution x simple and known in advance.

42. Exemplify the process of successive improvements, A being the Hilbert matrix of order n and the solution x simple and known in advance, for various values of n. When does the process work indeed to successively improve the solution?

43. For each of the following systems, check whether Jacobi iteration converges by calculating explicitly the sequence of errors $(e^{(t)})_{t=0}^{\infty}$, starting with an arbitrary initial error $e^{(0)}$:

(a) $\begin{array}{rcrcr} 6x & - & y & = & 4 \\ 2x & + & 5y & = & 12 \end{array}$

(b) $\begin{array}{rcrcr} 2x & - & 5y & = & -8 \\ 4x & - & y & = & 2 \end{array}$

(c) $\begin{array}{rcrcr} 4x & + & 3y & = & 10 \\ 5x & - & 4y & = & -3 \end{array}$

44.

(a) Under which conditions on a general 2×2 system does Jacobi iteration converge? (Assume that the matrix of coefficients is invertible and has non-zero diagonal elements.)

(b) Jacobi iteration may be applied to a 2×2 system in two ways, namely with the equations in the given order or in the inverse order. In view of the preceding part, what can be said usually (i.e., except perhaps for borderline cases) about the convergence of the method for each of these orders?

45. Same as Question 43 for Gauss-Seidel iteration.

46. Show that for a general 2×2 system the conditions for the convergence of Jacobi iteration and those for Gauss-Seidel iteration usually coincide, but the latter is roughly twice as fast as the first.

47. Consider the system in Question 43.a.

(a) Write down explicitly the transformation according to which SOR acts in this case.

(b) For what range of the over-relaxation parameter ω do we have convergence to the solution of the system?

(c) How does the rate of convergence depend on ω for this particular system?

48. Check experimentally, using MATLAB, possible and good values for the parameter ω used in the SOR (Successive Over-Relaxation) iterative method, as follows. Select 100 random 5×5 matrices of coefficients A, such that the diagonal entries lie in the interval $[1, 2]$ and the off-diagonal entries in the interval $[-0.2, 0.2]$. For each such A, put $b = Av$, where $v = (1, 2, \ldots, 5)^t$. Solve the linear system $Ax = b$ using SOR for parameter values $\omega = 1, 1.1, 1.2, \ldots, 2$, and some pre-determined number of iterations. How does the mean size of the error vector (measured by the norm of your liking) depend on ω? Present your findings on a suitable plot.

5 Numerical Differentiation

49. Suppose f is defined only to the right of the point x, so that the central difference approximation for $f'(x)$, namely

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2) \,,$$

is inapplicable. Can you suggest an alternative $O(h^2)$ approximation for the required (one-sided) derivative?

50. Prove the approximation, mentioned in class,

$$f^{(4)}(x) = \frac{-f_3 + 12f_2 - 39f_1 + 56f_0 - 39f_{-1} + 12f_{-2} - f_{-3}}{6h^4} + O(h^4) \,,$$

where $f_i = f(x + ih)$ for each i.

51. Prove that, for any fixed non-zero distinct numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, one can find an $O(h^k)$ approximation for $f'(x)$ in terms of $f(x), f(x+\lambda_1 h), f(x+\lambda_2 h), \ldots, f(x+\lambda_k h)$. (You may assume f to have as many derivatives as required.)

6 Numerical Integration

52. The integral $\int_0^1 \frac{e^x}{x+1}$ is to be evaluated to within an error of 0.001 by the rectangle method. Into how many subintervals n do you need to divide $[0,1]$ to obtain the required precision? (Try to minimize n. Ignore computer errors.)

53. The integral $\int_0^1 (1-\sqrt{2x-x^2})dx$ has to be approximated by the rectangle method, with division into n subintervals. Find an n for which the error is at most 0.001.

54.

(a) Suppose we approximate the integral $\int_a^b f(x)dx$ by dividing the interval $[a,b]$ into n subintervals $[x_{i-1},x_i], 1 \le i \le n$, and using the rectangle method on each of them. For each subinterval $[x_{i-1},x_i]$ we have an approximation I_i and some error E_i. Now for each E_i we have an expression, which depends on some inner point within the interval $[x_{i-1},x_i]$. Estimate the total error $E=\sum_{i=1}^n E_i$ by a certain Riemann integral $\int_a^b g(x)dx$, which can easily be evaluated (similarly to one of the ideas we presented in class for the integral $\int_0^1 e^x dx$, using the trapezoid rule). Add this estimate $\int_a^b g(x)dx$ on the error to the initial approximation $\sum_{i=1}^n I_i$ for $\int_a^b f(x)dx$ to obtain a (supposedly better) approximation. What approximation do we get?

