Probability Theory

Solutions to Selected Exercises

1 Review Questions in Combinatorics

3.

(a) k^n.

(b) Let A_i be the event that the ith processor is assigned at least one job, $i = 1, 2, \ldots, k$, and A the event in the question. Clearly, $A = \bigcap_{i=1}^{k} A_i$. By the principle of inclusion and exclusion, and using the symmetry of the events A_i, we have

$$\begin{aligned} P(\overline{A}) &= P\left(\bigcup_{i=1}^{k} \overline{A_i}\right) \\ &= kP(\overline{A_1}) - \binom{k}{2} P(\overline{A_1 A_2}) + \ldots + (-1)^{k-1} \binom{k}{k} P\left(\bigcap_{i=1}^{k} \overline{A_i}\right). \end{aligned}$$

Thus

$$\begin{aligned} P(A) &= 1 - P(\overline{A}) \\ &= 1 - k\left(1 - \frac{1}{k}\right)^n + \binom{k}{2}\left(1 - \frac{2}{k}\right)^n + \ldots + (-1)^{k-2}\binom{k}{k-1} \cdot \frac{1}{k^n}. \end{aligned}$$

4.

(a) r^n.

(b) The first letter of the word may be any of the letters in Σ. In each of the other $n-1$ places we may put any of the $r-1$ letters distinct from the one in the preceding place. Hence there are $r(r-1)^{n-1}$ possibilities in all.

(c) Let a_n be the number of words not containing two consecutive occurrences of σ. The set of words in question consists of two disjoint subsets – those starting with σ, and those starting with some other letter. In the first set, each word may have any letter distinct from σ in the second place, and the remaining $n-2$ letters must form a word of length $n-2$ satisfying our condition. In the second set, each word may have any letter distinct from σ as the first letter, and the remaining $n-1$ letters must form a word of length $n-1$ satisfying our condition. Thus we have:

$$a_n = (r-1)a_{n-2} + (r-1)a_{n-1}, \qquad a_0 = 1, a_1 = r.$$

Solving the above recurrence relation, we obtain:

$$x^2 + (r-1)x - (r-1) = 0.$$

Hence

$$a_n = c_1 b_1^n + c_2 b_2^n,$$

where $b_1 = \frac{r-1+\sqrt{(r-1)(r+3)}}{2}$ and $b_2 = \frac{r-1-\sqrt{(r-1)(r+3)}}{2}$. Substituting in the initial conditions $a_0 = 1, a_1 = r$, we obtain:

$$c_1 = \frac{b_2 - r}{b_2 - b_1}, \qquad c_2 = \frac{r - b_1}{b_2 - b_1}.$$

(d) $\binom{n}{n_1, n_2, \ldots, n_r}$.

(e) $r^{\left[\frac{n+1}{2}\right]}$.

5.

(b) The generating function is $f(x) = \frac{1}{1-2x}$, and we obtain $a_n = 2^n,\ n \geq 0$.

8. Since $n! = e^{\sum_{i=1}^{n} \ln i}$ the inequality

$$e\left(\frac{n}{e}\right)^n \leq n! \leq e\left(\frac{n+1}{e}\right)^{n+1}$$

is equivalent to

$$n \ln n - n + 1 \leq \sum_{i=1}^{n} \ln i \leq (n+1)\ln(n+1) - n.$$

Since $\int \ln x dx = x \ln x - x + c$ and the function $\ln x$ is increasing, we have

$$\int_1^n \ln x dx \leq \sum_{i=1}^{n} \ln i \leq \int_1^{n+1} \ln x dx,$$

which gives the required result.

9.

(b) Consider the $2n$th row of Pascal's triangle. The sum of all entries is 2^{2n}, and therefore each of them, in particular the middle entry $\binom{2n}{n}$, is less than 2^{2n}. On the other hand, it is easy to check that the binomial coefficients $\binom{2n}{j}$ increase as j increases from 0 to n, and decrease from that place on. In particular, $\binom{2n}{n}$ is the maximal entry in the row. Consequently:

$$\frac{2^{2n}}{2n+1} \leq \binom{2n}{n} \leq 2^{2n}.$$

11.

