Probability Theory

Exercises

1 Review Questions in Combinatorics

1. Car license plates consist of 5 digits and 5 letters. How many different license plates are possible

(a) if all digits precede the letters?

(b) if no two letters may be adjacent?

(c) if there is no restriction?

2. A certain lock has five buttons, numbered from 1 to 5. The combination of numbers that opens the lock is a sequence of seven numbers and is reset every week.

(a) How many combinations are possible if every button must be used at least once?

(b) Assume that the lock may also have combinations requiring to push first two buttons simultaneously and then the other five one at a time (still with the restriction that every button must be used). How many combinations does this add?

3. A computer with k processors receives n jobs.

(a) How many possibilities are there to assign the jobs?

(b) How many possibilities are there if each processor must be assigned at least one job?

(c) How many possibilities are there if exactly one processor has to remain idle?

4. A *word* over an alphabet Σ is a finite sequence of elements of Σ. Let $|\Sigma| = r$. How many words of length n are there over Σ

(a) if there are no restrictions?

(b) if adjacent letters must be distinct?

(c) if some specific $\sigma \in \Sigma$ is not allowed to appear twice in a row?

(d) if each letter σ_i has to appear some prescribed number n_i of times (where $\sum_{i=1}^{r} n_i = n$)?

(e) if the word has to be a palindrome (i.e., the last letter is the same as the first, the second last the same as the second, etc)?

5. Solve the following recurrences by the method of generating functions:

(a)

$$\begin{aligned} &a_{n+2} - 3a_{n+1} + 2a_n = 0, \qquad n \geq 0, \\ &a_0 = 1,\ a_1 = 6. \end{aligned}$$

(b)

$$\begin{aligned} &a_{n+2} - 2a_{n+1} + a_n = 2^n, \qquad n \geq 0, \\ &a_0 = 1,\ a_1 = 2. \end{aligned}$$

6. For fixed n, find the generating function of the sequence $\left(\binom{n}{k}\right)_{k=0}^{\infty}$.

7. Solve the following systems of recurrence relations by the method of generating functions:

(a)

$$\left.\begin{aligned} a_{n+1} &= -2a_n - 4b_n, \\ b_{n+1} &= 4a_n + 6b_n, \end{aligned}\right. \qquad n \geq 0$$

$$a_0 = 1,\ b_0 = 0.$$

(b)

$$\left.\begin{aligned} a_{n+1} &= 2a_n - b_n + 2, \\ b_{n+1} &= -a_n + 2b_n - 1, \end{aligned}\right. \qquad n \geq 0$$

$$a_0 = 0,\ b_0 = 0.$$

8. Recall that Stirling's formula provides the following approximation for $n!$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Use the idea of the integral test (for testing the convergence of infinite series) to prove the following weak form of Stirling's formula:

$$e\left(\frac{n}{e}\right)^n \le n! \le e\left(\frac{n+1}{e}\right)^{n+1}.$$

9. Consider the middle binomial coefficients $\binom{2n}{n}$.

(a) What asymptotics does Stirling's formula yield for them?

(b) Employing elementary means only, derive upper and lower bounds which are at most polynomially worse than the "precise" asymptotics found in the first part.

10. Show that the number C_n of binary trees on n vertices is $\dfrac{1}{n+1}\dbinom{2n}{n}$. Note that the C_n's are the *Catalan numbers*. (Hint: use the generating function $f(x) = \sum_{n=0}^{\infty} C_n x^n$.)

11. Two queues Q_1 and Q_2 and a stack S are given. Initially, Q_1 contains the integers $1, 2, \ldots, n$ (with 1 at the head, 2 after it, and so forth). At each stage we either move the element at the head of Q_1 to S, or the element at the top of S to Q_2. The process continues for $2n$ stages, at which time all integers are at Q_2. Let P_n be the number of possible permutations of $1, 2, \ldots, n$ at the end.

(a) Find a recurrence relation satisfied by the sequence (P_n).