(b) Now we use a similar approach, but with the midpoint rule instead. Write down the approximation you get for the integral, and prove that it is indeed asymptotically better than the approximation obtained from the midpoint rule. Namely, the error is $O(h^3)$ (where $h=(b-a)/n$) instead of $O(h^2)$.

55. Consider the function log on the interval $\left[\frac{1}{2}, n+\frac{1}{2}\right]$.

(a) Find the exact value of the integral $\int_{1/2}^{n+1/2} \log x dx$.

(b) Estimate the integral employing the midpoint rule, with division to n subintervals. Using the estimate for the error, obtain a weak form of Stirling's formula (providing an asymptotic expression for $n!$).

(c) What can you get in the previous part if you use the bound on the error for each subinterval separately, and not just for the whole interval?

(d) Repeat the above for the integral $\int_{1/2n}^{1+1/2n} \log x dx$.

56. It is well known that, denoting $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, we have

$$\lim_{n\to\infty} (H_n - \log n) = \gamma,$$

where γ is Euler's constant. How much of this do you obtain by approximating the integral $\int_{1/2}^{n+1/2} \frac{dx}{x}$ using the midpoint rule, with division to n intervals, and estimating the error?

57. Verify directly that, using the trapezoidal rule with division to n subintervals to estimate the integral $\int_a^b x^2 dx$, the resulting error is within the predicted bounds.

58. Verify directly that, using Simpson's rule with division to n subintervals to estimate the integral $\int_0^1 e^x dx$, the resulting error is within the predicted bounds.

59.

(a) Prove that there exists no approximation formula for $\int_a^b f(x)dx$, based on the values of f at finitely many points only, which is completely precise for all polynomials.

(b) Prove that, moreover, for any such approximation formula, based on the values of f at n points, there exists a polynomial of degree at most $2n$ for which the formula gives an imprecise answer.

(c) Explain why the bound $2n$ in the preceding part is best possible.

60. Show that Boole's rule gives the exact value for integrals of polynomials of degree not exceeding 5.

61. Let $(P_n(x))_{n=0}^{\infty}$ be a sequence of polynomials. Show that, if $\deg P_n = n$ for each n, then $\{P_n(x) : n = 0, 1, 2, \ldots\}$ is a basis of the space of all polynomials.

62. Verify the following properties of Legendre polynomials:

(a) $\phi_n'(x) = x\phi_{n-1}'(x) + n\phi_{n-1}(x)$.

(b) $\phi_n(x) = x\phi_{n-1}(x) + \frac{x^2-1}{n}\phi_{n-1}'(x)$.

(c) $(n+1)\phi_{n+1}(x) = (2n+1)x\phi_n(x) - n\phi_{n-1}(x)$.

(d) $\int_{-1}^{1} \phi_n(x)^2 dx = \frac{2n-1}{2n+1} \int_{-1}^{1} \phi_{n-1}(x)^2 dx$.

(e) $\int_{-1}^{1} \phi_n(x)^2 dx = \frac{2}{2n+1}$.

(f) The set of zeros of ϕ_n is symmetric around 0.

63. Carry out the Gram-Schmidt process for the system $\{1, x, x^2, \ldots\}$ on the interval $[-1, 1]$ to find the polynomials $\phi_n(x),\ 0 \leq n \leq 4$.

64.

(a) Use Gaussian integration with two and with three division points to estimate $\int_{-1}^{1} \frac{dx}{1+x^2}$. Compare with the correct value and verify, for two division points, that the error is within the allowed limits.

(b) Same for the integral $\int_{-1}^{1} \frac{dx}{x+2}$.

65. Use Gaussian integration with two and with three division points to estimate $\int_{-1}^{1} x^n dx$. What are the absolute and the relative errors? Explain!

66. Use Gaussian integration with two division points to estimate the integral $\int_{-1}^{1} \cos \pi n x \; dx$. Use MATLAB to check how good the estimate is for various values of n. Do you think that the points $\pm \frac{1}{\sqrt{3}}$ used for the estimate are more representative for these integrals (for large n) than random points in $[-1, 1]$? Explain!

