Final #1

Mark the correct answer in each part of the following questions.

1. The number of daughters of a random Tanzanian elephantess is distributed as $X - 1$, where $X \sim G(1/2)$.

   (a) The probability for a random Tanzanian elephantess to have 2 granddaughters (which here means **only** daughters of daughters, and **not** daughters of sons) is:

   (i) 1/32.
   (ii) 2/27.
   (iii) 3/16.
   (iv) 2/9.
   (v) None of the above.

   (b) If a random elephantess has 3 granddaughters (again, daughters of daughters), then the probability she has 2 daughters is

   (i) 37/156.
   (ii) 27/104.
   (iii) 119/417.
   (iv) 243/832.
   (v) None of the above.

   (c) Two random elephantesses have jointly 4 daughters. The probability that each has two daughters is

   (i) 1/16.
   (ii) 1/9.
   (iii) 1/5.
   (iv) 1/4.

(v) None of the above.

(d) In one of the parks in Tanzania there are altogether 1000 elephantesses. The probability that they will have altogether 1200 daughters is:

(i) $\frac{\binom{2199}{1200}}{2^{2200}}$.

(ii) $\frac{\binom{2199}{1200}}{2^{2199}}$.

(iii) $\frac{\binom{2200}{1200}}{2^{2200}}$.

(iv) $\frac{\binom{2200}{1200}}{2^{2199}}$.

(v) None of the above.

(e) Markov's Inequality implies that the probability that the 1000 elephantesses above will have at least 2500 daughters altogether is at most

(i) 0.1.

(ii) 0.2.

(iii) 0.3.

(iv) 0.4.

(v) None of the above.

Remark: We mean here the best bound that be reached. For example, if Markov's Inequality implies that the above probability is at most 0.1, hence it is also at most 0.2, and (0.3 and 0.4), but only (i) should be marked as the correct answer.

2. An experiment consists of $n \geq 1$ stages. In each stage i we select uniformly randomly i integers between 1 and i. (Thus, altogether $\frac{n(n+1)}{2}$ numbers are chosen.) For $1 \leq i \leq n$, let Y_i be the number of times the number i was selected at the i-th stage, S_i the sum of all i numbers selected at the i-th stage, and $S = \sum_{i=1}^{n} S_i$. (For example, if $n = 5$ and the selected numbers have been $1, 2, 1, 2, 3, 1, 2, 1, 4, 4, 5, 5, 5, 1, 5$, then $Y_1 = Y_2 = Y_3 = 1, Y_4 = 2, Y_5 = 4, S_1 = 1, S_2 = 3, S_3 = 6, S_4 = 11, S_5 = 21, S = 42$.) (Hint: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.)

(a) The ratio $P(Y_{1000} = 3)^2/P(Y_{1000} = 6)$ is approximately

(i) 5.
(ii) 10.
(iii) 20.
(iv) 40.
(v) None of the above.

(b) Suppose it is known that at the 100-th stage all numbers between 1 and 100 have been selected. The probability that at least one of them was equal to the number indicating its location within the selected numbers (namely, either the first number was 1, or the second was 2, ..., or the last one was 100) is

(i) approximately 0.37.
(ii) approximately 0.63.
(iii) approximately 0.76.
(iv) very close to 1.
(v) None of the above.

(c) If $n = 100$, then the probability for all three numbers $98, 99, 100$ not to be selected even once (out of all 5050 selected numbers) is approximately

(i) $1/e^2$.
(ii) $1/e^3$.
(iii) $1/e^4$.
(iv) $1/e^6$.
(v) None of the above.

(d) $\text{Cov}(S_{15}, Y_{15}) =$

(i) 7.
(ii) 7.5.
(iii) 8.
(iv) 8.5.
(v) None of the above.

(e) A direct application of Chebyshev's Inequality for $n = 24$ yields the following (where the bounds on the right-hand side are approximate):

(i) $P(2350 \leq S \leq 2850) \geq 0.78$.
(ii) $P(2350 \leq S \leq 2850) \geq 0.83$.
(iii) $P(2350 \leq S \leq 2850) \geq 0.88$.
(iv) $P(2350 \leq S \leq 2850) \geq 0.93$.
(v) None of the above.

Remark: We mean here the best bound, which can that be reached. For example, if Chebyshev's Inequality implies that the above probability is at least 0.93, hence it is also at least 0.88, and (0.83 and 0.78), but only (iv) should be marked as the correct answer.

