Final #2

Mark all correct answers in each of the following questions.

1. In an urn there are k balls, labelled by the numbers $1, 2, \dots, k$, where $k \geq 2$. We draw the balls from the urn with replacement until each ball has been drawn at least once. Define random variables as follows:

   - X – the number of stages of the experiment.
   - Y – the number of times the ball drawn at the first stage is drawn altogether.
   - N_j (for $1 \leq j \leq k$) – the number of stages that elapsed from the stage in which for the $(j-1)$-st time the ball to be drawn has not been drawn earlier until the j-th time this happened. (Thus, we agree that $N_1 = 1$.)

   (For example, if $k = 3$ and the balls drawn were $2, 2, 2, 3, 2, 2, 3, 3, 2, 1$, then $X = 10, Y = 6, N_1 = 1, N_2 = 3, N_3 = 6$.)

   (a) $E(X) = k\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)$.
   (b) N_j is hypergeometrically distributed for $2 \leq j \leq k$.
   (c) $E(Y) = 1 + \frac{1}{2} + \dots + \frac{1}{k-1}$.
   (d) The random variables X, Y are independent.
   (e) $P(Y = 1 \,|\, X = k+1) = \frac{1}{2} - \frac{1}{k}$ for $k \geq 2$.

2. We toss n dice. Those who show a “6” are tossed a second time. Let X be the number of dice showing “6” at the first stage, Y the number of dice showing “6” at the second stage, and S the sum of results at the

second stage. (For example, if $n = 7$ and the results at the first stage are $2, 2, 5, 6, 1, 6, 6$, then $X = 3$, and dice 4, 6 and 7 are tossed again. If the results at the second stage are $6, 4, 6$, then $Y = 2$ and $S = 16$.)

(a) The 3-dimensional random variable $(X, Y, n-X-Y)$ is distributed trinomially (i.e., multinomially in the 3-dimensional case).

(b) $P(X = n \,|\, Y = 0) = \frac{1}{6^n}$.

(c) Markov's inequality implies:

$$P(S \geq n) \leq \frac{7}{12}\,.$$

(d) Suppose that $n = 2$ and we repeat the experiment 1296 times. Let $S_1, S_2, \ldots, S_{1296}$ be the values of S at the various stages. The probability that there exist exactly 2 indices i, for which $S_i = 11$, is approximately e^{-2}.

(e) Under the assumptions of the preceding part:

$$P\left(\sum_{i=1}^{1296} S_i \leq 1512\right) \approx 0.84\,.$$

3. An urn contains 3 white balls and 2 black balls. All balls are drawn from the urn one by one without replacement. For $k = 1, 2 \ldots, 5$, let X_k denote the difference between the number of white balls and the number of black balls drawn throughout the first k steps. Put $Y = \min_{1 \leq k \leq 5} X_k$. (For example, if the first ball to be drawn is white, the following two are black, and the last two are white, then $X_1 = 1, X_2 = 0, X_3 = -1, X_4 = 0, X_5 = 1, Y = -1$.)

(a) $P(X_1 = -1 \,|\, Y = -1) = \frac{1}{2}\,.$

(b) Suppose we repeat the experiment 300 times. Let N_{300} denote the number of times Y assumes the value -2. Chebyshev's inequality implies:

$$P\left(N_{300} > 35\right) \leq \frac{3}{4}\,.$$

(c) Suppose we repeat the experiment 10000 times, and let N_{10000} be defined analogously to N_{300} of the preceding part. Then:

$$P(900 \leq N_{10000} \leq 970) \approx 0.16.$$

(d) Suppose we repeat the experiment indefinitely. Let $Y_1, Y_2, \ldots$ be the random variables corresponding to Y in the various experiments. For every $\varepsilon > 0$ there exist constants $L = L(\varepsilon)$ and $N = N(\varepsilon)$ such that for every $n > N$ we have:

$$P\left(\left|\sum_{i=1}^{n} Y_i + \frac{2n}{5}\right| \geq L\right) \leq \varepsilon.$$

(e) Consider again the original experiment, but with 39 white balls and 20 black balls. Define $X_k, 1 \leq k \leq 58$ and Y analogously to the original definitions. Then:

$$P(Y = 0) = \frac{1}{2}.$$

4. The variable (X, Y) is uniformly distributed in the triangle with vertices $(0,0), (1,0), (1,-1)$.

(a) The variable (X^2, Y^2) is distributed uniformly in the region $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

(b) $Y \sim U(-1, 0)$.

