# Final #2

Mark the correct answer in each part of the following questions.

1. We toss a coin over and over until it shows a head for the first time. At each stage we also select a random number (uniformly distributed) between 0 and 1. Consider the sequence of random numbers thus obtained.

   (a) The probability that the sequence is increasing is

   (i) 1/3.
   (ii) 1/2.
   (iii) $\sqrt{e} - 1$.
   (iv) $\ln 2$.
   (v) none of the above.

   (b) The probability that the last number in the sequence is the largest is

   (i) 1/3.
   (ii) 1/2.
   (iii) $\sqrt{e} - 1$.
   (iv) $\ln 2$.
   (v) none of the above.

2. Two drunkards – one positively-oriented and the other negatively-oriented – leave the WWW (Water⟶Wine⟶Whisky) bar, located at the origin of the $x$-axis, at the same time. The positively-oriented drunkard makes at every second either a step in the positive direction or in the negative direction, with probabilities 2/3 and 1/3, respectively. The other moves similarly, but with reversed probabilities.

(a) The probability that after $n$ seconds the two are at the same point is

(i) $\binom{2n}{n}(1/9)^n$.

(ii) $\binom{2n}{n}(1/8)^n$.

(iii) $\binom{2n}{n}(2/9)^n$.

(iv) $\binom{2n}{n}(1/4)^n$.

(v) none of the above.

(b) The probability that after 15 minutes the positively-oriented drunkard is at least 640 steps to the right of the negatively-oriented one lies in the interval:

(i) $[0, 0.2)$.

(ii) $[0.2, 0.4)$.

(iii) $[0.4, 0.6)$.

(iv) $[0.6, 0.8)$.

(v) $[0.9, 1]$.

(c) It is given that after 15 minutes the positively-oriented drunkard is at the point 200 on the axis. The probability that throughout his walk he never got to the negative axis is

(i) 201/901.

(ii) 201/551.

(iii) 201/351.

(iv) 201/301.

(v) none of the above.

3. Consider Banach's matchbox problem.

(a) Suppose that, unlike the version studied in class, the person does not have the same number of matches in his pockets, but rather $M$ matches in his right pocket and $N$ in his left. The probability that, when he realizes one of the pockets is empty, the other pocket contains exactly $k$ matches is

(i) $\left(\binom{M+N-k}{M} + \binom{M+N-k}{N}\right)\left(\frac{1}{2}\right)^{M+N-k+1}$.

(ii) $\left(\binom{M+N-k}{M}+\binom{M+N-k}{N}\right)\left(\frac{1}{2}\right)^{M+N-k}$.

(iii) $\left(\binom{M+N-k}{M+1}+\binom{M+N-k}{N+1}\right)\left(\frac{1}{2}\right)^{M+N-k+1}$.

(iv) $\left(\binom{M+N-k}{M+1}+\binom{M+N-k}{N+1}\right)\left(\frac{1}{2}\right)^{M+N-k}$.

(v) none of the above.

(b) Now suppose, as in class, that each pocket contains initially $N$ matches. However, when he looks for a match, he tries the right pocket with probability 2/3 and the left one with probability 1/3. The probability that, when he discovers one of the pockets is empty, the other pocket contains exactly $k$ matches is

(i) $\binom{2N-k}{N}2^{N+1}/3^{2N-k+1}$.

(ii) $\binom{2N-k}{N}\left(2^{N+1}+2^{N-k}\right)/3^{2N-k+1}$.

(iii) $\binom{2N-k}{N}\left(2^{N+2}-2^{N-k}\right)/3^{2N-k+1}$.

(iv) $\binom{2N-k}{N}2^{N+2}/3^{2N-k+1}$.

(v) None of the above.

(c) Now suppose that the person has three pockets with $N$ matches in each at the beginning, and he searches each of them with a probability of 1/3. The probability that, when he discovers one of the pockets is empty, the other two are empty as well, is

(i) $\binom{2N}{N}/3^{2N+1}$.