3. Reuven and Shimon play an infinite-stage game, as follows. At stage 0 they toss a coin. At each stage $n \geq 1$ they toss both a coin and a die. If the coin shows a head at both stages $n-1$ and n, Reuven gets 1 shekel from Shimon. If the die shows a 6 at the n-th stage, Shimon gets one shekel from Reuven. Let X_n be the net profit of Reuven at the n-th stage. (For example, if the coin showed T, T, H, H, H, H, T and the die $6, 3, 1, 6, 5, 6$, then $X_1 = X_6 = -1, X_2 = X_4 = 0, X_3 = X_5 = 1$.)

(a) For $n \geq 1$ we have $P(X_n = 0) =$

(i) $\frac{1}{2}$.
(ii) $\frac{2}{3}$.
(iii) $\frac{3}{4}$.
(iv) $\frac{5}{6}$.
(v) None of the above.

(b) For $n \geq 1$ $P(X_{n+1} = 0 | X_n = 0) =$

(i) 17/30.
(ii) 17/24.
(iii) 17/20.
(iv) 17/18.

(v) None of the above.

(c) The expectation and variance of X_n are:

(i) $E(X_n) = -1/12, V(X_n) = 29/144$.
(ii) $E(X_n) = 0, V(X_n) = 7/48$.
(iii) $E(X_n) = 1/12, V(X_n) = 47/144$.
(iv) $E(X_n) = 1/6, V(X_n) = 5/36$.
(v) None of the above.

(d) Consider the sequence of averages:

$$\bar{X}_n = \frac{X_1 + X_2 + \ldots + X_n}{n}.$$

(i) The random variables $\bar{X}_1, \bar{X}_2, \ldots$ are dependent, and with probability 1 their sequence of values is dense in the interval $[-1, 1]$. (That is, for every subinterval $[a, b] \subseteq [-1, 1]$, one of those values lies in the subinterval.)
(ii) The random variables $\bar{X}_1, \bar{X}_2, \ldots$ are independent and have the same distribution as the variables X_n.
(iii) The random variables $\bar{X}_1, \bar{X}_2, \ldots$ are dependent, and therefore the sequence does not satisfy the weak law of large numbers.
(iv) $P(|\bar{X}_n - 1/12| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$ for every $\varepsilon > 0$.
(v) None of the above.

(e) Now assume that only even rounds qualify for wins, namely odd rounds are still held but money is not moved on those rounds. Let X be the total profit (positive or negative) of Reuven in the first 8352 rounds. Then $P(290 \leq X \leq 377) \approx$

(i) 0.18.
(ii) 0.36.
(iii) 0.72.
(iv) 0.82.
(v) None of the above.

4. A wooden rod of unit length is broken into two pieces at a uniformly distributed point along it. The two pieces are put orthogonally to each

other to generate the two perpendiculars of a right triangle. A plastic rod of appropriate length is constructed to serve as the hypotenuse of the triangle, and a square is constructed on this hypotenuse. Let X be the length of the right part of the first rod, H – the length of the hypotenuse, $S_\triangle$ the area of the triangle, and $S_\square$ the area of the square. (For example, if the right part of the wooden rod is of length 3/7, then $X = 3/7, H = 5/7, S_\triangle = 6/49$, and $S_\square = 25/49$.)

(a) The density function $f_H(h)$ is given by:

(i)

$$f_H(h) = \begin{cases} \frac{2h}{\sqrt{2h^2-1}}, & \frac{\sqrt{2}}{2} \le h \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(ii)

$$f_H(h) = \begin{cases} \sqrt{h^2 + (1-h)^2}, & 0 \le h \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(iii)

$$f_H(h) = \begin{cases} \frac{1}{\sqrt{h^2+(1-h)^2}}, & 0 \le h \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(iv)

$$f_H(h) = \begin{cases} \frac{2h}{\sqrt{h^2+(1-h)^2}}, & \frac{\sqrt{2}}{2} \le h \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(v) None of the above.

(b) The distribution function $F_{S_\triangle}$ of $S_\triangle$ is given by:

(i)

$$F_{S_\triangle}(t) = \begin{cases} 0, & t < 0, \\ 8t, & 0 \le t \le 1/8, \\ 1, & t > 1/8. \end{cases}$$

(ii)

$$F_{S_\triangle}(t) = \begin{cases} 0, & t < 0, \\ 2\sqrt{2t}, & 0 \le t \le 1/8, \\ 1, & t > 1/8. \end{cases}$$

(iii)

$$F_{S_\triangle}(t) = \begin{cases} 0, & t < 0, \\ 1 - \sqrt{1-8t}, & 0 \le t \le 1/8, \\ 1, & t > 1/8. \end{cases}$$

(iv)

$$F_{S_\triangle}(t) = \begin{cases} 0, & t < 0, \\ 9t - 8t^2, & 0 \le t \le 1/8, \\ 1, & t > 1/8. \end{cases}$$

(v) None of the above.