(b) Using generating functions, arrive at a closed-form formula for P_n.

12. A *walk* in the plane consists of a sequence of moves, each one from a point (l, m) to either $(l+1, m)$ or to $(l, m+1)$.

(a) Find the number of all walks going from $(0, 0)$ to (l, m).

(b) Same, if a walk is not allowed to go above the main diagonal.

(c) Suppose we allow also to move directly from a point (l, m) to $(l+1, m+1)$. Find the generating function

$$f(x, y) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} W_{lm} x^l y^m$$

of the double sequence (W_{lm}), counting the number of such walks from $(0, 0)$ to (l, m).

(d) Suppose P is a finite set of vectors (a_i, b_i), each having non-negative coordinates, not both 0. Suppose a walk consists of moves from any point (l, m) to any of the points $(l + a_i, m + b_i)$. Can you guess what the generating function will be in this general case? (Hint: Start by trying to find a connection between the possible moves in the preceding part and the form of the generating function there.)

13. Find the number of binary trees of n vertices, such that

(a) the left subtree is empty.

(b) the left subtree has only one vertex.

(c) the left subtree has two vertices.

(d) the left and right subtrees have the same number of vertices.

14. An *r-ary tree* is defined similarly to a binary tree, but each node has r subtrees instead of 2.

(a) Find a recurrence relation satisfied by the sequence $(T_r(n))_{n=0}^{\infty}$ counting the number of r-ary trees on n vertices.

(b) Use the recurrence to show that $T_r(n) = \frac{1}{(r-1)n+1}\binom{rn}{n}$.

(c) Find an equation satisfied by the generating function of the sequence.

15. The following method is used to sort a list:

```
void BubbleSort(int [ ] array)
{
   int temp;
   n = array.length;
   for (int i = 0; i < n - 1; i++)
      for (int j = 0; j < n - 1 - i; j++)
         if (array[j] > array[j+1])
         {
            temp = array[j];
            array[j] = array[j+1];
            array[j+1] = temp;
         }
}
```

Find the number of comparisons the method does when sorting a list of length n.

16. Consider the following two classes:

```
class Father
{
   public static int count= 0;

   public Father() {}

   public Father(int num)
   {
      for (int i = 0; i < num; i++)
      {
         new Father(i);
         count++;
      }
   }
}

class Son extends Father
{
   public Son(int num)
   {
      for (int i = 0; i < num; i++)
         new Father(i);
   }
}
```

The main method has:

```
Son x = new Son(n);
System.out.println(x.count);
```

(a) What is the value of *x.count* for any given n?

(b) Same if the class Father is changed to:

```
class Father
{
   public static int count = 0;

   public Father() {}

   public Father(int num)
   {
      for (int i = 0; i < num; i++)
         new Father(i);
      count++;
   }
}
```

2 Elementary Probability Calculations

17. Consider an experiment having N possible outcomes, say $\{w_1, w_2, \dots\ w_N\}$, where it is known that each outcome w_{j+1} is twice as likely as outcome w_j, $j = 1, 2, \dots, N-1$. Find $P(A_k)$, where $A_k = \{w_1, w_2, \dots, w_k\}$.

18. A fair coin is tossed n times. Find the probability that

(a) the sequence of results is palindromic.

(b) no two consecutive tosses produce the same result.

(c) no three consecutive tosses produce the same result.

(d) the total number of heads equal that of tails.

19. A full deck contains 52 cards. Find the probability that:

(a) the four aces are located consecutively in the deck.

(b) the locations of the aces in the deck form an arithmetic sequence of difference 7.

(c) the locations of the aces in the deck form an arithmetic sequence.

20. Forty books stand on a shelf, among them the three volumes of "the seventh column" by Alterman. Find the probability that the three volumes are arranged on the shelf according to the correct order (not necessarily adjacent to each other).

21. A number a is chosen randomly from the set $\{1, 2, \ldots n\}$. Find the probability p_n that $a^2 \bmod 10 = 1$. Find $\lim_{n \longrightarrow \infty} p_n$.