(ii) $\binom{2N}{N}/3^{2N}$.

(iii) $\binom{3N}{N,N,N}/3^{3N+1}$.

(iv) $\binom{3N}{N,N,N}/3^{3N}$.

(v) none of the above.

(d) Now suppose he has two pockets, with an infinite number of matches in each. Let $X$ be the number of the trial at which he searches his right pocket for the first time and $Y$ the analogous quantity for the left pocket. Then $\rho(X, Y)$ lies in the interval

(i) $[-1, -0.6)$.
(ii) $[-0.6, -0.2)$.
(iii) $[-0.2, 0.2)$.
(iv) $[0.2, 0.6)$.
(v) $[0.6, 1]$.

4. (a) Consider the following four statements:

(A) If $X$ is a discrete uniform random variable, then so is $2X$.
(B) If $X$ is a continuous uniform random variable, then so is $2X$.
(C) If $X$ is an exponential random variable, then so is $2X$.
(D) If $X$ is a normal random variable, then so is $2X$.

(i) (B),(C), and (D) are true, but (A) is false.
(ii) Only (D) is true.
(iii) Only (B) and (D) are true.
(iv) All four statements are true.
(v) None of the above.

(b) Consider the following four statements, all relating to a random variable $X$ that assumes only non-negative values:

(A) If $X$ is memory-less, then so is $2X$.
(B) If $X$ is memory-less, then so is $X^2$.
(C) If $X \sim U[0, a]$, then $X$ is memory-less.
(D) If $X \sim U(0, a)$, then $X$ is memory-less.

(i) Only (A) is true.
(ii) Only (A) and (B) are true.
(iii) Only (A) and (D) are true.
(iv) (A),(C), and (D) are true, but (B) is false.
(v) None of the above.

5. Let us say (for the purpose of this question only) that a non-negative random variable $X$ *satisfies Markov's Inequality* if there exists a constant $C > 0$ such that $P(X \geq a) \leq C/a$ for every $a > 0$. Similarly, a (not necessarily non-negative) random variable $X$ with expectation $\mu$ *satisfies Chebyshev's Inequality* if there exists a constant $C > 0$ such that $P(|X - \mu| \geq \varepsilon) \leq C/\varepsilon^2$ for every $\varepsilon > 0$.

(a) $X_1, X_2, X_3$ are random variables with distribution functions $F_1, F_2, F_3$, respectively, given by:

$$F_1(x) = \begin{cases} 1 - \frac{1}{\sqrt{x}}, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$F_2(x) = \begin{cases} 1 - \frac{1}{x}, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$F_3(x) = \begin{cases} 1 - \frac{1}{x^2}, & x \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

(i) All three random variables have finite expectations, and in particular all of them satisfy Markov's Inequality.

(ii) Out of the three random variables, only $X_3$ has a finite expectation, and it is the only one satisfying Markov's Inequality.

(iii) Out of the three random variables, only $X_3$ has a finite expectation, yet $X_2$ also satisfies Markov's Inequality.

(iv) $X_2$ and $X_3$ have finite expectations. $X_1$ does not have a finite expectation, nor does it satisfy Markov's Inequality.

(v) None of the above.

(b) $X_1, X_2, X_3$ are random variables with density functions $f_1, f_2, f_3$, respectively, given by:

$$f_1(x) = \theta|x|e^{-x^2}, \qquad -\infty < x < \infty, \ (\theta > 0),$$
$$f_2(x) = \begin{cases} \frac{1}{|x|^3}, & |x| \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$f_3(x) = \begin{cases} \frac{1}{|x|^{5/2}}, & |x| \geq 1, \ (\theta > 0), \\ 0, & \text{otherwise}. \end{cases}$$

(i) All three random variables satisfy Chebyshev's Inequality.

(ii) $X_1$ and $X_2$ satisfy Chebyshev's Inequality, whereas $X_3$ does not.

(iii) $X_2$ and $X_3$ satisfy Chebyshev's Inequality, whereas $X_1$ does not.

