Midterm #2

Mark all correct answers in each of the following questions.

1. We are given an infinite supply of light bulbs, whose life-length in hours is distributed $\mathrm{Exp}(\theta)$. We light the first bulb at time 0, the second – an hour later, the third – an hour after the second, and so forth. Denote by T_1 the first time at which some bulb burns out, and by T_2 the second time this occurs.

   (a) For any positive integer n and $t \in [n-1, n)$ we have:

   $$P(T_1 \geq t) = e^{n(n-1)\theta - n\theta t}.$$

   (b) The probability that the n-th bulb to be lit burns out before all the bulbs lit before it is $\frac{1}{n}e^{-n(n-1)\theta/2}$.

   (c) The probability that the 20-th bulb to be lit burns out before the 30-th bulb to be lit is $e^{-10\theta}$.

   (d) For any positive integer n and $t \in [n-1, n)$, the expected number of bulbs which are on at time t is $\frac{e^{\theta n}-1}{e^{\theta}-1}e^{-\theta t}$.

   (e) The random variables T_1 and T_2 are independent.

2. Reuven tosses a coin repeatedly until he gets heads for the first time. Shimon tosses a die repeatedly until he gets a "6" for the first time. Let X be the number of tosses of Reuven, Y – that of Shimon, and S – the total number of tosses. At the end of the game, Shimon pays Reuven $Z = Y - 3X$ shekels.

   (a) $V(X) = 4$.

(b) $V(Y) = 30$.

(c) $P(Z = 0) = 25/432$.

(d) $\text{Cov}(Z, S) = 24$.

(e) $\rho(Z, S) = \sqrt{3/7}$.

3. (a) If X is Cauchy distributed, then $\text{arctg}\, X$ does not have a variance.

   (b) If X, Y are discrete random variables, then XY is discrete as well.

   (c) If X, Y are continuous random variables, then XY is continuous as well.

   (d) Let X be a discrete random variable, taking the values $x_1, x_2, \ldots$, with probabilities $p_1, p_2, \ldots$, respectively. Let Y be a discrete random variable, taking the values $y_1, y_2, \ldots$, with (the same) probabilities $p_1, p_2, \ldots$, respectively. If $y_i \geq x_i$ for each i, and $E(X)$ does not exist, then $E(Y)$ does not exist either.

   (e) Let X, Y be random variables. If $E(X) = E(Y) = 0$, then $E(XY)$ may not exist. However, if it is known to exist, then it is 0.

Solutions

1. (a) The probability for the first bulb to still operate at any time t is $e^{-\theta t}$, the probability for the second to still operate is $e^{-\theta(t-1)}$, and so forth. Hence:

$$P(T_1 \geq t) = e^{-\theta t} e^{-\theta(t-1)} \ldots e^{-\theta(t-n+1)} = e^{n(n-1)\theta/2 - n\theta t} .$$

   (b) For the n-th bulb to burn out before all its predecessors, we first need all the bulbs $1, 2, \ldots, n-1$ to still operate at time $n-1$, and then we need bulb n to burn out before all of them. By part (a), the first probability is $e^{-n(n-1)\theta/2}$. If indeed none of the first $n-1$ bulbs burns out by time $n-1$, then, due to the memorylessness property of the exponential distribution and symmetry, the probability for the n-th bulb to burn out first is $1/n$. It follows that the required probability is $\frac{1}{n} e^{-n(n-1)\theta/2}$.

(c) The probability for the 20-th bulb to still operate at the time the 30-th is lit is $e^{-10\theta}$. If this is the case, then (again due to memorylessness and symmetry) the two bulbs have the same probability of burning out first. Thus the required probability is $\frac{1}{2}e^{-10\theta}$.

(d) Clearly, the number X of bulbs which are on at time t may be written in the form $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ if the i-th bulb to be lit is on at time t and $X_i = 0$ otherwise. Clearly:

$$E(X_i) = P(X_i = 1) = e^{-(t-i+1)\theta}, \qquad 1 \le i \le n.$$

Consequently:

$$E(X) = e^{-\theta t}\left(1 + e^{\theta} + e^{2\theta} + \ldots + e^{(n-1)\theta}\right) = \frac{e^{\theta n} - 1}{e^{\theta} - 1} e^{-\theta t}.$$

(e) Obviously, the probabilities $P(T_1 > 3)$ and $P(T_2 < 2)$ are both positive, yet $P(T_1 > 3, T_2 < 2) = 0$. Hence T_1 and T_2 are dependent.