22. A number a is chosen randomly from $\{0, 1, 2, \ldots 10^n - 1\}$. Find the probability p that a is a k-digit number, i.e., $a = c_k 10^{k-1} + c_{k-1} 10^{k-2} + \ldots + c_1$, where $0 \leq c_i \leq 9$ for $i = 1, 2, \ldots k-1$ and $0 < c_k \leq 9$.

23. Two numbers X and Y are chosen randomly with replacement from the set $\{1, 2, \ldots n\}$, where $n \geq 4$. Let

$$p_2 = P\left(X^2 - Y^2 \bmod 2 = 0\right)$$

and

$$p_3 = P\left(X^2 - Y^2 \bmod 3 = 0\right).$$

Which of the probabilities p_2 and p_3 is bigger?

24. Two numbers X and Y are chosen randomly with replacement from the set $\{1, 2, \ldots n\}$.

(a) Find the probability

$$Q_n(p) = P\left(\left(X^{p-1} - Y^{p-1}\right) \bmod p = 0\right).$$

(Hint: Employ Fermat's little theorem, which states that, for a prime p and b relatively prime to p we have $b^{p-1} \bmod p = 1$.)

(b) Find $Q(p) = \lim_{n \longrightarrow \infty} Q_n(p)$.

(c) Find $Q = \lim_{p \longrightarrow \infty} Q(p)$.

25. Let $M = 2^{\{1,2,\ldots,n\}}$. Two elements A_1 and A_2 of M are chosen randomly (with replacement). Find the probability that $A_1 \bigcap A_2$ is empty.

26. An urn contains B blue and R red balls. A random sample of size n is drawn from the urn. Find the probability that the sample contains exactly b blue balls, if the sample is drawn:

(a) with replacement.

(b) without replacement.

27. An urn contains M_i balls of color i for $i = 1, 2 \ldots, N$. A random sample of size n is drawn from the urn without replacement. Find the probability that

(a) the sample contains exactly n_i balls of color i for $i = 1, 2 \ldots, N$.

(b) each of the colors is represented.

28. Itzik and Shmulik take a course with 3 possible grades: 0, 56 and 100. The probability to get 56 is 0.3 for Itzik and 0.4 for Shmulik. The probability that neither gets 0 but at least one gets 56 is 0.1. Find the probability that at least one gets 56 but neither gets 100.

29. Two numbers X_1 and X_2 are drawn randomly from the set $\{1, 2, \ldots, n\}$ without replacement. Find $P(X_2 > X_1)$.

30. Three numbers X_1, X_2 and X_3 are drawn randomly from the set $\{1, 2, \ldots, n\}$ without replacement. Find the probability that $X_1 < X_2 < X_3$.

31. X_i, $i = 1, 2, \ldots, n$, are drawn randomly from the set $\{1, 2, \ldots, N\}$ without replacement. Denote by Y_i their order statistics:

$$\min_{1 \le i \le n} X_i = Y_1 < Y_2 < Y_3 < \ldots < Y_n = \max_{1 \le i \le n} X_i.$$

(a) Find $P(Y_m \le M < Y_{m+1})$.

(b) Find the limit of the probability in part (a) if $N, M \longrightarrow \infty$ with $M/N \longrightarrow \theta \in [0, 1]$ (for constant m and n).

32. Thirteen cards are drawn randomly from a full deck of 52 cards. Find the probability of getting:

(a) a full hand.

(b) no aces.

(c) no aces and exactly one king and one queen.

(d) one ace, one king, one queen, etc.

33. Find the probability that, out of a set of n people, at least two have their birthdays in the same month. (Assume the months are equally likely.)

34. A sequence of length n over $\{0, 1, 2\}$ is chosen randomly. Find the probability that the sequence:

(a) starts with a 0.

(b) starts and ends with a 0, and contains exactly m additional 0's.

(c) contains digit i exactly n_i times, $i = 0, 1, 2$.

