Review Questions

Mark all correct answers in each of the following questions.

1. The first two parts below deal with fixed points of a given function g. We start at some point x_0 and continue according to the iteration $x_{n+1} = g(x_n)$ for $n \geq 0$. In the last four parts, we have various functions f, and want to find zeros of these functions. We start again from some initial point x_0, and continue according to Newton's method.

   (a) Let $g(x) = \operatorname{tg} x$. The function g has a fixed point at every interval of the form $((k-1/2)\pi, (k+1/2)\pi)$ with integer k. However, no fixed point of g is attracting.

   (b) Let $g(x) = x \operatorname{tg} x$. If x_0 is sufficiently close to the fixed point 0, then $x_n \underset{n\to\infty}{\longrightarrow} 0$. Moreover, for sufficiently large n, the number of correct digits in the expansion of x_{n+1} is approximately twice the number of correct digits in the expansion of x_n.

   (c) Let $f(x) = x \operatorname{tg} x$. If x_0 is sufficiently close to 0, then $x_n \underset{n\to\infty}{\longrightarrow} 0$. However, the rate of convergence is worse than is usually the case for Newton's method; it is roughly the same as the rate of convergence in the bisection method.

   (d) Let $f(x) = \operatorname{tg}^2 x$. Then for every $x_0 \in (0, \pi/2)$ we have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

   (e) Let $f(x) = \sqrt[3]{x}$. Then for every $x_0 \neq 0$ we have $x_n \underset{n\to\infty}{\longrightarrow} 0$.

   (f) Let $f(x) = \sqrt[3]{x^2}$. Then f is not differentiable at 0, and therefore there exists no $x_0 \neq 0$ for which $x_n \underset{n\to\infty}{\longrightarrow} 0$.

2. (a) Let I_{n1} be the approximation obtained for $\int_0^\pi \cos x dx$, when we divide the interval $[0, \pi]$ into n equal parts and use the midpoint rule for each. Let I_{n2} be the analogous quantity if we use Simpson's rule instead. Then, for every sufficiently large n, the approximation I_{n2} is closer to the exact value than is I_{n1}.

   (b) Let I_n be the approximation obtained for $\int_{-\pi/2}^{\pi/2} \sin x dx$, when we divide the interval $[-\pi/2, \pi/2]$ into n equal parts and use the rectangle rule for each. Then:

$$I_n \leq \int_{-\pi/2}^{\pi/2} \sin x dx \leq I_n + \frac{5}{n} .$$

   (c) Let I_n be the approximation obtained for $\int_0^1 \frac{dx}{1+x}$, when we divide the interval $[0, 1]$ into n equal parts and use the trapezoid rule for each. Then the error $E = \int_0^1 \frac{dx}{1+x} - I_n$ satisfies:

$$E = \frac{1}{12n^2} + o\left(\frac{1}{n^2}\right) .$$

   (d) Suppose we employ Simpson's rule, with division into n subintervals of equal lengths, to estimate integrals. For every sufficiently large integer d, there exist polynomials P, Q, R, all of degree exactly d, such that the error when estimating $\int_2^3 P(x)dx$ is positive, when estimating $\int_2^3 Q(x)dx$ – negative, and when estimating $\int_2^3 R(x)dx$ – 0. (Hint: You may use the fact that the error in Simpson's rule when estimating $\int_a^b f(x)dx$ is $-\frac{f^{(4)}(\eta)}{90}(b-a)^5$ for some $\eta \in [a, b]$.)

   (e) When estimating $\int_a^b f(x)dx$ using Simpson's rule, the estimate is in fact $\int_a^b P_2(x)dx$ for some polynomial P_2 of degree at most 2, determined by the fact that it coincides with f at the points $a, \frac{a+b}{2}, b$. If f is continuously differentiable, then $\int_a^b P_2'(x)dx$ is the estimate for $\int_a^b f'(x)dx$, obtained by employing the trapezoid rule.

3. (a) Consider Taylor's polynomial of order n around 0 of the function $f(x) = e^x$:

$$P(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} .$$

There exist points $x_0, x_1, \ldots, x_n \in [0,1]$ such that $P(x)$ is the interpolation polynomial of degree at most n, coinciding with f at the points $x_0, x_1, \ldots, x_n$.