(c) The random variable X has the same distribution as the variable $\min(U_1, U_2)$, where U_1, U_2 are independent $U(0,1)$-distributed random variables.

(d) $\rho(X, Y) < 0$. Moreover, in every situation where (as in our case) X assumes only non-negative values and Y only non-positive values, the correlation coefficient $\rho(X, Y)$, if it exists, is non-positive.

(e) Let $W = X^3$. The two events $\{X \geq 3/4\}$ and $\{W \geq 27/64\}$ coincide, and in particular $P(X \geq 3/4) = P(W \geq 27/64)$. However, if we use Markov's inequality to bound these two probabilities, we obtain distinct bounds.

5. In a certain bank branch there are k tellers. The time in minutes a teller spends with a random customer is distributed Exp(1). On a certain day, when the branch was closed, there were still $k+m$ customers inside, k of whom were being served by the tellers, and the other m waiting (in a common queue) for their turn. When one of the tellers finishes serving a customer, the next customer in line (if there is still one) turns to him for service. Let $X_i, 1 \le i \le k$, denote the number of customers out of the latter m to be served by teller i. (Thus, $\sum_{i=1}^{k} X_i = m$.)

   (a) The time until all customers will finish is distributed as a sum of $k+m$ independent random variables, one of which is Exp(1)-distributed, one is Exp(2)-distributed, one is Exp(3)-distributed, ..., one is Exp($k-1$)-distributed, and all the other $m+1$ are Exp(k)-distributed.

   (b) If $m \ge k$, then

$$\begin{aligned} P(X_1 > 0, \ldots, X_k > 0) = k\left(1 - \frac{1}{k}\right)^m &- \binom{k}{2}\left(1 - \frac{2}{k}\right)^m \\ &+ \binom{k}{3}\left(1 - \frac{3}{k}\right)^m - \ldots \\ &+ (-1)^{k-2}\binom{k}{k-1}\left(1 - \frac{k-1}{k}\right)^m . \end{aligned}$$

   (c) For $m = k$:

$$P(X_1 > 0, \ldots, X_k > 0) = \frac{k!}{k^k} .$$

   (d) $(X_1, X_2, \ldots, X_m)$ is multinomially distributed.

   (e) Denote by T_n the time until all customers finish, if there are n customers at the branch when it closes. (For example, the time considered in part (a) is T_{m+k}.) There exists a constant C such that $E(T_n) = Cn$ for each $n \ge k$.

6. (a) If $(X_1, X_2, \ldots, X_9)$ is multinomially distributed, then $X_3+X_6+X_9$ is binomially distributed.

   (b) If $X, Y \sim \text{Exp}(\theta)$ are independent, then $|X - Y| \sim \text{Exp}(2\theta)$.

(c) If X, Y are independent normal variables and $P(X > Y) = 1/2$, then $E(X) = E(Y)$ and $V(X) = V(Y)$.

(d) Let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables with expectations, satisfying the weak law of large numbers. Then for every sequence $(a_n)_{n=1}^{\infty}$ of real numbers, which grows sufficiently fast, the sequence $(X_n + a_n)_{n=1}^{\infty}$ does not satisfy the weak law of large numbers.

(e) Let X be a random variable with moment generating function $\psi_X(t) = e^{50t^2+t}$. Then $P(X > 1) = 1/2$.