35. A coin is tossed until the first time a head turns up. Find the probability that the coin is tossed

(a) an even number of times.

(b) exactly k times.

(c) at least k times.

36. A coin is tossed until the same result appears twice in a row. Find the probability that the coin is tossed

(a) less than 6 times.

(b) an even number of times.

37. A drug is assumed to be effective with unknown probability p. To estimate p, the drug is tried on n different partients. It is found to be effective on m of them. The *method of maximum likelihood* for estimating p suggests that we choose the value of p yielding the highest probability to the result actually obtained at the experiment. Show that the maximum likehood estimate for p is $\frac{m}{n}$.

38. Prove the following claims:

(a) For sufficiently small $x \geq 0$ we have

$$e^{-x-x^2} \leq 1 - x \leq e^{-x}.$$

(b) If $(\alpha_n)_{n=1}^{\infty}$ is a sequence of numbers in the open interval $(0, 1)$, then $\prod_{n=1}^{\infty}(1 - \alpha_n) = 0$ if and only if $\sum_{n=1}^{\infty} \alpha_n = \infty$.

39. Numbers $X_1, X_2, \dots, X_k$ are chosen randomly from the set $\{1, 2, \dots n\}$ with replacement.

(a) Find the probability that all chosen numbers are distinct.

(b) Find the limit of the probability calculated in part (a) if $n \longrightarrow \infty$ and $\frac{k}{\sqrt{n}} \longrightarrow \theta$ for some $0 < \theta < \infty$. (Hint: Employ Exercise 38.a.)

40. Numbers $X_1, X_2, \dots, X_k$ are chosen randomly without replacement from the set $\{1, 2, \dots 2n\}$.

(a) Find the probability that $|X_i - X_j| \neq n$ for $i \neq j$.

(b) Find the limit of the expression you found if $n \longrightarrow \infty$ and $\frac{k}{\sqrt{n}} \longrightarrow \theta$ for some $0 < \theta < \infty$. (Hint: Employ Exercise 38.a.)

41. Let $\Omega = [0, 1]$ and suppose the probability of each sub-interval of Ω is equal to its length. Find

(a) $P\left(\bigcup_{i=1}^{\infty}[\frac{1}{2i+1}, \frac{1}{2i}]\right)$.

(b) the probabilty of the set of points whose infinite decimal expansion does not contain the digit 7.

(c) the probabilty of the set of points whose infinite hexadecimal expansion does not contain the digit D.

(d) the probabilty of the set of points whose infinite decimal expansion contains infinitely many occurrences of the digit 7. (Hint: Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers increasing arbitrarily fast. Our set contains the set of points whose expansion contains at least one 7 in the first a_1 places, at least one 7 in the next a_2 places, etc. Employ the idea of the proof of Exercise 38.a to show that the probabilty of the latter set may be made arbitrarily close to 1.)

42. Prove that for any events $A_1, A_2, \dots, A_n$ we have $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$.

43. Given any sequence $(A_n)_{n=1}^{\infty}$ of events, denote by $\limsup_{n\to\infty} A_n$ the set of points in the sample space belonging to infinitely many of the sets A_n, and by $\liminf_{n\to\infty} A_n$ the set of points belonging to all sets A_n from some place on.

(a) Let

$$A_n = \begin{cases} [0, 1+1/n], & n \text{ even}, \\ [1/2, 2+1/n], & n \text{ odd}. \end{cases}$$

Find the sets $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

(b) Prove that, for any sequence $(A_n)_{n=1}^{\infty}$ of events, $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ are also events.

44. For each of the following collections $\mathcal{B}$ of subsets of Ω, determine whether $\mathcal{B}$ is a σ-field:

(a) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of sets containing the number 1.

(b) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of all finite sets of even size and complements of such sets.

(c) $\Omega = \mathbf{N}$, $\mathcal{B}$ is the collection of all infinite sets and $\emptyset$.

(d) $\Omega = [0, 1]$, $\mathcal{B}$ is the collection of all finite sets and their complements.