(c) $E(S_\triangle) =$

(i) 1/48.

(ii) 1/30.

(iii) 1/24.

(iv) 1/12.

(v) None of the above.

(d) $\rho(S_\triangle, S_\square) =$

(i) -1.

(ii) $-1/2$.

(iii) 0.

(iv) 1.

(v) None of the above.

(e) Let ψ be the moment generating function of $S_\square$. Then $\psi(0.003) \approx$

(i) 0.996.

(ii) 0.998.

(iii) 1.002.

(iv) 1.004.

(v) None of the above.

Solutions

1. (a) Let A be the event whereby a random Tanzanian elephantess has 2 granddaughters, and $H_k,\ \ k \ge 0$, the event whereby she has k

daughters. Then:

$$
\begin{aligned}
P(A) &= \sum_{k=0}^{\infty} P(A|H_k) \cdot P(H_k) \\
&= \sum_{k=0}^{\infty} P(A|H_k) \cdot P(X = k+1) \\
&= \sum_{k=0}^{\infty} P(A|H_k) \cdot \left(\frac{1}{2}\right)^{k+1}.
\end{aligned} \tag{1}
$$

Let A_1 be the event whereby the elephantess has two granddaughters, both daughters of the same mother, and A_2 be the event whereby it still has two granddaughters, but they have different mothers. Clearly, $A = A_1 \cup A_2$, and the union is disjoint. In these terms:

$$P(A|H_k) = P(A_1|H_k) + P(A_2|H_k), \qquad k \geq 0.$$

Obviously, for $k \geq 0$ we have

$$P(A_1|H_k) = \binom{k}{1} \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{k-1} = \binom{k}{1} \left(\frac{1}{2}\right)^{k+2}, \tag{2}$$

and

$$P(A_2|H_k) = \binom{k}{2} \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{k-2} = \binom{k}{2} \left(\frac{1}{2}\right)^{k+2}. \tag{3}$$

Substituting (2) and (3) into (1) we obtain:

$$
\begin{aligned}
P(A) &= \sum_{k=0}^{\infty} \left(\binom{k}{1} + \binom{k}{2}\right) \cdot \left(\frac{1}{2}\right)^{2k+3} \\
&= \frac{1}{8} \cdot \left(\sum_{k=0}^{\infty} \binom{k}{1} \cdot \left(\frac{1}{4}\right)^k + \sum_{k=0}^{\infty} \binom{k}{2} \cdot \left(\frac{1}{4}\right)^k\right) \\
&= \frac{1}{8} \cdot \left(\frac{1/4}{(1-1/4)^2} + \frac{(1/4)^2}{(1-1/4)^3}\right) = 2/27.
\end{aligned}
$$

Thus, (ii) is true.

(b) Let C be the event whereby a random Tanzanian elephantess has 3 granddaughters, and $H_k,\ \ k \geq 0$, the event whereby she has k daughters. Then:

$$
\begin{aligned}
P(H_2|C) &= \frac{P(C|H_2) \cdot P(H_2)}{\sum_{k=0}^{\infty} P(C|H_k) \cdot P(H_k)} \\
&= \frac{P(C|H_2) \cdot P(X=3)}{\sum_{k=0}^{\infty} P(C|H_k) \cdot P(X=k+1)} \qquad (4) \\
&= \frac{P(C|H_2) \cdot \frac{1}{8}}{\sum_{k=0}^{\infty} P(C|H_k) \cdot \left(\frac{1}{2}\right)^{k+1}}.
\end{aligned}
$$

Let C_1, C_2, C_3 be the subevents of C whereby the three granddaughters have the same mother, two distinct mothers, three mothers, respectively. Obviously, $C = C_1 \cup C_2 \cup C_3$, and the union is disjoint. In these terms:

$$P(C|H_k) = P(C_1|H_k) + P(C_2|H_k) + P(C_3|H_k), \qquad k \geq 0.$$

Obviously, for $k \geq 0$ we have:

$$P(C_1|H_k) = \binom{k}{1}\left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{k-1} = \binom{k}{1}\left(\frac{1}{2}\right)^{k+3}, \qquad (5)$$

$$P(C_2|H_k) = 2 \cdot \binom{k}{2} \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{k-2} = 2 \cdot \binom{k}{2}\left(\frac{1}{2}\right)^{k+3}, \qquad (6)$$

and

$$P(C_3|H_k) = \binom{k}{3} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{k-3} = \binom{k}{3}\left(\frac{1}{2}\right)^{k+3}, \qquad (7)$$