(a) The set of all permutations which may be obtained by the system in the question decomposes into a union of disjoint subsets as follows. Consider the number of the step, between 2 and $2n$ at which the number 1 was moved from S to Q_2. The step number may be any even number in this range. Consider the set of permutations in which this move took place at the $2j$th step. Thus, we moved the number 1 from Q_1 to S at the first step, moved the next $j-1$ integers $2, 3, \ldots, j$ from Q_1 to Q_2 (via S) in the next $2j-2$ steps, moved 1 from S to Q_2 at the $2j$th step, and then moved the remaining integers $j+1, j+2, \ldots, n$ from Q_1 to Q_2 in the next $2(n-j)$ steps. We had P_{j-1} possibilities for moving $2, 3, \ldots, j$ and P_{n-j} possibilities for moving $j+1, j+2, \ldots, n$. Consequenly, the sequence (P_n) satisfies the recurrence:

$$P_{n+1} = P_0P_n + P_1P_{n-1} + P_2P_{n-2} + \ldots + P_{n-1}P_1 + P_nP_0.$$

The sequence is completely determined by this recurrence and the initial condition $P_0 = 1$.

(b) Let $f(x) = \sum_{i=0}^{\infty} P_i x^i$. Solving the recurrence relation in the preceding part we have:

$$f(x) - P_0 = x \sum_{n=0}^{\infty} \sum_{i=0}^{n} P_i P_{n-i} x^n = x f^2(x),$$

which yields the solutions

$$f_1(x) = \frac{1 + \sqrt{1-4x}}{2x}$$

and

$$f_2(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Now $\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n = 1 - \sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n}x^n$. Since the P_n's are positive, we have to select $f_2(x)$ as the relevant solution. The generating function is then

$$f(x) = \frac{1}{2x}\left(1 - \left(1 - \sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n}x^n\right)\right) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n,$$

which gives

$$P_n = \frac{1}{n+1}\binom{2n}{n}.$$

2 Elementary Probability Calculations

18.

(a) For the event in question to occur, the first $\lceil n/2 \rceil$ tosses may have any outcomes, and then the other $\lfloor n/2 \rfloor$ tosses are uniquely determined by them. Hence the required probability is $2^{\lceil n/2 \rceil}/2^n = 1/2^{\lfloor n/2 \rfloor}$.

(b) There are only two sequences satisfying the property, and hence the probability is $\frac{2}{2^n} = \frac{1}{2^{n-1}}$.

(c) For a non-negative integer n and a word w over $\{0,1\}$, denote by $a_{n,w}$ the number of words of length n starting with w and satisfying the requiremnts. It is easy to see that

$$a_{n,0} = a_{n,00} + a_{n,01} = a_{n,001} + a_{n,01} = a_{n-2,1} + a_{n-1,1}.$$

Due to symmetry, $a_{n,w} = a_{n,\overline{w}}$, where $\overline{w}$ is the word obtained from w upon replacing each 0 by 1 and each 1 by 0. Therefore:

$$a_{n,0} = a_{n-2,0} + a_{n-1,0}.$$

The initial conditions are $a_{0,0} = a_{1,0} = 1$. It follows that $a_{n,0}$ is Fibonacci's sequence. Therefore the required probability is $\frac{2F_n}{2^n} = \frac{F_n}{2^{n-1}}$.

(d) If n is even the probability is $= \binom{n}{n/2}/2^n$, while if n is odd there are no sequences satisfying the condition, so that the probability is 0.

25. The sets A_1 and A_2 may be chosen in $2^n \cdot 2^n = 4^n$ ways altogether. To satisfy the condition $A_1 \bigcap A_2 = \emptyset$, we have to require that each $j \in \{1, 2, \ldots, n\}$ belongs to at most one of the sets A_1 and A_2. Thus we have 3 possibilities for each j, namely either $j \in A_1 \bigcap \overline{A}_2$

or $j \in \overline{A}_1 \bigcap A_2$ or $j \in \overline{A}_1 \bigcap \overline{A}_2$. Hence the number of possibilities satisfying the requirement is 3^n. It follows that the probability of the event in question is $(3/4)^n$.

30. Due to symmetry, all $3! = 6$ possible orderings of X_1, X_2 and X_3 are equi-probable, whence each has probability $1/6$.

41.

(a) Since all the events in the union are disjoint, the probability is the sum of probabilities. Consequently:

$$P\left(\bigcup_{i=1}^{\infty}\left[\frac{1}{2i+1},\frac{1}{2i}\right]\right) = \sum_{i=1}^{\infty}\left(\frac{1}{2i}-\frac{1}{2i+1}\right) = 1-\ln 2$$

.

(b) For any n, the set in question is contained in the set of numbers whose infinite decimal expansion does not contain the digit 7 in any of the first n places. The latter set is clearly of probability $(9/10)^n$. Thus the probability of our set is less than $(9/10)^n$ for each n, and therefore it vanishes.

(c) As in the preceding part, the probability is 0.

43.