(e) $\Omega = [0, 1]$, $\mathcal{B}$ is the collection of all sets which are at most countable and their complements.

45. A sequence $(A_n)_{n=1}^{\infty}$ of events is *increasing* if $A_n \subseteq A_{n+1}$ for each n and *decreasing* if $A_n \supseteq A_{n+1}$ for each n. Prove that

(a) if (A_n) is increasing, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$.

(b) if (A_n) is decreasing, then $P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$.

46. Let A be a finite set of size n. A subset R of A is chosen randomly. Find the probability that:

(a) $|R|$ (the size of R) is even.

(b) $|R|$ is divisible by 3; is 1 modulo 3. (Hint: Let $\omega = e^{2\pi i/3}$ be a root of unity of order 3. What is the value of $1 + \omega^k + \omega^{2k}$ as a function of k?)

(c) $|R|$ is divisible by 4; is 1,2,3 modulo 4.

47. (Requires Algebra II) Let F be a finite field with q elements. A polynomial $P(x)$ of degree d over F is chosen randomly, with the same probability to each such polynomial. Find the probability that:

(a) Some fixed $a \in F$ is a root of $P(x)$.

(b) Some fixed $a_1, a_2, \ldots, a_k \in F$ are roots of $P(x)$.

48. (Requires Algebra II) Let F be a finite field with q elements. An $n \times n$ matrix A over F is chosen randomly, each entry being chosen with equal probabilities to all field elements, independently of all other entries. Find the probability that:

(a) $A = A_0$ for some arbitrary fixed $A_0 \in M_n(F)$.

(b) A is invertible.

(c) $\det(A) = a$ for some arbitrary fixed $a \in F$.

(d) A is of rank 1.

(e) A is of rank $n-1$.

(f) all eigenvalues of A belong to F and are distinct from each other.

(g) $Av = 0$ for some arbitrary fixed non-zero vector $v \in F^n$.

(h) $Av = v$ for some arbitrary fixed non-zero vector $v \in F^n$.

(i) the sum of all eigenvalues of A (counting multiplicities) is a, where $a \in F$ is arbitrary and fixed.

49. A finite sequence of length n consisting of distinct numbers is permuted randomly (all $n!$ orderings being equi-probable). The numbers are read one by one, and put into a binary search tree T. Find the probability that:

(a) T is of height $n-1$.

(b) the left subtree and the right subtree of T contain $\frac{n-1}{2}$ nodes each (assuming n is odd).

(c) T is perfectly balanced (namely, contains all 2^k possible nodes at level k for each $k \leq h$, where h is the height of the tree; we assume here that $n = 2^{h+1} - 1$).

50. We choose a random graph $G = (V, E)$ on n vertices as follows. V is any set of size n (say, $V = \{1, 2, \ldots, n\}$). For any two distinct $v_1, v_2 \in G$, the edge (v_1, v_2) belongs to E with probability p and does not belong to it with probability $q = 1 - p$, where $0 < p < 1$ is arbitrary fixed. Show that the probability of G being connected converges to 1 as $n \longrightarrow \infty$.

51. An (abstract) urn contains n left brackets "(" and n right brackets ")".

(a) The brackets are drawn randomly from the urn without replacement. Let w denote the "word" consisting of the brackets, according to the order in which they were drawn. Find the probability that w is a legal expression in brackets. (An expression in brackets is *legal* if each prefix contains at least as many left brackets as right brackets, and altogether the same number of both types.)

(b) The same if the brackets are drawn with replacement. (Assume that altogether $2n$ are drawn, so the sequence is again of length $2n$.)

52. Consider the problem of $2n$ people waiting to buy movie tickets. Find the probability that:

(a) all of the people with 100-shekel bills will have to wait for their change.

(b) exactly one person will have to wait for change.