67. Let F denote the vector space of all Riemann-integrable functions over a finite interval $[a, b]$, and let $w \in F$ be strictly positive throughout the interval. Show that the function $\langle \cdot, \cdot \rangle : F \times F \longrightarrow \mathbf{R}$, defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx, \qquad f, g \in F,$$

forms an inner product on F.

68. Verify that each of the following defines an inner product on the space of polynomials:

(a) $\langle P, Q \rangle = \int_0^\infty P(x)Q(x)e^{-x}dx$.

(b) $\langle P, Q \rangle = \int_{-\infty}^{\infty} P(x)Q(x)e^{-x^2}dx$.

(c) $\langle P, Q \rangle = \int_{-1}^{1} \frac{P(x)Q(x)}{\sqrt{1-x^2}}dx$.

69.

(a) Using the equality $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, show that one can calculate $\int_{-\infty}^{\infty} e^{-x^2}P(x)dx$ precisely for any polynomial P.

(b) Calculate $\int_0^\infty \sqrt{x}e^{-x}dx$.

70. Suppose we are interested in integrals of the form $\int_0^1 f(x) \ln \frac{1}{x} dx$.

(a) Write down the equations obtained by trying to find an approximation of the form

$$I = w_1 f(x_1) + w_2 f(x_2) + \dots w_k f(x_k)$$

for this integral, which will be precise if f is a polynomial of degree not exceeding $2k-1$.

(b) Define an appropriate inner product on the space of continuous real-valued functions on $[0,1]$, which will be of use for finding the division points $x_1, x_2, \dots, x_k$.

(c) Find the division points for $k = 2$.

7 Interpolation

71. Write down the Taylor polynomial of order n (around 0) of the function $\frac{1}{x^3+1}$. At what interval can it be expected to form a good approximation to the function?

72.

(a) Prove that there does not exist a polynomial P approximating the function e^x to within a fixed error on the whole line (namely, such that $|e^x - P(x)| < C$ for every $x \in \mathbf{R}$, where C is a constant).

(b) Prove that, if $\varepsilon > 0$ is sufficiently small, then there does not exist a polynomial P approximating the function $\sin x$ to within an error of ε on the whole line.

73. Let f be a continuous function on the interval $[-a, a]$. Use Weierstrass's Theorem to prove the following:

(a) If f is an even function (i.e., $f(-x) = f(x)$ for every x), then for every $\varepsilon > 0$ there exists a polynomial P of the form $P(x) = \sum_{i=0}^{n} c_i x^{2i}$, satisfying $|f(x) - P(x)| < \varepsilon$ for every $x \in [-a, a]$.

(b) If f is odd, then there exist approximating polynomials of the form $P(x) = \sum_{i=0}^{n} c_i x^{2i+1}$.

74. Use Weierstrass's Theorem to prove the following:

(a) If f is a continuous function on $[1,\infty)$ and $f(x) \underset{x\to\infty}{\longrightarrow} a$, then for every $\varepsilon > 0$ there exists a polynomial P such that $|f(x) - P(1/x)| < \varepsilon$ for every $x \in [1,\infty)$.

(b) If f is a continuous function on $[0,\infty)$ and $f(x) \underset{x\to\infty}{\longrightarrow} a$, then for every $\varepsilon > 0$ there exists a polynomial P such that $|f(x) - P(e^{-x})| < \varepsilon$ for every $x \in [0,\infty)$.

75. Let f be a continuous function defined on $[a,b]$ and $(x_k)_{k=1}^{\infty}$ a sequence of distinct points in the interval. For each n, let P_n denote the interpolation polynomial of degree at most $n-1$ coinciding with f at the points $x_1, x_2, \ldots, x_n$.

(a) Prove that, if f is a polynomial, then $P_n = f$ for sufficiently large n.

(b) Show by an example that, in general, the sequence $(P_n)_{n=1}^{\infty}$ does not even need to converge to f as $n \longrightarrow \infty$.