Solutions

1. (a-b) N_j counts the number of times we draw a ball until a new one emerges. Since we are at a position where $j-1$ balls have already been drawn and $k-j+1$ are yet to be drawn, the probability of success is $\frac{k-j+1}{k}$. Hence $N_j \sim G\left(\frac{k-j+1}{k}\right)$. Now $X = \sum_{j=1}^{k} N_j$, and therefore:

$$\begin{aligned} E(X) &= \sum_{j=1}^{k} E(N_j) \\ &= 1 + \sum_{j=2}^{k} \frac{k}{k-j+1} \\ &= k\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right). \end{aligned}$$

(c) Let us write $Y = \sum_{j=1}^{k} Y_j$, where Y_j is the number of times the first drawn ball is drawn between the $(j-1)$-st time a new ball is drawn and the j-th time this happens. In the course of these drawings, all drawings are of the $j-1$ balls already drawn, except for the last which is a new ball. Due to symmetry we have therefore:

$$E(Y_j) = E\left(\frac{N_j - 1}{j-1}\right) = \frac{1}{k-j+1}.$$

(Alternatively, we may obtain this equality as follows. Consider the drawings between the $(j-1)$-st time a new ball is drawn and the j-th time this happens. We may ignore drawings of old balls in this interval, except for drawings of the first drawn ball. Then, at each drawing, the probability for the first ball is $\frac{1}{k-j+2}$ and for a new ball $\frac{k-j+1}{k-j+2}$. Hence $Y_j = Y_j' - 1$, where $Y_j' \sim G\left(\frac{k-j+1}{k-j+2}\right)$. Thus,

$$E(Y_j) = \frac{k-j+2}{k-j+1} - 1 = \frac{1}{k-j+1}$$

as before.) It follows that:

$$E(Y) = 1 + \sum_{j=2}^{k} \frac{1}{k-j+1} = 2 + \frac{1}{2} + \ldots + \frac{1}{k-1} \,.$$

(d) Obviously, X may assume (with a positive probability) every integer value from k and above. Y may assume every positive integer value. However, for example, $P(X=k, Y=2) = 0$. Hence X, Y are dependent.

(e) We have:

$$P(Y = 1 \,|\, X = k+1) = \frac{P(X = k+1, Y = 1)}{P(X = k+1)} \,.$$

The event $\{X = k+1\}$ occurs if all drawings but one are of new balls. Partitioning this event according to the step j between 2 and k at which we draw an already drawn ball, we obtain:

$$\begin{aligned} P(X = k+1) = {} & \sum_{j=2}^{k} \frac{k-1}{k} \cdot \frac{k-2}{k} \cdot \ldots \cdot \frac{k-j+1}{k} \\ & \qquad \cdot \frac{j-1}{k} \cdot \frac{k-j}{k} \cdot \frac{k-j-1}{k} \cdot \ldots \cdot \frac{1}{k} \\ = {} & \frac{k!}{k^{k+1}} \cdot \frac{k(k-1)}{2} = \frac{(k-1)k!}{2k^k} \,. \end{aligned}$$

Similarly:

$$P(X = k+1, Y = 1) = \sum_{j=2}^{k} \frac{k-1}{k} \cdot \frac{k-2}{k} \cdot \ldots \cdot \frac{k-j+1}{k} \cdot \frac{j-2}{k} \cdot \frac{k-j}{k} \cdot \frac{k-j-1}{k} \cdot \ldots \cdot \frac{1}{k}$$

$$= \frac{k!}{k^{k+1}} \cdot \frac{(k-2)(k-1)}{2}.$$

Finally:

$$P(Y = 1 \,|\, X = k+1) = \frac{k-2}{k}.$$

Thus, only (a) is true.

2. (a) If a multi-dimensional random variable $(T_1, T_2, \ldots, T_k)$ is distributed multinomially, then in particular each of the components T_i assumes only non-negative values. In our case, $n - X - Y$ may assume negative values (for example, if all dice show a "6" on both tosses).