53. The following algorithm has been proposed for selecting a random tree on n labeled vertices $1, 2, \ldots, n$ out of the n^{n-2} such trees. First we choose a random vertex v_1 (with equal probabilities $1/n$ to each vertex). Next we choose a random vertex v_2 out of the remaining vertices and connect it to v_1. In general, for each $1 \leq k \leq n$, we choose a random vertex v_{k+1} out of the remaining vertices and connect it to a random vertex out of those chosen already.

(a) Explain why each tree has a positive probability of being selected when using this algorithm.

(b) Show that for $n \leq 3$ the algorithm assigns equal probabilities $1/n^{n-2}$ to all trees.

(c) Show in two ways that for sufficiently large n the algorithm does not assign equal probabilities to all trees:

  (i) by finding a specific tree whose probability is too large.

  (ii) by finding a specific tree whose probability is too small.

(d) Prove the result of the preceding part without calculating the probability of any specific tree. (Hint: Show that, even after cancellation, the rational numbers you obtain cannot possibly have n^{n-2} in the denominator.)

54. A drunkard moves randomly along the x-axis. At time 0 he is at the point 0, and then at each time unit he moves either 1 distance unit to the right or 1 distance unit to the left. The probability for him to move right (independently of his current location and past movements) is p and to move left is $q = 1 - p$.

(a) Show that, if $p > 1/2$, then there is positive probability for him to stay from time 1 on in the positive x-axis.

(b) Show that, if $p < 1/2$, then the probability for him to stay from time 1 on in the positive x-axis is 0.

(c) The same as (b) for $p = 1/2$.

3 Conditional Probability

55. Let the probability p_n that a family has exactly n children be αp^n for $n \geq 1$. Suppose that all sex distributions of n children have the same probability. Find the probabily that a family contains exactly k boys.

56. Given $N+1$ urns, each containing N blue and white balls. The kth urn contains k blue and $N-k$ white balls, $k = 0, 1, 2, \ldots \mathbf{N}$. An urn is chosen at random and a random ball is drawn from it $n+1$ times with replacement. Let $A = \{$the first n balls turn out to be blue$\}$ and $B = \{$the $(n+1)$'st ball is blue$\}$.

(a) Find $P(B \mid A)$.

(b) Show that for large N the probability in part (a) is approximatly $\frac{n+1}{n+2}$. (Hint: Use $\int_0^1 x^n dx$.)

57. Two players are playing a game until one of them is ruined. At each round one of them wins 1 dollar from the other. At the beginning of the game the first player has x dollars and the second has y dollars. The probabilities for the first and second players to win each round are p and q, respectively (where $p+q=1$).

(a) Find the probabilities P_1 and P_2 for the first and the second player, respectively, to win the whole match. (Hint: Distinguish between the cases $p = q = 1/2$ and $p \neq q$.)

(b) What happens to the probabilities you calculated in part (a) as $y \longrightarrow \infty$ (assuming x remains constant)?

58. An urn contains initially w white and b black balls. A ball is drawn at random. It is replaced and, moreover, another ball of the same color is put into the urn. The procedure is repeated indefinitely. Let $P_k(n)$ be the probability of drawing k white (and $n-k$ black) balls in the first n trials.

(a) Given that the second ball was black, what is the probability that the first was black?

(b) Prove the recurrence relation

$$P_k(n+1) = P_k(n)\frac{b+n-k}{w+b+n} + P_{k-1}(n)\frac{w+k-1}{w+b+n},$$

where we set $P_{-1}(k) = 0$.

(c) Using the preceding part (or directly) prove that:

$$P_k(n) = \binom{n}{k} \cdot \frac{w+b-1}{w+b+n-1} \cdot \frac{\binom{w+b-2}{w-1}}{\binom{w+b+n-2}{w+k-1}}.$$

59. Programming languages usually have random number generators. Such a generator provides numbers $U \in [0,1]$ with the property that $P(U \leq x) = x$ for every $0 \leq x \leq 1$. In Java such a generator is implemented by the method `Math.random()`.