(a) Let us show that:

$$\limsup_{n\to\infty} A_n = [0,2], \qquad \liminf_{n\to\infty} A_n = [1/2,1].$$

Indeed, if $x \in [0,1]$, then $x \in A_n$ for each even n, while if $x \in [1,2]$, then $x \in A_n$ for each odd n, so that $\limsup_{n\to\infty} A_n \supseteq [0,2]$. On the other hand, if $x < 0$, then $x \notin A_n$ for any n, while if $x > 2$ then $x \notin A_n$ for $n > \frac{1}{x-2}$. This gives the inverse inclusion $\limsup_{n\to\infty} A_n \subseteq [0,2]$.
If $x \in [1/2,1]$, then $x \in A_n$ for each n, and in particular $\liminf_{n\to\infty} \supseteq [1/2,1]$. If $x < 1/2$, then $x \notin A_n$ for any odd n, while if $x > 1$ then $x \notin A_n$ for odd $n > \frac{1}{x-1}$. Therefore $\liminf_{n\to\infty} \subseteq [1/2,1]$.

46.

(a) The number of all subsets of A is of size 2^n. Thus Equivalently, we have to calculate the sum of those binomial coefficients $\binom{n}{k}$ with even k. Since the expression $\frac{1+(-1)^k}{2}$ takes the value 1 for even k and vanishes for odd k, we have:

$$\sum_{2|k}\binom{n}{k} = \sum_{k=0}^{n}\frac{1+(-1)^k}{2}\binom{n}{k}$$

$$\begin{aligned} &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} + \frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k} \\ &= \frac{1}{2}\cdot 2^n + \frac{1}{2}\cdot(1-1)^n = 2^{n-1}. \end{aligned}$$

Consequently the required probability is $\frac{1}{2}$.

(b) A simple calculation yields:

$$1+\omega^k+\omega^{2k} = \begin{cases} 3, & 3|k, \\ 0, & 3\nmid k. \end{cases}$$

Consequently:

$$\begin{aligned} \sum_{3|k}\binom{n}{k} &= \sum_{k=0}^{n}\frac{1+\omega^k+\omega^{2k}}{3}\binom{n}{k} \\ &= \frac{1}{3}\left[2^n+(1+\omega)^n+(1+\omega^2)^n\right] \\ &= \frac{2^n+(-\omega^2)^n+(-\omega)^n}{3}. \end{aligned}$$

Hence the probability for $|R|$ to be divisible by 3 is

$$\frac{1}{3}\left[1+\frac{(-\omega^2)^n+(-\omega)^n}{2^n}\right].$$

Similarly, to find the probability for $|R|$ to be 1 modulo 3, we calculate:

$$1+\omega^2\omega^k+\omega\omega^{2k} = \begin{cases} 3, & k\equiv 1 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \sum_{k\equiv 1 \pmod 3}\binom{n}{k} &= \sum_{k=0}^{n}\frac{1+\omega^2\omega^k+\omega\omega^{2k}}{3}\binom{n}{k} \\ &= \frac{1}{3}\left[2^n+\omega^2(1+\omega)^n+\omega(1+\omega^2)^n\right] \\ &= \frac{2^n+\omega^2(-\omega^2)^n+\omega(-\omega)^n}{3}. \end{aligned}$$

and the probability is $\frac{1}{3}\left[1-\frac{(-\omega^2)^{n+1}+(-\omega)^{n+1}}{2^n}\right]$.

51.

(a) The problem is equivalent to the problem regarding $2n$ people, n with \$10 bills and n with \$5 bills, waiting in line to but tickets for a show, which has been solved in class. Hence the required probability is $\dfrac{1}{n+1}$.

(b) The number of legal expressions is the same as in the preceding part, namely $\dfrac{\binom{2n}{n}}{n+1}$,. However, this time the total number of possibilities is 2^{2n}. Thus the required probability is $\dfrac{\binom{2n}{n}}{(n+1)2^{2n}}$.

3 Conditional Probability

4 Discrete Distributions

87.

(c) For any $c > 0$ the values assumed by $p(x)$ are non-negative. The value of c is determined by the requirement that their sum be 1. First let us decompose the given rational function. Namely, we are looking for constants a, b and d for which:

$$\frac{1}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{d}{x+2} .$$

This gives:

$$a(x+1)(x+2) + bx(x+2) + dx(x+1) = 1.$$

Making the substitutions $x = 0$, $x = -1$ and $x = -2$ we obtain:

$$2a = 1, \qquad -b = 1, \qquad 2d = 1,$$

and therefore

$$a = \frac{1}{2}, \qquad b = -1, \qquad d = \frac{1}{2} .$$

Hence:

$$\sum_{x=1}^{\infty} \frac{1}{x(x+1)(x+2)} = \sum_{x=1}^{\infty} \left(\frac{1/2}{x} - \frac{1}{x+1} + \frac{1/2}{x+2} \right) = \frac{1/2}{1} - \frac{1}{2} + \frac{1/2}{2} = \frac{1}{4} .$$

Thus $c = 4$.