(iv) $X_1$ satisfies Chebyshev's Inequality, whereas $X_2$ and $X_3$ do not.

(v) None of the above.

6. The two-dimensional density function of a continuous random variable $(X, Y)$ is defined by:

$$f_{XY}(x,y) = \begin{cases} C(3+2x-y), & -1 \le x \le 1, -1 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $C =$

(i) 1/24.

(ii) 1/18.

(iii) 1/16.

(iv) 1/12.

(v) none of the above.

(b) $P(X > 0|Y < 0) =$

(i) 3/7.

(ii) 1/2.

(iii) 4/7.

(iv) 9/14.

(v) none of the above.

(c) $P(XY > 0) =$

(i) $2C$.

(ii) $3C$.

(iii) $6C$.

(iv) $9C$.

(v) none of the above.

(d) $\rho(X, Y) =$

(i) $-1/\sqrt{299}$.

(ii) 0.

(iii) $\sqrt{2/299}$.

(iv) $2/\sqrt{299}$.

(v) none of the above.

(e) The value of the moment generating function of $X$ at the point 1 is

(i) $C(4e + 4/e)$.

(ii) $C(4e + 2/e)$.

(iii) $C(6e + 4/e)$.

(iv) $C(6e + 2/e)$.

(v) none of the above.

# Solutions

1. (a) Let $X$ be the number of tosses of the coin until it shows a head for the first time. Obviously, $X \sim G\left(\frac{1}{2}\right)$. Denote by $A$ the event whereby the sequence of random numbers is increasing. By the law of total probability:

$$P(A) = \sum_{k=1}^{\infty} P(A|X=k) \cdot P(X=k).$$

Since at each stage we select a random number from the continuous distribution, then the probability of the two (or more) random numbers being equal is 0. Hence, by symmetry, $P(A|X=k) = \frac{1}{k!}$. Therefore:

$$P(A) = \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \left(\frac{1}{2}\right)^k = e^{1/2} - 1.$$

Thus, (iii) is true.

(b) Let $X$ be as in the previous part. Denote by $B$ the event whereby the last number in the sequence is the largest. By the law of total probability

$$\begin{aligned} P(B) &= \sum_{k=1}^{\infty} P(B|X=k) \cdot P(X=k) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \left(\frac{1}{2}\right)^k \\ &= -\ln 1/2 = \ln 2. \end{aligned}$$

Thus, (iv) is true.

2. (a) Suppose during $n$ seconds the positively-oriented drunkard makes $k$ steps in the positive direction and $n-k$ in the negative direction. To arrive at the same point on the $x$-axis, the negatively-oriented

drunkard should also make $k$ steps in the positive direction and $n-k$ in the negative one.

Therefore, denoting by $A$ the event whereby after $n$ seconds the two will be at the same point:

$$\begin{aligned} P(A) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{n-k} \cdot \binom{n}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k} \\ &= \left(\frac{2}{9}\right)^n \sum_{k=0}^{n} \binom{n}{k}^2 \\ &= \left(\frac{2}{9}\right)^n \binom{2n}{n}. \end{aligned}$$

Thus, (iii) is true.

(b) For $1 \leq i \leq 900$, let $X_i = 1$ if the $i$-th step of the positively-oriented drunkard is in the positive direction and $X_i = -1$ otherwise. Let $Y_i$ be the analogous random variable for the negatively-oriented drunkard. Obviously, the variables $X_1, Y_1, \ldots, X_{900}, Y_{900}$ are independent and

$$P(X_i = -1) = \frac{1}{3}, \quad P(X_i = 1) = \tfrac{2}{3},$$
$$P(Y_i = -1) = \frac{2}{3}, \quad P(Y_i = 1) = \tfrac{1}{3}.$$

Obviously, for each $i$ we have:

$$E(X_i) = \frac{1}{3}, \quad E(Y_i) = -\frac{1}{3}, \quad V(X_i) = V(Y_i) = \frac{8}{9}.$$