(a) Show that the instruction

```
int k = (int) (n+1) * Math.random();
```

generates an integer between 0 and n, each such integer having the same probability $\frac{1}{n+1}$ to be selected.

(b) The method `choose(n,k)` calculates the binomial coefficient $\binom{n}{k}$. Consider the method:

```
public static double Gen(int n)
{
   int k;
   do
      k = (int) ((n+1) * Math.random());
   while (Math.random() > choose(n,k) / choose(n,n/2));
   return k;
}
```

Prove that it generates an integer between 0 and n, such that the probability for any k in this range to be selected is $\binom{n}{k}/2^n$.

60. Let `anyint` be a (theoretical) type of integers, which includes all integers. Consider the following code fragment:

```
double x,y;
anyint k=1;
do
{
   x = Math.random();
   y = Math.random();
   k++;
}
while (Math.abs(x-y) > 1/k);
```

(a) Find the probability of entering an infinite loop with this code. Ignore rounding errors of the computer. (Hint: Consider a sample space Ω which is a unit square.)

(b) Same if the last line is changed to:

```
while (Math.abs(x-y) > 1/(k*k));
```

(c) Same if the last line is changed to:

```
while (Math.abs(x-y) > 1/Math.sqrt(k));
```

(d) Same if the last line is changed to:

```
while (Math.abs(x-y) > 1/(4*k*k));
```

(Hint: In the last part you may use Wallis's formula

$$\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot \ldots \cdot 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) \cdot (2n+1)} \xrightarrow[n\to\infty]{} \frac{\pi}{2}$$

in your calculations.)

61. Let `anyint` be as in Problem 60. Suppose the method `add(anyint k)` adds the element k to a set and `deleteRandom()` deletes a randomly chosen element from it. Consider the following code fragment:

```
set mySet;
for (anyint k = 1; ; k++)
{
   mySet.add(2*k-1);
   mySet.add(2*k);
   mySet.deleteRandom();
}
```

Find the probability that after the execution of this (infinite) loop, starting with an empty set `mySet`,

(a) the number 1 belongs to `mySet`.

(b) `mySet` is empty.

62. Three people toss a die repeatedly until all three obtain the same result. Then they continue tossing their dice as many times as before.

(a) What is the probability that, in the course of the second stage of the experiment, they received three different results exactly k times?

(b) Assuming that at the second stage they received three different results k times, what is the probability that the first stage took n steps?

63. A coin of radius $r \leq a$ is dropped at random on an infinite grid (a system of straight lines parallel to the x-axis and the y-axis) with distance $2a$ between the parallel lines. Find the probability that the coin crosses

(a) none of the lines.

(b) at most one line.

64. A point P is selected at random in the circle $S = \{(x, y) : x^2 + y^2 \leq R^2\}$. Find the probability that the interval of length R centered at P and parallel to the x-axis is contained in S.

65. A right angle with sides of length b and c is dropped at random on a plane divided by a system of parallel lines of distance $2a$ apart. Suppose $b^2 + c^2 \leq 4a^2$. Find the probability that the angle crosses a line.

66. Solve Buffon's needle problem if the length $2l$ of the needle is larger than the distance $2a$ between the lines.

67. Consider the version of Buffon's needle problem (known as Laplace's problem) where we have a grid with distance $2a$ between the parallel lines. Find the probability that the needle crosses none of the lines.

68. A semicirle of radius $r \leq a$ is dropped at random on a plane divided by a system of parallel lines of distance $2a$ apart. Find the probability that the semicirle meets

a. none of the lines.

b. exactly one of the lines.

c. two of the lines.

69. A parent particle splits at the end of its life into 0, 1 or 2 particles, with probabilities 1/4, 1/2 and 1/4, respectively. Starting with a single particle and denoting by X_i the number of partices in the i-th generation, find

a. $P(X_2 > 0)$.

b. $P(X_1 = 2 | X_2 = 1)$.

c. $P(X_1 = 1 | X_3 > 0)$.