(b) We have:

$$P(X = n \,|\, Y = 0) = \frac{P(X = n, Y = 0)}{P(Y = 0)}.$$

The event $\{X = n, Y = 0\}$ occurs if all dice show a "6" on the first toss, but none does so on the second. Thus:

$$P(X = n, Y = 0) = \left(\frac{1}{6}\right)^n \cdot \left(\frac{5}{6}\right)^n = \left(\frac{5}{36}\right)^n.$$

To calculate $P(Y = 0)$, we may assume that all dice are tossed twice, and the question is about the probability that none of the dice shows a "6" on both tosses. Hence:

$$P(Y = 0) = \left(\frac{35}{36}\right)^n.$$

It follows that:

$$P(X = n \,|\, Y = 0) = \frac{(5/36)^n}{(35/36)^n} = \frac{1}{7^n}.$$

(c) Denote by $R_i, 1 \leq i \leq n$, the contribution of the i-th die to S. Clearly, $P(R_i = 0) = 5/6$, and $P(R_i = k) = 1/36$ for $k = 1, 2, \ldots, 6$. Then:

$$E(R_i) = 0 \cdot \frac{5}{6} + 1 \cdot \frac{1}{36} + \ldots + 6 \cdot \frac{1}{36} = \frac{7}{12}\,.$$

Since $S = \sum_{i=1}^{n} R_i$, this gives $E(S) = 7n/12$. Markov's inequality implies therefore:

$$P(S \geq n) \leq \frac{7n/12}{n} = \frac{7}{12}\,.$$

(d) To have $S_i = 11$ for some specific i, both dice need to show a "6" at stage 1 of the i-th experiment, and then one needs to show a "6" and the other a "5" at stage 2. Hence $P(S_i = 11) = 2/6^4$. It follows that the number of indices i satisfying $S_i = 11$ is distributed $B(1296, 2/6^4)$, which is approximately $P(2)$. Hence the probability in question is approximately

$$\frac{2^2}{2!} \cdot e^{-2} = 2e^{-2}\,.$$

(e) By part (c):

$$E(S_i) = 2 \cdot \frac{7}{12} = \frac{7}{6}, \qquad 1 \leq i \leq 1296.$$

Denote $V(S)$ by σ^2. The normal approximation yields:

$$\begin{aligned} P\left(\sum_{i=1}^{1296} S_i \leq 1512\right) &= P\left(\bar{S}_{1296} \leq \frac{7}{6}\right) \\ &= P\left(\frac{\bar{S}_{1296} - 7/6}{36\sigma} \leq 0\right) \\ &\approx P(Z \leq 0), \end{aligned}$$

where $Z \sim N(0, 1)$. Hence the required probability is approximately $1/2$.

Thus, only (c) is true.

3. (a) We have:

$$P(X_1 = -1 \,|\, Y = -1) = \frac{P(X_1 = -1, Y = -1)}{P(Y = -1)}\,.$$

There are $\binom{5}{2}$ possibilities as to the locations of the black balls among the drawn balls. Out of these, 4 constitute the event in the denominator of the right-hand side, and only 3 constitute the event in the numerator. Therefore:

$$P(X_1 = -1 \,|\, Y = -1) = \frac{3/10}{4/1} = \frac{3}{4}\,.$$

(b) The event $\{Y = -2\}$ occurs when the first two drawings are of the black balls, and only then are the white balls drawn. Hence $P(Y = -2) = 1/10$. Thus, $N_{300} \sim B(300, 1/10)$, so that

$$E(N_{300}) = 300 \cdot 1/10 = 30$$

and

$$V(N_{300}) = 300 \cdot 1/10 \cdot 9/10 = 27.$$

By Chebyshev's inequality:

$$\begin{aligned} P\left(N_{300} > 35\right) &= P\left(N_{300} \geq 36\right) \\ &\leq P\left(|N_{300} - 30| \geq 6\right) \\ &\leq \frac{27}{6^2} = \frac{3}{4}\,. \end{aligned}$$