(b) Since $f(x) = e^x$ grows asymptotically faster than every polynomial, there does not exist a sequence of polynomials $(P_n)_{n=0}^{\infty}$ such that $P_n(x) \underset{n\to\infty}{\longrightarrow} e^x$ uniformly over the whole real line. However, since $g(x) = \sin x$ and all its derivatives are uniformly bounded over the whole real line, we can find a sequence $(x_k)_{k=0}^{\infty}$ with the following property: If P_n denotes for each n the interpolation polynomial of degree at most n, coinciding with g at the points $x_0, x_1, \ldots, x_n$, then $P_n(x) \underset{n\to\infty}{\longrightarrow} \sin x$ uniformly over the whole real line.

(c) For each n we can find interpolation polynomials P_n and Q_n, of degrees at most n, coinciding with the functions $g(x) = \sin x$ and $h(x) = \cos x$, respectively, at some points $x_0, x_1, \ldots, x_n \in (0, \pi/2)$, such that $P_n(x) \underset{n\to\infty}{\longrightarrow} \sin x$ and $Q_n(x) \underset{n\to\infty}{\longrightarrow} \cos x$ uniformly over the interval $(0, \pi/2)$. Consequently, we can find a sequence of interpolation polynomials for the function $l(x) = \operatorname{tg} x$, which converges to $\operatorname{tg} x$ uniformly over the interval $(0, \pi/2)$.

(d) Let P be the interpolation polynomial of degree at most 2, coinciding with the function $m(x) = \log_2 x$ at the points $1, 2, 4$. Then $P(3) > m(3)$.

(e) The estimate $\frac{f(x_0-h)-2f(x_0)+f(x_0+h)}{h^2}$ for $f''(x_0)$ is actually the value of $P''(x_0)$, where P is the interpolation polynomial of degree at most 2, coinciding with f at the points $x_0 - h, x_0, x_0 + h$.

4. (a) We want to fit a curve of the form $y = ax^2 + bx$ to given data points $(x_1, y_1), \ldots, (x_n, y_n)$. Then the parameters a, b, for which the least-squares curve of that form is obtained, are determined by a system of two linear equations.

(b) Suppose we linearize the data by replacing y with the variable $Y = y/x$. The values of a, b, for which we obtain the least-squares line for the corresponding data points are the same as the values calculated in the preceding part.

(c) We defined the least-squares line as the line for which the sum of squares of vertical distances between the given data points and the line is minimal. The line, for which the sum of squares of horizontal distances between the given data points and the line is minimal, coincides with the least-squares line.

(d) Suppose we have calculated the least-squares line for given data points $(x_1, y_1), \ldots, (x_n, y_n)$. Now a new data point (x_{n+1}, y_{n+1}) is adjoined to the set. Then the least-squares line for the augmented data set coincides with the old line if and only if the newly added point (x_{n+1}, y_{n+1}) lies on the old line.

Solutions

1. (a) Since g is continuous in every interval of the form $((k-1/2)\pi, (k+1/2)\pi)$, and it converges to $-\infty$ and ∞ as the argument converges to $(k-1/2)\pi$ and $(k+1/2)\pi)$, respectively, it must have a fixed point in each such interval. Since $g'(x) = 1/\cos^2 x \geq 1$ for each x, no fixed point may be attracting.

   (b) Since $g'(x) = x/\cos^2 x + \operatorname{tg} x$, we have $g'(0) = 0$. Hence the fixed point 0 is attracting, and the convergence is quadratic, which means that the number of correct digits in the expansion of x_{n+1} is indeed approximately twice the number of correct digits in the expansion of x_n. (In fact, in this particular case, one can readily verify by induction that, if $|x_0| < \pi/4$, then the sequence converges monotonically to 0. Moreover, since $\operatorname{tg} x$ behaves asymptotically as x near 0, we have $g(x) \approx x^2$, so that the convergence is clearly quadratic.)

   (c) Usually, Newton's method yields quadratic convergence. However, in our case $f'(0) = 0$ (as shown for the function g of part (b)), which is a problem in Newton's method. Indeed, we iterate for the function

$$g(x) = x - \frac{x \operatorname{tg} x}{x/\cos^2 x + \operatorname{tg} x}.$$

   Hence:

$$g(x) = x - \frac{\cos x \cdot \sin x/x}{1 + \cos x \cdot \sin x/x} \cdot x\,.$$

Since $\lim_{x\to 0} \sin x/x = 1$, the function $g(x)$ behaves asymptotically as $x/2$ near 0. Hence, for x_0 sufficiently close to 0, the sequence (x_n) behaves roughly as $C/2^n$, so that the convergence is at a rate similar to that obtained when using the bisection method.