Substituting (5), (6) and (7) into (4), we obtain:

$$\begin{aligned}
P(H_2|C) &= \frac{\left(\binom{2}{1}+2\cdot\binom{2}{2}\right)\cdot\frac{1}{2^8}}{\sum_{k=0}^{\infty}\left(\binom{k}{1}+2\cdot\binom{k}{2}+\binom{k}{3}\right)\cdot\left(\frac{1}{2}\right)^{2k+4}} \\
&= \frac{1/4}{\sum_{k=0}^{\infty}\binom{k}{1}\cdot\left(\frac{1}{4}\right)^k+2\cdot\sum_{k=0}^{\infty}\binom{k}{2}\cdot\left(\frac{1}{4}\right)^k+\sum_{k=0}^{\infty}\binom{k}{3}\cdot\left(\frac{1}{4}\right)^k} \\
&= \frac{1/4}{\frac{1/4}{(1-1/4)^2}+2\cdot\frac{(1/4)^2}{(1-1/4)^3}+\frac{(1/4)^3}{(1-1/4)^4}} = \frac{81}{256}.
\end{aligned}$$

Thus, (v) is true.

(c) For $i = 1, 2$ denote by Y_i the number of daughters of i-th elephantesses. Hence:

$$\begin{aligned}
P(Y_1 = 2, Y_2 = 2 | Y_1 + Y_2 = 4) &= \frac{P(Y_1 = 2, Y_2 = 2)}{P(Y_1 + Y_2 = 4)} \\
&= \frac{P(Y_1 = 2)\cdot P(Y_2 = 2)}{\sum_{i=0}^{4} P(Y_1 = i)\cdot P(Y_2 = 4-i)} \\
&= \frac{P(Y_1 = 2)\cdot P(Y_2 = 2)}{\sum_{i=0}^{4} P(Y_1 = i)\cdot P(Y_2 = 4-i)} \\
&= \frac{1/2^3\cdot 1/2^3}{\sum_{i=0}^{4} 1/2^{i+1}\cdot 1/2^{4-i+1}} \\
&= \frac{1/2^6}{5/2^6} = \frac{1}{5}.
\end{aligned}$$

Thus, (iii) is true.

(d) Let Y_i be the number of daughters of the i-th elephantess and $X_i = Y_i + 1$ for $1 \le i \le 1000$. The variables X_i are independent and $G(1/2)$-distributed. We have:

$$\begin{aligned}
P(Y_1 + Y_2 + \ldots + Y_{1000} = 1200) &= P\left(\sum_{i=1}^{1000}(X_i - 1) = 1200\right) \\
&= P\left(\sum_{i=1}^{1000} X_i - 1000 = 1200\right) \\
&= P\left(\sum_{i=1}^{1000} X_i = 2200\right).
\end{aligned}$$

Clearly, $\sum_{i=1}^{1000} X_i \sim \overline{B}(1000, 1/2)$. In particular,

$$P\left(\sum_{i=1}^{1000} X_i = 2200\right) = \binom{2200-1}{1200} \cdot 1/2^{1000} \cdot 1/2^{1200}$$
$$= \binom{2199}{1200} \cdot 1/2^{2200},$$

so that:

$$P(Y_1 + Y_2 + \ldots + Y_{1000} = 1200) = \frac{\binom{2199}{1200}}{2^{2200}}.$$

Thus, (i) is true.

(e) According to the solution of the preceding part, the total number S of daughters is distributed as $W - 1000$, where $W \sim \overline{B}(1000, 1/2)$. Hence

$$E(S) = E(W) - 1000 = 2 \cdot 1000 - 1000 = 1000.$$

Markov's Inequality implies:

$$P(S \geq 2500) \leq \frac{E(S)}{2500} = 0.4.$$

Thus, (iv) is true.

2. (a) Obviously, $Y_i \sim B(i, 1/i)$, $1 \leq i \leq n$, and in particular $Y_{1000} \sim B(1000, 1/1000)$. By the Poissonian approximation of the binomial distribution, Y_{1000} is distributed approximately $P(1)$. It follows that the required probability is approximately

$$\frac{P^2(Y_{1000} = 3)}{P(Y_{1000} = 6)} \approx \frac{\left(\frac{e^{-1}}{3!}\right)^2}{\frac{e^{-1}}{6!}} = \frac{20}{e}.$$

Thus, (v) is true.