(c) We have $N_{10000} \sim B(10000, 1/10)$, so that by the normal approximation:

$$\begin{aligned} P(900 \leq N_{10000} \leq 970) &= P\left(-\frac{10}{3} \leq \frac{N_{10000} - 10000 \cdot 1/10}{\sqrt{10000 \cdot 1/10 \cdot 9/10}} \leq -1\right) \\ &= P\left(-\frac{10}{3} \leq Z \leq -1\right), \end{aligned}$$

where $Z \sim N(0, 1)$. Hence:

$$P(900 \leq N_{10000} \leq 970) \approx \Phi(-1) - \Phi(-10/3) = 0.16.$$

(d) A routine calculation shows that the probability function of Y is given by the following table:

y	-2	-1	0	1
p	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

Hence

$$E(Y) = -2 \cdot \frac{1}{10} - 1 \cdot \frac{4}{10} + 0 \cdot \frac{3}{10} + 1 \cdot \frac{2}{10} = -\frac{2}{5}$$

and

$$E\left(Y^2\right) = (-2)^2 \cdot \frac{1}{10} + (-1)^2 \cdot \frac{4}{10} + 0^2 \cdot \frac{3}{10} + 1^2 \cdot \frac{2}{10} = 1,$$

so that $V(Y) = 21/25$. Hence (and since $V(Y)$ is certainly finite), the sequence $(Y_n)_{n=1}^{\infty}$ satisfies the weak law of large numbers, namely

$$P\left(\left|\bar{Y}_n + \frac{2}{5}\right| \geq \delta\right) \underset{n\to\infty}{\longrightarrow} 0, \qquad \delta > 0.$$

Namely:

$$P\left(\left|\sum_{i=1}^{n} Y_i + \frac{2n}{5}\right| \geq \delta n\right) \underset{n\to\infty}{\longrightarrow} 0, \qquad \delta > 0.$$

Thus, the convergence claimed in the question is much faster than that guaranteed by the law of large numbers. In fact, this faster convergence does not hold, as from the Central Limit Theorem we obtain:

$$P\left(\left|\bar{Y}_n + \frac{2}{5}\right| < \frac{L}{n}\right) = P\left(\left|\frac{Y_n - (-2/5)}{\sqrt{21/25}/\sqrt{n}}\right| < \frac{5L}{\sqrt{21n}}\right) \underset{n\to\infty}{\longrightarrow} P(Z = 0),$$

where $Z \sim N(0,1)$. Hence the probability in question converges to 1 as $n \to \infty$.

(e) This part is very similar to the ballot problem, with 39 votes for the first candidate and 20 for the second. The event $\{Y = 0\}$ in the question corresponds to the event that the second candidate never leads, but at some point during the process there is a tie. Now the event $\{Y \geq 0\}$ corresponds to the event that the second candidate never leads, so that by the solution of the ballot problem

$$P(Y \geq 0) = \frac{39 - 20 + 1}{39 + 1} = \frac{1}{2}\,.$$

Clearly, $P(Y > 0) > 0$, and therefore $P(Y = 0) < 1/2$.

Thus, (b) and (c) are true.

4. We first note that, since the area of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,-1)$ is $1/2$, we have

$$P((X,Y) \in D) = 2 \cdot \mathrm{area}(D)$$

for every subset D of that triangle.

(a) If (X^2, Y^2) was distributed uniformly in $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$, then we would have in particular

$$P\left(Y^2 > (X^2)^2\right) = 0.$$

However:

$$P\left(Y^2 > (X^2)^2\right) = P(Y < -X^2) = 2\int_0^1 (-x^2 + x)dx = 1/3.$$

(b) We have:

$$P(Y > y) = 2 \cdot \left(\frac{1}{2} - \frac{(1+y)^2}{2}\right) = 1 - (1+y)^2, \qquad -1 \leq y \leq 0.$$

Hence in the "interesting" interval $[-1, 0]$

$$F_Y(y) = (1+y)^2, \qquad f_Y(y) = 2(1+y),$$

and in particular Y is not uniformly distributed in $(-1, 0)$.