(d) The function f is increasing and convex in the interval $(0, \pi/2)$, Therefore, the tangent to the graph at any point is below the graph throughout this interval, which shows that the sequence (x_n) is positive and strictly decreasing. Its limit is therefore non-negative, belongs to the interval $[0, \pi/2)$, and is a zero of f. Hence this limit must be 0.

(e) We iterate the function

$$g(x) = x - \frac{x^{1/3}}{\frac{1}{3} \cdot x^{-2/3}} = -2x.$$

Thus, $x_n = (-2)^n x_0$, which does not converge for any $x_0 \neq 0$.

(f) The function f is indeed not differentiable at 0. However, we iterate the function

$$g(x) = x - \frac{x^{2/3}}{\frac{2}{3} \cdot x^{-1/3}} = -\frac{x}{2},$$

which yields $x_n = (-1/2)^n x_0 \underset{n\to\infty}{\longrightarrow} 0$.

Thus, (a), (b), (c) and (d) are true.

2. (a) Since $\cos x$ is anti-symmetric with respect to the line $x = \pi/2$, it is easy to see that the contributions to I_{n1} of every two sub-intervals of the form $[(k-1)\pi/n, k\pi/n]$ and $[(n-k)\pi/n, (n-k+1)\pi/n]$ cancel each other, and for odd n the contribution of the middle interval $[(n-1)\pi/2n, (n+1)\pi/2n]$ is 0. Hence I_{n1} is 0, which is the value of the integral under investigation. Namely, I_{n1} is completely accurate for each n. (It is easily verified that the same holds for I_{n2}.)

(b) Since the integrand increases throughout the interval, the error is positive on any sub-interval, and hence the total error is positive as well, namely $I_n \leq \int_{-\pi/2}^{\pi/2} \sin x dx$. According to the theoretical bound on the error

$$E \leq \frac{1}{2} \max_{x\in[-\pi/2,\pi/2]} \cos x \cdot (\pi/2 - (-\pi/2)) \cdot \frac{\pi/2-(-\pi/2)}{n} = \frac{\pi^2}{2n} < \frac{5}{n}\,.$$

(c) The integrand is convex in the interval (as can be verified by differentiating it twice and noticing that the obtained expression is positive throughout the interval). Hence the error when using the trapezoid rule is negative.

(d) Let $d \geq 6$. According to the estimate for the error in Simpson's rule, this error is negative for both $f(x) = x^d$ and $f(x) = x^{d-1}$. Consider the polynomials $f(x) = x^d + ax^{d-1}$ for real a. According to the above, the error is clearly negative if a is non-negative (or negative of sufficiently small absolute value), and is positive for sufficiently negative a. Since the error is clearly linear as a function of a, it must vanish for some intermediate value of a.

(e) The estimate for $\int_a^b f'(x)dx$ is $(b-a)\cdot\frac{f'(a)+f'(b)}{2}$, whereas $\int_a^b P_2'(x)dx = P_2(b) - P_2(a) = f(b) - f(a)$ (which is actually accurate).

Thus, (b) and (d) are true.

3. (a) Since

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots, \qquad x \in \mathbf{R},$$

we have $e^x > P(x)$ for all $x > 0$. In other words, the only point in $[0,1]$ where $e^x = P(x)$ is 0, and therefore $P(x)$ cannot be an interpolation polynomial of f for $n \geq 1$.

(b) The claim regarding f is indeed correct for the reason stated in the text. However, the claim regarding g is false for a similar reason. In fact, if P_n is not a constant polynomial, then $P_n(x)$ goes to $\pm\infty$ as $x \to \pm\infty$, and hence P_n cannot be uniformly close to g on the whole real line. On the other hand, since g assumes all values between -1 and 1, if P_n is constant, then $P_n(x)$ is at a distance of at least 1 from $g(x)$ for some values of x.

(c) The claims regarding the existence of P_n and Q_n indeed follow immediately from the observation that, if all derivatives of some function are uniformly bounded throughout the interval by some constant (same constant for all derivatives), then any sequence of interpolation polynomials for that function, based on more and more interpolation points, converges uniformly to the function. However, as l is not bounded in the interval, no sequence of polynomials can possibly converge to l uniformly.