(b) Let A be the event in question. Clearly,

$$P(A) = 1 - P(\overline{A}).$$

The event $\overline{A}$ is equivalent to the event studied in the absent-minded secretary problem, and its probability for large values of n is approximately $1/e$. Hence:

$$P(A) \approx 1 - 1/e \approx 0.63.$$

Thus, (ii) is true.

(c) Let B be the event in question. Clearly, the number 100 can appear only at the 100-th stage, the number 99 – at the 99-th or 100-th stages, and the number 98 – at the 98-th, 99-th or 100-th stages. For $n = 98, 99, 100$, let E_n be the event whereby none of the above three numbers have been selected at the n-th stage. Thus,

$$\begin{aligned} P(B) &= P(E_{100}) \cdot P(E_{99}) \cdot P(E_{98}) \\ &= \left(\frac{97}{100}\right)^{100} \cdot \left(\frac{97}{99}\right)^{99} \cdot \left(\frac{97}{98}\right)^{98} \\ &= \left(1 - \frac{3}{100}\right)^{100} \cdot \left(1 - \frac{2}{99}\right)^{99} \cdot \left(1 - \frac{1}{98}\right)^{97} \\ &\approx e^{-3} \cdot e^{-2} \cdot e^{-1} = e^{-6}. \end{aligned}$$

Thus, (iv) is true.

(d) Let X_i be the i-th number selected at the 15-th stage and

$$I_i = \begin{cases} 1, & X_i = 15, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the variables X_i, $1 \le i \le 15$, are independent $U[1, 15]$-distributed, and the variables I_i, $1 \le i \le 15$, are independent $B(1, 1/2)$-distributed. In terms of these variables, $S_{15} = \sum_{i=1}^{15} X_i$ and $Y_{15} = \sum_{j=1}^{15} I_j$. Hence:

$$\begin{aligned} \mathrm{Cov}(S_{15}, Y_{15}) &= \mathrm{Cov}\left(\sum_{i=1}^{15} X_i, \sum_{j=1}^{15} I_j\right) \\ &= \sum_{i=1}^{15} \mathrm{Cov}(X_i, I_i) + \sum_{i=1}^{15} \sum_{1 \le j \ne i \le 15} \mathrm{Cov}(X_i, I_j). \end{aligned}$$

Since for $i \neq j$ the variables X_i and I_j are independent, $\text{Cov}(X_i, I_j) = 0$. Therefore:

$$\text{Cov}(S_{15}, Y_{15}) \;=\; \sum_{i=1}^{15} \text{Cov}(X_i, I_i) = 15 \cdot \text{Cov}(X_1, I_1). \qquad (8)$$

Since $E(X_1) = (1+15)/2 = 8$, and $E(I_1) = 1/15$, and

$$E(X_1 \cdot I_1) = 15 \cdot \frac{1}{15} = 1,$$

we obtain

$$\text{Cov}(X_1, I_1) = E(X_1 \cdot I_1) - E(X_1)E(I_1) = 1 - 8/15 = 7/15.$$

Substituting $\text{Cov}(X_1, I_1)$ in (8), we get

$$\text{Cov}(S_{15}, Y_{15}) = 15 \cdot 7/15 = 7.$$

Thus, (i) is true.

(e) For $1 \leq i \leq 24$, let $X_{ji},\ \ 1 \leq j \leq i$, be the j-th number selected at the i-th stage. For an arbitrary fixed stage i, the random variables $X_{ji},\ \ 1 \leq j \leq i$, are independent and $U[1, i]$-distributed. In these terms, $S_i = \sum_{j=1}^{i} X_{ji},\ \ 1 \leq i \leq 24$, and thus

$$E(S_i) = \sum_{j=1}^{i} E(X_{ji}) = i \cdot E(X_{1i}) = \frac{i(i+1)}{2} = \binom{i+1}{2},$$

and

$$V(S_i) = \sum_{j=1}^{i} V(X_{ji}) = i \cdot V(X_{1i}) = i \cdot \frac{(i-1+1)^2 - 1}{12} = \frac{1}{2}\binom{i+1}{3}.$$

Since

$$S = \sum_{i=1}^{24} S_i, \qquad (9)$$

we obtain

$$E(S) = \sum_{i=1}^{24} E(S_i) = \sum_{i=1}^{24} \binom{i+1}{2} = \binom{26}{3} = 2600,$$

and

$$V(S) = \sum_{i=1}^{24} V(S_i) = \frac{1}{2}\sum_{i=1}^{24}\binom{i+1}{3} = \frac{1}{2}\binom{26}{4} = 7475.$$

Chebyshev's Inequality yields:

$$P(|S-2600| \le \varepsilon) \ \ge \ 1 - \frac{7475}{\varepsilon^2}.$$

In particular, for $\varepsilon = 250$:

$$\begin{aligned} P(2350 \le S_{1000} \le 2850) &= P(|S-2600| \le 250) \\ &\ge 1 - \frac{7475}{250^2} \approx 0.88. \end{aligned}$$

Thus, (iii) is true.

