תרגילים נוספים

פתרון

.1 $X \sim U(a,b) \Leftarrow f_X(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{אחרת} \end{cases}$

$Y \sim \exp(\lambda) \Leftarrow f_Y(y) = \begin{cases} \lambda \cdot e^{-\lambda y}, & y \ge 0 \\ 0 & \text{אחרת} \end{cases}$

$f_{X+Y}(t)\ ?$

X, Y ב"ת לכן $\quad f_{X+Y}(t) = \int_{-\infty}^{+\infty} f_X(t-y) f_Y(y)\,dy$

$$f_{X+Y}(t) = \int_0^{+\infty} f_X(t-y)\,\lambda \cdot e^{-\lambda y}\,dy =$$

$$= \left\{ \begin{matrix} u = t-y & y=0 \Rightarrow u=t \\ du = -dy & y=+\infty \Rightarrow u=-\infty \end{matrix} \right\} =$$

$$= -\int_t^{-\infty} \lambda \cdot e^{-\lambda(t-u)} f_X(u)\,du = \int_{-\infty}^{t} \lambda \cdot e^{-\lambda(t-u)} f_X(u)\,du =$$

$$= \left\{ f_X(u) = \begin{cases} \frac{1}{b-a} & a<u<b \\ 0 & \text{אחרת} \end{cases} \right\} =$$

$$= \begin{cases} 0 & t<a \\ \int_a^t \frac{\lambda}{b-a} e^{-\lambda(t-u)}\,du & a \le t < b \\ \int_a^b \frac{\lambda}{b-a} e^{-\lambda(t-u)}\,du & t \ge b \end{cases} =$$

$$= \begin{cases} 0 & t<a \\ \frac{1}{b-a}\cdot\left(1-e^{-\lambda(t-a)}\right) & a<t<b \\ \frac{1}{b-a}\left(e^{-\lambda(t-b)}-e^{-\lambda(t-a)}\right) & t>b \end{cases}$$

.2

$a>0$, $b>0$, $f_X(x) = \begin{cases} \frac{2}{b^2-a^2}x & a<x<b \\ 0 & \text{אחרת} \end{cases}$

$$f_Y(y) = \begin{cases} \frac{1}{b-a} & a<y<b \\ 0 & \text{אחרת} \end{cases}$$

$$f_{X+Y}(t) - ?$$

X, Y בלתי תלויים לכן

$$f_{X+Y}(t) = \int_{-\infty}^{+\infty} f_X(t-y)\cdot f_Y(y)\,dy = \int_a^b f_X(t-y)\frac{1}{b-a}\,dy =$$

$$= \left\{ \begin{matrix} u = t-y & y=a \Rightarrow u = t-a \\ du = -dy & y=b \Rightarrow u = t-b \end{matrix} \right\} =$$

$$= -\int_{t-a}^{t-b} \frac{1}{b-a} f_X(u)\,du = \int_{t-b}^{t-a} \frac{1}{b-a} f_X(u)\,du =$$

$$= \begin{cases} 0 & t<2a \\ \int_a^{t-a} \frac{2}{(b^2-a^2)(b-a)} u\,du & 2a<t<a+b \\ \int_{t-b}^{b} \frac{2}{(b^2-a^2)(b-a)} u\,du & a+b<t<2b \end{cases}$$

$f_{x+y}(t)=0 \qquad t>2b \qquad$ עבור

$$f_{x+y}(t)=\begin{cases}0 & t<2a \ \vee \ t>2b \\ \frac{1}{b^2-a^2}\cdot\frac{1}{b-a}\left(t^2-2at\right) & 2a<t<a+b \\ \frac{1}{b^2-a^2}\cdot\frac{1}{b-a}\cdot\left(2tb-t^2\right) & a+b<t<2b\end{cases}$$

.3 $f_X(x)=\begin{cases}1 & 1<x<2 \\ 0 & \text{אחרת}\end{cases} \Leftarrow X\sim U(1,2)$

$Y\sim U(1,2)$

$$f_Y(y)=\begin{cases}1 & 1<y<2 \\ 0 & \text{אחרת}\end{cases}$$

X, Y בלתי תלויים.

(א).

$$f_{-y}(t)=f_y(-t)=\begin{cases}1 & -2<t<-1 \\ 0 & \text{אחרת}\end{cases}$$

$X, -Y$ בלתי תלויים. $X-Y=X+(-Y)$

$$f_{x-y}(t)=\int_{-\infty}^{+\infty} f_x(t-y)\,f_{-y}(y)\,dy=\int_{-2}^{-1} f_x(t-y)\,dy=$$

$$=\left\{\begin{matrix}u=t-y & y=-2 \Rightarrow u=t+2 \\ du=-dy & y=-1 \Rightarrow u=t+1\end{matrix}\right\}=\int_{t+1}^{t+2} f_x(u)\,du=$$