(d) According to the theorem regarding the error of the interpolation polynomial, taking $x_0 = 1, x_1 = 2, x_3 = 4$ and $x = 3$, we obtain

$$E_2(3) = \frac{(3-1)(3-2)(3-4)}{(2+1)!} \cdot \frac{2}{\eta^3 \ln 2}, \qquad (1 \le \eta \le 4).$$

As the expression on the right-hand side is negative, we get $P(3) > m(3)$.

(e) Writing down Lagrange's interpolation polynomial, and differentiating it twice, we indeed obtain the expression $\frac{f(x_0-h)-2f(x_0)+f(x_0+h)}{h^2}$.

Thus, (d) and (e) are true.

4. (a) The parameters a, b of the required curve are determined by the condition that the function

$$D(a,b) = \sum_{i=1}^{n} \left(ax_i^2 + bx_i - y_i\right)^2$$

assumes for them its minimum. Now:

$$\frac{1}{n}D(a,b) = \overline{x^4}a^2 + \overline{x^2}b^2 + \overline{y^2} + 2\overline{x^3}ab - 2\overline{x^2y}a - 2\overline{xy}b.$$

Since D is quadratic both as a function of a and as a function of b, with positive leading coefficients for both, the optimal a, b are determined by:

$$a = \frac{-\overline{x^3}b + \overline{x^2y}}{\overline{x^4}},$$

$$b = \frac{-\overline{x^3}a + \overline{xy}}{\overline{x^2}}.$$

Simplifying the equations, we obtain:

$$\begin{aligned} \overline{x^4}a &+ \overline{x^3}b &= \overline{x^2y}, \\ \overline{x^3}a &+ \overline{x^2}b &= \overline{xy}. \end{aligned}$$

(b) For the new variables, the data points are $(x_1, y_1/x_1), \ldots, (x_n, y_n/x_n)$. The parameters a, b are given by the formula for the least-squares line for these data points:

$$a = \frac{\overline{y} - \overline{x} \cdot \overline{y/x}}{\overline{x^2} - \overline{x}^2},$$

$$b = \frac{\overline{x^2} \cdot \overline{y/x} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2}.$$

Obviously, these values of a and b do not satisfy the linear equations satisfied by the a and b of the preceding part.

(c) Consider the line $y = ax + b$, for which the sum of squares of horizontal distances between the given data points and the line is minimal. Rewriting the equation of the line in the form $x = \frac{1}{a}y - \frac{b}{a}$, we see that the requirement is for parameter values a, b for which the function

$$D(a, b) = \sum_{k=1}^{n} \left(\frac{1}{a} y_k - \frac{b}{a} - x_k \right)^2$$

attains its minimum. Employing the formula for the least-squares line, we obtain the values

$$\frac{1}{a} = \frac{\overline{yx} - \overline{y} \cdot \overline{x}}{\overline{y^2} - \overline{y}^2}$$

$$-\frac{b}{a} = \frac{\overline{y^2} \cdot \overline{x} - \overline{y} \cdot \overline{yx}}{\overline{y^2} - \overline{y}^2}.$$

We see that the expression obtained for a, for example, is not the inverse of the expression for a in the least-squares line. (Indeed, if one takes random data with at least 3 points, the values of a obtained in the two ways will be distinct.)

(d) The parameter values a, b of the least-squares line, corresponding to the initial data, are those minimizing the function

$$D_1(a, b) = \sum_{k=1}^{n} (ax_k + b - y_k)^2,$$

while those of the updated line are those minimizing the function

$$D_2(a,b) = \sum_{k=1}^{n+1} (ax_k + b - y_k)^2 .$$

For any a, b we have $D_2(a,b) \geq D_1(a,b)$. If (x_{n+1}, y_{n+1}) lies on the old line, then the extra term in the sum for $D_2(a,b)$ vanishes for the values of a, b minimizing $D_1(a,b)$, so that those values clearly minimize $D_2(a,b)$ as well. Hence the two lines coincide in this case. Conversely, suppose the two lines coincide. Substituting in the formula for b, we get:

$$\frac{1}{n+1}\sum_{k=1}^{n+1} y_k - a \cdot \frac{1}{n+1}\sum_{k=1}^{n+1} x_k = \frac{1}{n}\sum_{k=1}^{n} y_k - a \cdot \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Routine calculations yield:

$$y_{n+1} - ax_{n+1} = \frac{1}{n}\sum_{k=1}^{n} y_k - a \cdot \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Thus, the additional point lies on the initial least-squares line.

Thus, (a) and (d) are true.