70. An urn contains 5 balls, r of them red and the others green, the value of r being unknown. (The possible values of r between 0 and 5 are *apriori* equally reasonable.) Mr. X is to draw a ball at random from the balls in the urn, and you are required to guess the color of the ball he will draw. You are allowed to hold one of the following three experiments prior to making your guess:

(1.) Draw a single ball from the urn.

(2.) Draw three balls with replacement from the urn.

(3.) Draw three balls without replacement from the urn.

(The balls are returned to the urn after the experiment.) What is the probability of making the right guess if you

a. use option 1 and guess that Mr. X will draw a ball of the same color as you did?

b. use option 2 and guess that Mr. X will draw a ball of the same color as the majority among the three balls you have drawn?

c. use option 3 and guess that Mr. X will draw a ball of the same color as the majority among the three balls you have drawn?

71. A and B throw alternately a pair of dice. A wins if he scores a total of 6 before B scores a total of 7; otherwise B wins. If A starts the game, what is his probability of winning?

72. N players $A_1, A_2, \ldots, A_N$ toss a biased coin whose probability of a head is p. A_1 starts the game, A_2 plays next, etc. The first player to get a head wins. Find the probability of each of the players to be the winner.

73. An urn contains initially W white and B black balls. A random sample of size n is drawn. Find the probability that the jth ball in the sample is black, given that the sample contains b black balls. Consider sampling

(a) with replacement.

(b) without replacement.

74. The game of craps is played as follows. The player tosses repeatedly a pair of dice. If he scores a total of 7 or 11 at his first toss he wins, whereas if his total is 2, 3 or 12 he loses. In any other case the game continues until for the first time he scores a total equal to either his first total or to 7. In the first of these cases he wins, and in the second – loses.

(a) Find the player's total probability of winning.

(b) Write a computer program estimating the same probability.

75. Construct examples of three events A, B and C

(a) which are pairwise independent but not independent.

(b) satisfying $P(A \cap B \cap C) = P(A)P(B)P(C)$, without any two of them being independent.

76. For events A, B and C, prove or disprove:

(a) If A and B are independent, then $P(A \cap B|C) = P(A|C)P(B|C)$.

(b) If $P(A) > P(B)$, then $P(A|C) > P(B|C)$.

(c) If $P(A) = 0$, then $P(A \cap B) = 0$.

(d) If $P(A|B) \geq P(A)$, then $P(B|A) \geq P(B)$.

(e) If $P(B|\overline{A}) = P(B|A)$, then A and B are independent.

(f) If $P(A) = a$ and $P(B) = b$, then $P(A|B) \geq \frac{a+b-1}{b}$.

(g) If $P(A) = P(B) > 0$, then $P(A|B) = P(B|A)$.

(h) If $P(A|B) = P(B|A)$, then $P(A) = P(B)$.

(i) If $P(A|B) = P(B|A)$, with $P(A \cup B) = 1$ and $P(A \cap B) > 0$, then $P(A) > \frac{1}{2}$.

(j) If $P(A) = a$ and $P(B) = b$, then $P(\overline{A} \cap \overline{B}) \geq 1 - a - b$.

77. A coin with probability p for a head is tossed n times. Suppose $A = \{$a head is obtained in the first toss$\}$ and $B_k = \{$exactly k heads are obtained$\}$. For which n and k are A and B_k independent?

78. It is known that each of four given people A, B, C and D tells the truth in any given instance with probability $\frac{1}{3}$. Suppose that A makes a statement, and then D says that C says that B says that

A is telling the truth. What is the probability that A is actually telling the truth?

79. A coin is tossed twice. Consider the following events:

A : Head on the first toss.
B : Head on the second toss.
C : Same outcomes in the two tosses.

(a) Are $A,\ B,\ C$ pairwise independent?

(b) Are A, B, C independent?

(c) Show that C is independent of A and B but not of $A \cap B$.

80. Same as Problem 46 if the set R is selected by including each element of A in it with probability p and excluding it with probability $1 - p$ (distinct elements being independent).