76. Let f be a function defined on some interval $[a,b]$ and $x_0, x_1, \ldots, x_n$ be distinct points of the interval, such that f is differentiable (at least) k_i times at x_i for $0 \le i \le n$ (where $k_i \ge 0$ for each i). Prove that there exists a polynomial P having the following property:

$$P^{(k)}(x_i) = f^{(k)}(x_i), \qquad 0 \le i \le n,\ 0 \le k \le k_i\,.$$

What upper bound can you ensure for the degree of P?

77. Suppose f is defined on an interval $[a,b]$ symmetric around 0 (i.e., $a = -b$), and the points $x_0, x_1, \ldots, x_n$ are symmetric around 0 (i.e., $x_{n-i} = -x_i$ for each i). Show that, if f is an even (or odd) function, then so is the interpolation polynomial.

78. For each of the following functions determine for which values of n the Taylor polynomial of order n is the interpolation polynomial of the function for appropriately selected points:

(a) $f(x) = e^x, \qquad x \in [0,2]$.

(b) $f(x) = \frac{1}{1+x}, \qquad x \in [-\frac{1}{2}, \frac{1}{2}]$.

(c) $f(x) = \ln(1+x), \qquad x \in [-\frac{1}{2}, \frac{1}{2}]$.

79.

(a) Find the polynomial $P_3(x)$ interpolating the function $(\frac{x}{\pi} + 1)\cos x$ at the points $0, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$.

(b) Compare the actual error obtained by predicting the value of the function at the point $\frac{\pi}{6}$ using the interpolation polynomial with the bound on the error guaranteed by the theory.

80. For each of the following functions, check whether every sequence $(P_n)_{n=0}^{\infty}$ of interpolation polynomials, interpolating the function at $x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}$, converges uniformly to the function:

(a) $f(x) = xe^x, \qquad x \in [a,b]$.

(b) $f(x) = x^2 e^x, \qquad x \in [a,b]$.

(c) $f(x) = \frac{1}{1+x}, \qquad x \in [0, \frac{1}{2}]$.

(d) $f(x) = \frac{1}{1-x}, \qquad x \in [0,1)$.

81. For an arbitrary fixed sequence $(x_n)_{n=0}^{\infty}$, let P_n interpolate the function $\sin x$ at the points $x_0, x_1, \ldots, x_n$.

(a) Show that the sequence $(x_n)_{n=0}^{\infty}$ can be constructed so that the sequence P_n converges to the function pointwise on the whole real line (that is, $P_n(x) \underset{n\to\infty}{\longrightarrow} \sin x$ for every $x \in \mathbf{R}$).

(b) Show that $(x_n)_{n=0}^{\infty}$ can be selected so that P_n does not converge to the function pointwise on the whole line (i.e., there exists at least one point at which the convergence does not hold).

82. Same as the preceding question for the function e^x.

83. Same as the preceding question for the function e^{x^2}.

84.

(a) Use MATLAB to plot the first several interpolation polynomials, based on interpolating Runge's function

$$f(x) = \frac{1}{1+x^2}, \qquad -5 \le x \le 5,$$

at equally spaced points, and verify that they form poor approximations to the function.

(b) Do the same, interpolating at Chebyshev's points

$$x_j^{(n)} = 5\cos\frac{j\pi}{n}, \qquad j = 0, 1, \ldots, n\ .$$

Verify that this time the polynomials form good approximations to the function.

85.

(a) Prove the theorem stated in class regarding the error in the interpolation polynomial for equally spaced points in the cases $n = 2$ and $n = 3$.

(b) State and prove the analogue for $n = 4$.

86. Let P_n interpolate f at the points $x_0, x_1, \ldots, x_n$. Find a simple expression for the polynomial interpolating the function $xf(x)$ at the same points.

87. Let P_n interpolate f at the points $x_0, x_1, \ldots, x_n$. Find a simple expression for the polynomial interpolating f at the points $x_1, x_2, \ldots, x_n$.

88. Recalculate the polynomial P_3 of Problem 79 using Newton's divided differences.

89. Write a MATLAB function implementing Horner's rule for evaluating the value of a polynomial at a given point.

90.

(a) Write a MATLAB function which, given a function f and interpolation points $x_0, x_1, \ldots, x_n$, finds the Newton form of the interpolation polynomial. Minimize the time and space requirements.

(b) Write a MATLAB function which, given a polynomial P in Newton form, finds its usual representation. Minimize time and space.

(c) Write a MATLAB function which, given a polynomial P in Newton form and a point t, calculates the value of P at t. Minimize time and space.