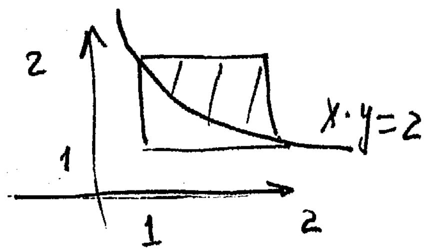
$$= \begin{cases} 0 & t < -1 \\ t+1 & -1 < t < 0 \\ 1-t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

$$E[\sqrt{x+y}\,] - ? \qquad .(ב)$$

$$E[g(\bar{X}, \bar{Y})] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f(x,y)\,dx\,dy$$

$$E[\sqrt{x+y}\,] = \int_0^1\int_0^1 \sqrt{x+y}\,dx\,dy$$

$$P(x \cdot y \geq 2) = \int_1^2 \int_{2/x}^2 dy\,dx = \qquad .(ג)$$



$$= \int_1^2 \left(2 - \frac{2}{x}\right) dx =$$

$$= 2 - 2\ln|x| \Big|_1^2 = 2 - 2\ln 2$$

$$\operatorname{COV}\left(\frac{X}{Y}, \frac{Y}{X}\right) = E\left[\frac{X}{Y}\cdot\frac{Y}{X}\right] - E\left[\frac{X}{Y}\right]\cdot E\left[\frac{Y}{X}\right] = \qquad .(ה)$$

$$= 1 - E\left[\frac{X}{Y}\right]\cdot E\left[\frac{Y}{X}\right].$$

$$E\left[\frac{X}{Y}\right] = \int_1^2 \int_1^2 \frac{x}{y}\,dx\,dy = \int_1^2 \frac{1}{y} \frac{x^2}{2}\Big|_1^2 dy =$$

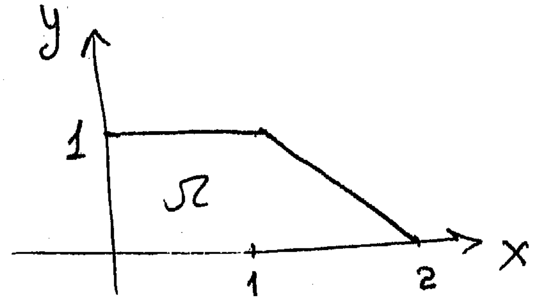
$$= \int_1^2 \frac{1}{y}\cdot\left(2-\frac{1}{2}\right) dy = \frac{3}{2}\ln 2$$

$$E\left[\frac{Y}{X}\right] = \int_1^2 \int_1^2 \frac{y}{x}\,dx\,dy = \frac{3}{2}\ln 2$$

$$\operatorname{cov}\left(\frac{X}{Y}, \frac{Y}{X}\right) = 1 - \frac{9}{4}\cdot(\ln 2)^2$$

.4

$$f_{X,Y}(x,y) = \begin{cases} c & x>0,\ 0<y<1,\ x+y\le 2 \\ 0 & \text{אחרת} \end{cases}$$

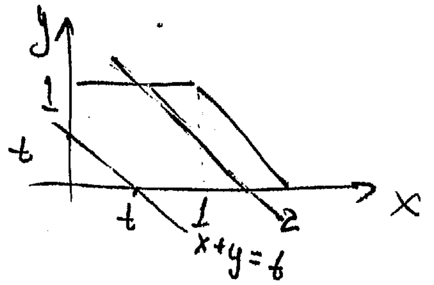


$$c = \frac{1}{S(\Omega)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$f(x) = \begin{cases} \int_0^1 \frac{2}{3}\,dy & 0<x<1 \\ \int_0^{2-x} \frac{2}{3}\,dy & 1<x<2 \\ 0 & \text{אחרת} \end{cases} = \begin{cases} \frac{2}{3}, & 0<x<1 \\ \frac{2}{3}(2-x), & 1<x \\ 0 & \text{אחרת} \end{cases}$$

$$f(y)=\begin{cases}\int_0^{2-y}\frac{2}{3}\,dx & 0<y<1\\ 0 & \text{אחרת}\end{cases}=\begin{cases}\frac{2}{3}(2-y) & 0<y<1\\ 0 & \text{אחרת}\end{cases}$$

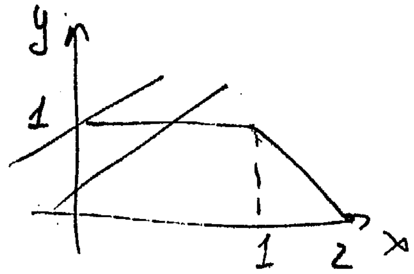
ג). X, Y תלויים לכן אנו לא יכולים להשתמש בקונבולוציה.



$$F_{x+y}(t)=\begin{cases}0 & t<0\\ \int_0^t\int_0^{t-x}\frac{2}{3}\,dy\,dx & 0<t<1\\ \frac{2}{3}\left[t-1+\frac{1}{2}\right] & 1<t<2\\ 1 & t>2\end{cases}=$$

$$=\begin{cases}0 & t<0\\ \frac{t^2}{3} & 0<t<1\\ \frac{2}{3}t-\frac{1}{3} & 1<t<2\\ 1 & t>2\end{cases}$$

$$f_{x+y}(t)=\begin{cases}\frac{2}{3}t & 0<t<1\\ \frac{2}{3} & 1<t<2\\ 0 & \text{אחרת}\end{cases}$$



$$F_{x-y}(t) = \begin{cases} 0 & t < -1 \\ \frac{2}{3}\,\frac{(t+1)^2}{2} & -1 \le t < 0 \\ \frac{2}{3}\left(\frac{3}{2} - 2\cdot\frac{1}{2}\cdot\frac{(2-t)^2}{4}\right) & 0 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

$$f_{x-y}(t) = \begin{cases} \frac{2}{3}\,(t+1) & -1 \le t \le 0 \\ \frac{1}{3}\,(2-t) & 0 < t \le 2 \\ 0 & \text{אחרת} \end{cases}$$