3. (a) For an arbitrary fixed stage $n \ge 1$, let A_1 be the event whereby both Reuven gets 1 shekel from Shimon and Shimon gets 1 shekel from Reuven at this stage, and let A_2 be the event whereby neither Reuven nor Shimon get money at this stage. In these terms

$$\{X_n = 0\} = A_1 \cup A_2, \qquad n \ge 1.$$

Clearly, $A_1 \cap A_2 = \emptyset$ and therefore

$$P(X_n = 0) = P(A_1) + P(A_2) = \frac{1}{4}\cdot\frac{1}{6} + \frac{3}{4}\cdot\frac{5}{6} = \frac{2}{3}.$$

Thus, (ii) is true.

(b)

$$P(X_{n+1} = 0 | X_n = 0) = \frac{P(X_n = X_{n+1} = 0)}{P(X_n = 0)}.$$

The values of X_n and X_{n+1} depend on 5 tosses $-$ the tosses of the coin at stages $n-1$, n, $n+1$, and those of the die at stages n and $n+1$. Denote by (c_1, c_2, c_3, a, b) the event whereby the coin shows $c_1, c_2, c_3 \in \{H, T\}$ and the die shows $a, b \in \{6, \overline{6}\}$ at the relevant stages. (Here $\overline{6}$ means any result but 6.) A component will assume the value $*$ when the event consists of both options for it. It is easy to verify that

$$\{X_n = X_{n+1} = 0\} = (*, T, *, \overline{6}, \overline{6}) \cup (T, H, T, \overline{6}, \overline{6}) \cup (H, H, T, 6, \overline{6}) \cup \\ \cup (T, H, H, \overline{6}, 6) \cup (H, H, H, 6, 6).$$

Therefore:

$$P(X_n = X_{n+1} = 0) = \frac{1}{2} \cdot \frac{25}{36} + \frac{1}{8} \cdot \frac{25}{36} + \frac{1}{8} \cdot \frac{5}{36} + \frac{1}{8} \cdot \frac{5}{36} + \frac{1}{8} \cdot \frac{1}{36} = \frac{17}{36}.$$

Hence, by the previous part:

$$P(X_{n+1} = 0 | X_n = 0) = \frac{17/36}{2/3} = \frac{17}{24}.$$

Thus, (ii) is true.

(c) One can easily verify that the probability function of each X_n is given by the following table:

x	-1	0	1
p	$1/8$	$2/3$	$5/24$

Thus,

$$E(X_n) = (-1) \cdot \frac{1}{8} + 0 \cdot \frac{2}{3} + 1 \cdot \frac{5}{24} = \frac{1}{12},$$

and

$$V(X_n) = E(X_n^2) - E^2(X_n) = \frac{1}{8} + \frac{5}{24} - \left(\frac{1}{12}\right)^2 = \frac{47}{144}.$$

Thus, (iii) is true.

(d) $\bar{X}_1, \bar{X}_2, \ldots$ are dependent. Indeed, for example, $P(\bar{X}_1 = 1, \bar{X}_2 = -1) = P(X_1 = 1, X_1 = X_2 = -1) = 0$, while $P(\bar{X}_1 = 1) = P(X_1 = 1) = 1/8$, and $P(\bar{X}_2 = -1) = P(X_1 = X_2 = -1) > 0$. In fact, to show the last inequality, we calculate the left-hand side. The values of X_1 and X_2 depend on 5 tosses – the tosses of the coin at stages 0, 1, 2, and those of the die at stages 1 and 2. As in part (b), denote by (c_1, c_2, c_3, a, b) the event whereby the coin shows $c_1, c_2, c_3 \in \{H, T\}$ and the die shows $a, b \in \{6, \overline{6}\}$ at the relevant stages. It is easy to verify that

$$\{X_1 = X_2 = -1\} \;=\; (*, T, *, 6, 6) \cup (T, H, T, 6, 6).$$

Therefore:

$$P(\bar{X}_2 = -1) = P(X_1 = X_2 = -1) = \frac{1}{2} \cdot \frac{1}{36} + \frac{1}{8} \cdot \frac{1}{36} = \frac{5}{288}.$$