8 Curve Fitting

91. Prove that the error measures presented in class for testing the compatibility of a function to given data always satisfy $E_1(f) \leq E_2(f) \leq E_\infty(f)$, with equalities if and only if $f(x_1) - y_1 = f(x_2) - y_2 = \ldots = f(x_n) - y_n$.

92. Find the least-squares line for the following data points:

(a) $(-1,2),(0,3),(1,5),(2,4)$.

(b) $\{(k,k^2) : k = 0,1,\ldots,n\}$.

93. Prove that:

(a) If the data points are symmetric with respect to the origin, then the least-squares line passes through the origin.

(b) If the data points are symmetric with respect to the y-axis, then the least-squares line is horizontal.

94. Prove that $\bar{x^2} \geq \bar{x}^2$, with equality if and only if $x_1 = x_2 = \ldots = x_n$.

95. Let $y = ax + b$ be the least-squares line corresponding to the points $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. Given constants c_1,c_2,d_1,d_2, find the least-squares line corresponding to the points $(c_1x_k+d_1, c_2y_k+d_2)$, $k = 1,2,\ldots,n$. Explain your result intuitively.

96. Given data points $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$, find the equations defining the least-squares fit for each of the following families of curves:

(a) Lines through the origin $y = ax$.

(b) Parabolas of the form $y = ax^2$.

(c) Parabolas of the form $y = ax^2 + b$.

(d) The family of curves $y = a\sin x + b\cos x$.

97. Consider the example given in class of the least-squares line corresponding to the points $(0,0),(1,1),(2,4),(3,9)$. Suppose the y-coordinates of these points are perturbed in such a way that the sum total of all changes does not exceed 1. That is, if the new y-coordinates are y_0 through y_3, then $|y_0 - 0| + |y_1 - 1| + |y_2 - 4| + |y_3 - 9| \leq 1$.

(a) Find the range of possible slopes a of the corresponding least-squares lines $y = ax + b$.

(b) Same for the range of possible values of b.

(c) Find the set of all possible values of the pair (a,b).

98. For each of the following families of curves use data linearization to find a general formula for the fit corresponding to given data points:

(a) $y = \frac{a}{x+b}$.

(b) $y = ax^b$.

(c) $y = \frac{1}{(ax+b)^2}$.

(d) $y = \frac{1}{ax^2+b}$.

(e) $y = \frac{1}{1+be^{ax}}$.

99. For each of the following families of curves, write down the conditions on the parameters necessary for the curve to be the least-squares fit. Obtain explicit expressions wherever possible, or at least try to simplify the equations you got:

(a) $y = \frac{a}{x+b}$.

(b) $y = x^a + b$.

(c) $y = a \sin x + b$.

(d) $y = a + be^x + ce^{-x}$.

(e) $y = (x+a)^2$.

(f) $y = a \sin^2 x + b \cos^2 x$.

100. The *least-squares plane* $z = ax + by + c$ corresponding to the n data points $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$ is the plane obtained by minimizing

$$E_2(a,b,c) = \left(\frac{1}{n} \sum_{k=1}^{n} (ax_k + by_k + c - z_k)^2 \right)^{1/2} .$$

(a) Find the equations determining the parameters of the least-squares plane in the general case.

(b) Find the least-squares plane corresponding to the data points $(1,4,5)$, $(2,7,9)$,$(3,9,12)$, and $(4,12,10)$.

(c) Find the least-squares plane for the data points $\{(k, k^2, k^3) : k = 0, 1, \ldots, n\}$.

9 Differential Equations

101. Consider the initial value problem:

$$y' = 1 - y, \qquad y(0) = 2 \ .$$

(a) Show that $y(t) = 1 + e^{-t}$ forms a solution of the equation on the whole real line.

(b) Show that the approximate solution obtained by Euler's method converges on the whole real line to the above solution as the step size converges to 0.

102. Find an explicit expression for $y^{(4)}(t)$, analogous to those found in class for the first three derivatives of $y(t)$.

103. Consider the initial value problem:

$$y' = \sqrt{1 - y^2}, \qquad y(0) = 0 \ .$$

Note that $y(t) = \sin t$ forms a solution of the equation on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Estimate the error when applying Runge-Kutta's method (of order 4) for one step, starting at 0.