Denote by $X = \sum_{i=1}^{900} X_i$ and $Y = \sum_{i=1}^{900} Y_i$ the location of the positively-oriented and the negatively-oriented drunkard, respectively, after 900 seconds. With these notations:

$$P(X - Y \geq 640) \quad = \quad P\left(\sum_{i=1}^{900} (X_i - Y_i) \geq 640\right).$$

Clearly, $E(X-Y) = \sum_{i=1}^{900} (E(X_i) - E(Y_i)) = 900 \cdot \frac{2}{3} = 600$ and $V(X-Y) = \sum_{i=1}^{900} (V(X_i) + V(Y_i)) = 900 \cdot \left(\frac{8}{9} + \frac{8}{9}\right) = 1600$. Now by the Central Limit Theorem:

$$
\begin{aligned}
P(X-Y \geq 640) &= P\left(\frac{\sum_{i=1}^{900}(X_i - Y_i) - 600}{\sqrt{1600}} \geq \frac{640-600}{\sqrt{1600}}\right) \\
&\approx 1 - \Phi(1) \\
&\approx 0.1587.
\end{aligned}
$$

Thus, (i) is true.

(c) Let $L$ be event whereby the positively-oriented drunkard never gets to the negative axis, given that after 900 steps he is located at the point 200 on the axis. $L$ corresponds to the event considered in the Ballot Problem, with total number of votes for both candidates $m+n=900$, while the first candidate obtains $m-n=200$ votes more than the second. Namely, if in the ballot the first candidate gets $m=550$ votes and the second gets $n=350$ votes, then the required probability is the probability that the second candidate never leads throughout the counting process. Hence the required probability it is $\frac{m-n+1}{m+1} = \frac{201}{551}$.
Thus, (ii) is true.

3. (a) Let $A_L$ be event whereby the left pocket will be found empty at the moment when the right one contains exactly $k$ matches. In this case let us identify a "success" with choosing the left pocket. Hence $A_L$ occurs if and only if exactly $M-k$ failures precede the $(N+1)$-st success. Hence:

$$
\begin{aligned}
P(A_L) &= \binom{M-k+N+1-1}{M-k}\left(\frac{1}{2}\right)^{M+N-k+1} \\
&= \binom{M+N-k}{N}\left(\frac{1}{2}\right)^{M+N-k+1}.
\end{aligned}
$$

Similarly, let $A_R$ be event whereby the right pocket will be found empty at the moment when the left one contains exactly $k$ matches. Now let us identify a "success" with choosing the right pocket.

Hence $A_R$ occurs if and only if exactly $N-k$ failures precede the $(M+1)$-st success. Hence:

$$\begin{aligned} P(A_R) &= \binom{N-k+M+1-1}{N-k}\left(\frac{1}{2}\right)^{M+N-k+1} \\ &= \binom{M+N-k}{M}\left(\frac{1}{2}\right)^{M+N-k+1}. \end{aligned}$$

Therefore, the probability that, when the person realizes one of the pockets is empty, the other pocket contains exactly $k$ matches, is

$$P(A_L)+P(A_R) = \frac{\binom{M+N-k}{M}+\binom{M+N-k}{N}}{2^{M+N-k+1}}.$$

Thus, (i) is true.

(b) Let $A_L$ and $A_R$ be as defined in the previous part. Since $M=N$:

$$\begin{aligned} P(A_L) &= \binom{2N-k}{N}\left(\frac{1}{3}\right)^{N+1}\cdot\left(\frac{2}{3}\right)^{N-k} \\ &= \binom{2N-k}{N}\frac{2^{N-k}}{3^{2N-k+1}}. \end{aligned}$$

Similarly,

$$\begin{aligned} P(A_R) &= \binom{2N-k}{N}\left(\frac{2}{3}\right)^{N+1}\cdot\left(\frac{1}{3}\right)^{N-k} \\ &= \binom{2N-k}{N}\frac{2^{N+1}}{3^{2N-k+1}}. \end{aligned}$$

Therefore the probability that, when the person realizes one of the pockets is empty, the other pocket contains exactly $k$ matches, is

$$P(A_L)+P(A_R) = \binom{2N-k}{N}\frac{2^{N+1}+2^{N-k}}{3^{2N-k+1}}.$$

Thus, (ii) is true.