Thus, (b) and (d) are true.

2. (a) Clearly, $X \sim G(1/2)$, and therefore $E(X) = 1/(1/2) = 2$ and $V(X) = (1 - 1/2)/(1/2)^2 = 2$.

(b) $Y \sim G(1/6)$, and therefore $E(Y) = 6$ and $V(Y) = 30$.

(c) The event $\{Z = 0\}$ occurs if, for some positive integer n, we have both $X = n$ and $Y = 3n$. Therefore:

$$\begin{aligned} P(Z=0) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot \left(\frac{5}{6}\right)^{3n-1} \cdot \frac{1}{6} \\ &= \frac{1}{5} \sum_{n=1}^{\infty} \left(\frac{1}{2} \cdot \frac{125}{216}\right)^n \\ &= \frac{25}{432} \cdot \frac{1}{1 - 125/432} = \frac{25}{307}. \end{aligned}$$

(d)

$$\begin{aligned}
\text{Cov}(Z,S) &= E(ZS)-E(Z)E(S) \\
&= E((Y-3X)(X+Y)) \\
&\quad -(E(Y)-3E(X))\cdot(E(X)+E(Y)) \\
&= E(Y^2-2XY-3X^2)-0 \\
&= V(Y)+E^2(Y)-2E(X)E(Y) \\
&\quad -3V(X)-3E^2(X) \\
&= 30+6^2-2\cdot 2\cdot 6-6-12=24.
\end{aligned}$$

(e) We have

$$V(Z)=V(Y-3X)=V(Y)+9V(X)=48,$$

and

$$V(S)=V(X+Y)=V(X)+V(Y)=32,$$

and therefore $\rho(Z,S)=\text{Cov}(Z,S)/\sqrt{V(Z)V(S)}=24/\sqrt{48\cdot 32}=\sqrt{3/8}$.

Thus, (b) and (d) are true.

3. (a) For any X, the variable $\operatorname{arctg} X$ is bounded, and therefore $E(X)$ and $V(X)$ exist. In the particular case where X is Cauchy distributed, the definition yields $\operatorname{arctg} X \sim U(-\pi/2,\pi/2)$, and consequently $V(\operatorname{arctg} X)=\pi^2/12$.

   (b) Let X,Y be discrete random variables. Suppose they assume (with positive probability) the values $\{x_i : i\in I\}$ and $\{y_j : j\in J\}$, respectively, where I,J are at most countable index sets. Then XY assumes only values belonging to the countable set $\{x_iy_j : i\in I, j\in J\}$ (perhaps not all these values), and thus it is discrete as well.

   (c) Let X be a continuous random variables. Then $Y=1/X$ is continuous as well, but the product XY is identically 1, and in particular discrete.

(d) Let X be a discrete random variable, assuming negative values only, so that $E(X)$ does not exist. (For example, $X = -X'$, where X' is the random variable discussed in the St. Petersburg game.) Let Y be a discrete random variable, assuming positive values only with corresponding probabilities, such that $E(Y)$ does exist (say, $Y \sim G(1/2)$). Then the required condition holds, but the desired conclusion does not.

(e) Let X be distributed according to the probability function given by

$$P(X = n) = c/|n|^3, \qquad n = \pm 1, \pm 2, \dots,$$

where c is a suitable constant. Then:

$$E(X) = \sum_{n=-\infty}^{-1} \frac{-c}{n^3} \cdot n + \sum_{n=1}^{\infty} \frac{c}{n^3} \cdot n = 0.$$

Now let $Y = X$. We have

$$E(XY) = E(X^2) = \sum_{n=-\infty}^{-1} \frac{-c}{n^3} \cdot n^2 + \sum_{n=1}^{\infty} \frac{c}{n^3} \cdot n^2,$$

which diverges. On the other hand, letting $X = \pm 1$ with probabilities $1/2$ each, and $Y = X$, we obtain $E(X) = E(Y) = 0$, but $E(XY) = E(X^2) = 1$.

Thus, only (b) is true.