Hence:

$$P(\bar{X}_1 = 1, \bar{X}_2 = -1) \neq P(\bar{X}_1 = 1) \cdot P(\bar{X}_2 = -1).$$

In general, any two adjacent random variables X_n and X_{n+1}, $n \geq 1$, are dependent, but X_n and X_{n+j}, $\;n \geq 1, \;\; j \geq 2$, are independent, since they depend on distinct tosses. Now, rewriting $\bar{X}_n$ in the form

$$\bar{X}_n = \frac{1}{2} \cdot \frac{X_1 + X_3 + \ldots + X_{n-1}}{n/2} + \frac{1}{2} \cdot \frac{X_2 + X_4 + \ldots + X_n}{n/2} \tag{10}$$

for even n and in the form

$$\bar{X}_n = \frac{n+1}{2n} \cdot \frac{X_1 + X_3 + \ldots + X_n}{(n+1)/2} + \frac{n-1}{2n} \cdot \frac{X_2 + X_4 + \ldots + X_{n-1}}{(n-1)/2} \tag{11}$$

for odd n, one can easily prove that $\bar{X}_n$ converges to $E(X_1) = 1/12$, and therefore (iv) is true. We now prove that $(\bar{X}_n)_{n=1}^{\infty}$ satisfies the law of large numbers. Denote:

$$Y_n = \frac{1}{n} \cdot \left(\bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_n\right).$$

Clearly, $E(Y_n) = E(X_1) = 1/12$ and $Y_n = \frac{\sum_{i=1}^{n} c_i X_i}{n}$, where $c_i = \sum_{j=i}^{n} \frac{1}{i}, \;\; 1 \leq i \leq n$. Since $\mathrm{Cov}(X_n, X_{n+j}) = 0$ for $n \geq 1, \; j \geq 2$, and $c_1 \geq c_2 \geq \ldots \geq c_n$,

$$\begin{aligned}
V(Y_n) &= \frac{1}{n^2}\left(\sum_{i=1}^{n} c_i^2 V(X_i) + 2 \sum_{1 \leq i < j \leq n} c_i c_j \mathrm{Cov}(X_s, X_k)\right) \\
&\leq \frac{c_1^2}{n^2}\left(nV(X_1) + 2\sum_{i=1}^{n-1} \mathrm{Cov}(X_i, X_{i+1})\right) \\
&\leq \frac{c_1^2}{n^2}\left(nV(X_1) + 2(n-1)\mathrm{Cov}(X_1, X_2)\right).
\end{aligned}$$

Now $c_1 = \sum_{j=1}^{n} \frac{1}{i} \leq 1 + \ln n$, and therefore:

$$V(Y_n) \leq \frac{(1+\ln n)^2}{n} \left(V(X_1) + 2 \cdot \mathrm{Cov}(X_1, X_2) \right).$$

Thus

$$V(\bar{S}_n) \longrightarrow 0, \qquad n \longrightarrow \infty,$$

and Chebyshev's Inequality implies that $(\bar{X}_n)$ satisfies the weak law of large numbers. Thus, (iv) is the only true claim.

(e) Denote $Y_i = X_{2i}, \quad 1 \leq i \leq 4176$. With this notation, $X = \sum_{i=1}^{4176} Y_i$. Obviously, $Y_i, \quad 1 \leq i \leq 4176$, are independent random variables with $E(Y_i) = E(X_2) = 1/12$ and $V(Y_i) = V(X_2) = 47/144$. Hence

$$\begin{aligned} P(290 \leq X \leq 377) &= P\left(\frac{290 - 4176 \cdot 1/12}{\sqrt{4176 \cdot 47/144}} \leq \frac{X - 4176 \cdot 1/12}{\sqrt{4176 \cdot 47/144}} \leq \frac{377 - 4176 \cdot 1/12}{\sqrt{4176 \cdot 47/144}} \right) \\ &\approx P(-1.57 \leq Z \leq 0.78), \end{aligned}$$

where Z is a standard normal variable. Therefore:

$$\begin{aligned} P(290 \leq X \leq 377) &\approx \Phi(0.78) - \Phi(-1.57) \\ &= 0.7823 - 0.0582 = 0.7241. \end{aligned}$$

Thus, (iii) is true.