(c) Let us first find the distribution of X:

$$F_X(x) = P(X \le x) = 2 \cdot \frac{x^2}{2} = x^2, \qquad 0 \le x \le 1.$$

Now let $U = \min(U_1, U_2)$, where U_1, U_2 are independent $U(0,1)$-distributed. Then for $0 \le u \le 1$:

$$\begin{aligned} F_U(u) &= P(U \le u) = 1 - P(U > u) \\ &= 1 - P(U_1 > u, U_2 > u) = 1 - (1-u)^2 = 2u - u^2. \end{aligned}$$

(d) Routine calculations yield:

$$\begin{array}{rclcrcl} E(X) & = & \frac{2}{3}, & & V(X) & = & \frac{1}{18}, \\ E(Y) & = & \frac{-1}{3}, & & V(Y) & = & \frac{1}{18}, \end{array}$$

and

$$E(XY) = -\frac{1}{4},$$

so that:

$$\text{Cov}(X,Y) = -\frac{1}{36}.$$

Hence:

$$\rho(X,Y) = \frac{-1/36}{1/18} = -\frac{1}{2}.$$

(The fact that $\rho(X,Y) < 0$ is intuitively clear if we notice that Y assumes values between $-X$ and 0. Thus, as X grows from 0 to 1, the values Y assumes tend to be smaller and smaller.)

However, the situation described later in the question is irrelevant to the value of $\rho(X,Y)$, as it does not relate changes in one of the variables to changes in the second. For example, let $U \sim U(0,1)$ and $X = U, Y = U-1$. Then X assumes only non-negative values and Y only non-positive values, yet $\rho(X,Y) = 1$.

(e) Since the mapping $x \mapsto x^3$ is one-to-one, the events $\{X \ge 3/4\}$ and $\{W \ge 27/64\}$ indeed coincide. For the first, Markov's inequality yields:

$$P(X \ge 3/4) \le \frac{E(X)}{3/4} = \frac{2/3}{3/4} = \frac{8}{9}.$$

To bound the second probability, we need to calculate $E(W)$ first. Now

$$F_W(w) = P(W \le w) = P(X \le \sqrt[3]{w}) = w^{2/3}\,, \qquad 0 \le w \le 1.$$

Therefore

$$f_W(w) = \frac{2}{3}w^{-1/3}\,, \qquad 0 \le w \le 1,$$

so that:

$$E(W) = \int_0^1 w \cdot \frac{2}{3}w^{-1/3}dw = \left[\frac{2}{5}w^{5/3}\right]_0^1 = \frac{2}{5}.$$

By Markov's inequality:

$$P(W \ge 27/64) \le \frac{E(W)}{27/64} = \frac{128}{135}\,.$$

Thus, only (e) is true.

5. A fact that will be used several times in the course of our calculations is that, if we are given independent random variables $T_i \sim \text{Exp}(\theta_i), 1 \le i \le r$, then $\min_{1\le i\le r} T_i \sim \text{Exp}(\sum_{i=1}^r \theta_i)$.

   (a) The time until the first customer, out of those currently being served, will finish is the minimum of k independent random variables, all Exp(1)-distributed, and is therefore Exp(k)-distributed. Due to the memorylessness property of the exponential distribution, the time from that point until the next customer will finish is also distributed the same. In general, as long as all tellers still serve customers, the time between consecutive service completions is distributed Exp(k). After the $(m+1)$-st customer is served, only $k-1$ customers are being served in parallel, so that the time until another customer finishes is Exp($k-1$)-distributed. Similarly, the time until another customer finishes is Exp($k-2$)-distributed, and so forth. Hence the total time is a sum of $k+m$ independent random variables, $m-k+1$ of which are Exp(k)-distributed, and the other $k-1$ are Exp($k-1$)-distributed, Exp($k-2$)-distributed, ..., Exp(1)-distributed.