5 Expectation

106. Denote by D the distance between v_1 and v_2. Since $P(D = 1) = p$:

$$E(D) \geq 1 \cdot P(D=1) + 2 \cdot P(D \geq 2) = p + 2(1-p) = 2 - p.$$

On the other hand, by the solution of Problem 50 we have

$$P(D \geq 3) \leq (n-2) \cdot \left(1-p^2\right)^{n-2},$$

and consequently:

$$\begin{aligned} E(D) &\leq 1 \cdot P(D=1) + 2 \cdot P(D \geq 2) + n \cdot P(D \geq 3) \\ &\leq p + 2(1-p) + n(n-2) \cdot \left(1-p^2\right)^{n-2} \underset{n\to\infty}{\longrightarrow} 2-p. \end{aligned}$$

Thus $E(D) \underset{n\to\infty}{\longrightarrow} 2-p$.

112. For $0 \leq i \leq n$, denote by X_i the diameter of the graph obtained after the ith stage of the construction process. With this notation, we have to bound $E(X_n)$ from below. Denote:

$$D_i = X_i - X_{i-1}, \qquad 1 \leq i \leq n.$$

Obviously $D_1 = 0$ and $D_2 = 1$. For $i \geq 3$ we have $D_i = 1$ if at the ith stage we connect the selected vertex to a vertex which is at a distance X_{i-1} from some vertex of the graph we have after the $(i-1)$st stage; otherwise – $D_i = 0$. Since the diameter after the $(i-1)$st stage is X_{i-1}, there are at least two vertices at that point satisfying this condition, so that

$$P(D_i = 1) \geq \frac{2}{i-1}.$$

Hence:

$$\begin{aligned} E(X_n) &= \sum_{i=1}^{n} E(D_i) \geq 0 + 1 + \sum_{i=3}^{n} \frac{2}{i-1} \\ &= 2(1 + 1/2 + 1/3 + \ldots + 1/(n-1)) - 1. \end{aligned}$$

6 Continuous Distributions

119. Obviously, X is distributed Cauchy.

7 Variance and Covariance

134. Let X denote the number of ones. Then $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the outcome of the ith roll is 1 and $X_i = 0$ otherwise. Let Y and Y_i, $1 \le i \le n$, be defined similarly for the sixes. Obviously, $X, Y \sim B(n, 1/6)$, so that:

$$E(X) = E(Y) = \frac{n}{6} .$$

Now

$$\begin{aligned} E(XY) &= E\left(\sum_{i=1}^{n} X_i \cdot \sum_{j=1}^{n} Y_j\right) \\ &= \sum_{i,j=1}^{n} E(X_i Y_j) = \sum_{i \neq j} E(X_i)E(Y_j) + \sum_{i=1}^{n} E(X_i Y_i) \\ &= n(n-1) \cdot \frac{1}{6} \cdot \frac{1}{6} + \sum_{i=1}^{n} 0 = \frac{n(n-1)}{6} , \end{aligned}$$

and therefore

$$\mathrm{Cov}(X, Y) = \frac{n(n-1)}{6} - \frac{n}{6} \cdot \frac{n}{6} = -\frac{n}{36} .$$

135.

(a) Obviously, $X \sim H(m, a, b)$, $Y \sim H(n, a, b)$, and therefore

$$E(X) = \frac{ma}{a+b} , \qquad V(X) = \frac{mab}{(a+b)^2}\left(1 - \frac{m-1}{a+b-1}\right) ,$$

and

$$E(Y) = \frac{na}{a+b} , \qquad V(Y) = \frac{nab}{(a+b)^2}\left(1 - \frac{n-1}{a+b-1}\right) .$$

(b) Write $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the ith ball is white and $X_i = 0$ otherwise. Write $Y = \sum_{i=1}^{n} Y_i$, analogously for the second batch. Then

$$\begin{aligned} E(XY) &= E\left(\sum_{i=1}^{m} X_i \cdot \sum_{j=1}^{n} Y_j\right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} E(X_i Y_j) = mn \frac{a(a-1)}{(a+b)(a+b-1)} , \end{aligned}$$

so that

$$\begin{aligned}\mathrm{Cov}(X, Y) &= mn\frac{a(a-1)}{(a+b)(a+b-1)} - \frac{ma}{a+b} \cdot \frac{na}{a+b} \\ &= -\frac{mnab}{(a+b)^2(a+b-1)} \, .\end{aligned}$$

(c) The covariance is negative since the more white balls there are in the first batch the less we should expect to have in the second.