4. (a) Let us first find the distribution function F_H of H. Clearly, $X \sim U(0,1)$ and $H = \sqrt{X^2 + (1-X)^2}$. Obviously, $\frac{\sqrt{2}}{2} \leq H \leq 1$. Hence, $F_H(t) = 0$ for $t < \frac{\sqrt{2}}{2}$ and $F_H(t) = 1$ for $t > 1$. For

$t \in [\sqrt{2}/2, 1]:$

$$\begin{aligned}
F_H(t) &= P(\sqrt{X^2 + (1-X)^2} \le t) \\
&= P(X^2 + (1-X)^2 \le t^2) \\
&= P\left(X^2 - X + \frac{1-t^2}{2} \le 0\right) \\
&= P\left(\frac{1-\sqrt{2t^2-1}}{2} \le X \le \frac{1+\sqrt{2t^2-1}}{2}\right) \\
&= F_X\left(\frac{1+\sqrt{2t^2-1}}{2}\right) - F_X\left(\frac{1-\sqrt{2t^2-1}}{2}\right) \\
&= \frac{1+\sqrt{2t^2-1}}{2} - \frac{1-\sqrt{2t^2-1}}{2} \\
&= \sqrt{2t^2-1}.
\end{aligned}$$

Therefore, the density function of H is

$$f_H(t) = (F_H(t))' = \begin{cases} \frac{2t}{\sqrt{2t^2-1}}, & \frac{\sqrt{2}}{2} \le t \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, (i) is true.

(b) Since $S_\triangle = \frac{X(1-X)}{2}$ and $X \sim U(0,1)$, we have $0 \le S_\triangle \le \frac{1}{8}$. Hence,

$F_{S_\triangle}(t) = 0$ for $t < 0$ and $F_{S_\triangle}(t) = 1$, for $t > \frac{1}{8}$. For $t \in [0, 1/8]$:

$$\begin{aligned}
F_{S_\triangle}(t) &= P\left(\frac{X(1-X)}{2} \leq t\right) \\
&= P\left(X^2 - X + 2t \geq 0\right) \\
&= P\left(X \leq \frac{1-\sqrt{1-8t}}{2}\right) + P\left(X \geq \frac{1+\sqrt{1-8t}}{2}\right) \\
&= F_X\left(\frac{1-\sqrt{1-8t}}{2}\right) + 1 - F_X\left(\frac{1+\sqrt{1-8t}}{2}\right) \\
&= \frac{1-\sqrt{1-8t}}{2} + 1 - \frac{1+\sqrt{1-8t}}{2} \\
&= 1 - \sqrt{1-8t}.
\end{aligned}$$

Thus, (iii) is true.

(c) First of all, it will be convenient to rewrite $S_\triangle$ in the following form:

$$\begin{aligned}
S_\triangle &= \frac{X(1-X)}{2} \\
&= \frac{1}{2} \cdot \left(-\left(X - \frac{1}{2}\right)^2 + \frac{1}{4}\right) \\
&= \frac{1}{2} \cdot \left(-\left(X - E(X)\right)^2 + \frac{1}{4}\right).
\end{aligned}$$

Thus:

$$\begin{aligned}
E(S_\triangle) &= E\left(\frac{1}{2} \cdot \left(-\left(X - E(X)\right)^2 + \frac{1}{4}\right)\right) \\
&= \frac{1}{2} \cdot \left(-E\left((X - E(X))^2\right) + \frac{1}{4}\right) \\
&= \frac{1}{2} \cdot \left(-V(X) + \frac{1}{4}\right) \\
&= \frac{1}{2} \cdot \left(-\frac{1}{12} + \frac{1}{4}\right) = \frac{1}{12}.
\end{aligned}$$

Note that by part (b) we can also calculate $E(S_\triangle)$ by the definition.

Thus, (iv) is true.

(d) One can easily see that

$$S_\square = H^2 = X^2 + (1-X)^2 = 1 - 4 \cdot \frac{X(1-X)}{2} = 1 - 4S_\triangle,$$

which yields

$$\rho(S_\triangle, S_\square) = -1.$$

Thus, (i) is true.

(e) A linear approximation near 0 yields:

$$\begin{aligned}
\psi(0.003) &\approx \psi(0) + \psi'(0) \cdot 0.003 \\
&= 1 + E(S_\square) \cdot 0.003 \\
&= 1 + (1 - 4 \cdot E(S_\triangle)) \cdot 0.003 \\
&= 1 + (1 - 4 \cdot 1/12) \cdot 0.003 = 1.002.
\end{aligned}$$

Thus, (iii) is true.