(b) Employing again the memorylessness property of the exponential distribution, we see that each of the customers in line has the same probability of $1/k$ of being served by any of the tellers, and distinct customers are independent. Consider the events:

$$A_i = \{X_i = 0\}, \qquad i = 1, 2, \ldots, k.$$

Obviously:

$$P(X_1 > 0, X_2 > 0, \ldots, X_k > 0) = P\left(\overline{\bigcup_{i=1}^{k} A_i}\right).$$

Now for any distinct indices $i_1, i_2, \ldots, i_l$ between 1 and k we have:

$$P\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_l}\right) = \left(1 - \frac{l}{k}\right)^m.$$

By the formula for the probability of a union of events:

$$\begin{aligned} P(X_1 > 0, \ldots, X_k > 0) = 1 &- k\left(1 - \frac{1}{k}\right)^m + \binom{k}{2}\left(1 - \frac{2}{k}\right)^m \\ &- \binom{k}{3}\left(1 - \frac{3}{k}\right)^m + \ldots \\ &+ (-1)^{k-1}\binom{k}{k-1}\left(1 - \frac{k-1}{k}\right)^m. \end{aligned}$$

(c) If $m = k$, then the event $\{X_1 > 0, \ldots, X_k > 0\}$ occurs only if each teller gets exactly one of the customers in line. There are altogether k^k possibilities as to which teller will serve each customer. The possibilities comprising our event correspond to all possibilities of matching the tellers with the customers, which is the number of permutations of k items. Hence:

$$P(X_1 > 0, \ldots, X_k > 0) = \frac{k!}{k^k}.$$

(d) As explained above, each customer has equal probabilities of $1/k$ of being served by each of the tellers, independently of which tellers serve the other customers. Hence $(X_1, X_2, \ldots, X_m)$ is multinomially distributed, where the parameter indicating the number of trials is m and the vector of probabilities is $(1/k, 1/k, \ldots, 1/k)$.

(e) The claim is false already for $k = 2$. In fact, using part (a) we see that

$$E(T_2) = \frac{1}{2} + \frac{1}{1} = \frac{3}{2},$$

while

$$E(T_3) = \frac{1}{2} + \frac{1}{2} + \frac{1}{1} = 2.$$

Thus, (a), (c) and (d) are true.

6. (a) In general, it follows from the definition of the multinomial distribution that any sum of some of the components of a multinomially distributed random variable is binomially distributed. Specifically, if $(X_1, X_2, \ldots, X_k)$ is multinomially distributed with parameters n (number of trials) and $(p_1, p_2, \ldots, p_k)$ (vector of probabilities), then $\sum_{j=1}^{l} X_{i_j} \sim B\left(n, \sum_{j=1}^{l} p_{i_j}\right)$.

(b) View X and Y as measuring the life lengths of two independent items with the memorylessness property. Then $|X - Y|$ measures the time difference between the two life lengths. Since when the first item "died", the second was "as new", this difference is again $\mathrm{Exp}(\theta)$-distributed.

(c) Let $X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2)$. Then $X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$. Hence:

$$\begin{aligned} P(X > Y) &= P(X - Y > 0) \\ &= P\left(\frac{(X - Y) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}} > \frac{-(\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \\ &= P\left(Z > \frac{-(\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right), \end{aligned}$$

where $Z \sim N(0, 1)$. The probability on the right-hand side is clearly $1/2$ if and only if $\mu_1 = \mu_2$.

(d) Denote $Y_n = X_n + a_n$ and $\nu_n = E(Y_n)$ for $n \geq 1$. One easily verifies that

$$\bar{Y}_n = \bar{X}_n + \bar{a}_n$$

and

$$\bar{\nu}_n = \bar{\mu}_n + \bar{a}_n$$

for each n. Thus:

$$\bar{Y}_n - \bar{\nu}_n = \bar{X}_n - \bar{\mu}_n.$$

Since $(X_n)_{n=1}^{\infty}$ satisfies the weak law of large numbers, so does $(Y_n)_{n=1}^{\infty}$.

(e) The moment generating function of X is, by the formula for the moment generating function of normal random variables, also the moment generating function of an $N(1, 100)$-distributed random variable. Since the moment generating function determines the distribution uniquely, $X \sim N(1, 100)$. Hence $P(X > 1) = 1/2$.

Thus, (a) and (e) are true.