(c) To discover that a pocket is empty, when the other two are empty as well, the person needs first to do $3N$ searches, $N$ in each pocket, and then at the $(3N+1)$-st search he will in any case find an empty pocket. Hence the required probability is $\frac{\binom{3N}{N,N,N}}{3^{3N}}$.
Thus, (iv) is true.

(d) Obviously, $X \sim G\left(\frac{1}{2}\right)$ and $Y \sim G\left(\frac{1}{2}\right)$. Therefore

$$E(X) = E(Y) = 2$$

and

$$V(X) = V(Y) = 2.$$

However, $X$ and $Y$ are not independent. In fact, $P(X = 1, Y = 1) = 0$ and $P(X = 1, Y = i) = P(X = i, Y = 1) = \left(\frac{1}{2}\right)^i$ for $i > 1$, and $P(X = i, Y = j) = 0$ for $i, j > 1$. Hence:

$$\begin{aligned} E(X \cdot Y) &= 2\sum_{i=2}^{\infty} i \cdot \left(\frac{1}{2}\right)^i \\ &= \sum_{i=1}^{\infty} i \cdot \left(\frac{1}{2}\right)^{i-1} - 1 = 3. \end{aligned}$$

Therefore

$$\rho(X, Y) = \frac{E(X \cdot Y) - E(X) \cdot E(Y)}{\sqrt{V(X) \cdot V(Y)}} = \frac{3 - 2 \cdot 2}{2} = -\frac{1}{2}.$$

Thus, (ii) is true.

4. (a) Obviously, (A) is wrong. For example, if $X \sim U[0, 1]$ then $P(X = 0) = P(X = 1) = \frac{1}{2}$, while $P(2X = 0) = P(2X = 2) = \frac{1}{2} \neq 0 = P(2X = 1)$, so that $2X$ is not a discrete uniform random variable. However, all other parts (B)-(D) are correct. These follows from the properties of the relevant distributions studied in class.
Thus, (i) is true.

(b) Part (A) is correct. Indeed, suppose that $X$ is memory-less, namely, $P(X > t + s|X > t) = P(X > s)$ for $t, s \geq 0$. Therefore:

$$\begin{aligned} P(2X > t + s \mid 2X > t) &= P\left(X > (t+s)/2 \mid X > t/2\right) \\ &= P\left(X > s/2\right) \\ &= P\left(2X > s\right). \end{aligned}$$

Therefore $2X$ is also memory-less.
However, all other parts (B)-(D) are wrong. In particular, if $X \sim \text{Exp}(1)$ then (as was shown in class) $X$ is memory-less. However, for arbitrary $t, s \geq 0$:

$$\begin{aligned} P(X^2 > t + s \mid X^2 > t) &= \frac{P(X > \sqrt{t+s})}{P(X > \sqrt{t})} \\ &= \frac{e^{-\sqrt{t+s}}}{e^{-\sqrt{t}}} \\ &= e^{-\sqrt{t+s}+\sqrt{t}} \\ &\neq e^{-\sqrt{s}} \\ &= P(X > \sqrt{s}) = P(X^2 > s). \end{aligned}$$

Thus, (B) is false.
Now, if $X \sim U[0, a]$, then, in particular, for $t = a - 2$ and $s = 1$ we have:

$$\begin{aligned} P(X > a - 1 \mid X > a - 2) &= \frac{P(X > a-1)}{P(X > a-2)} \\ &= \frac{1/(a+1)}{2/(a+2)} = \frac{1}{2} \\ &\neq \frac{a-1}{a+1} = P(X > 1). \end{aligned}$$

Thus, (C) is also false.
Similarly, if $X \sim U(0, a)$, then

$$P(X > 3a/4 \mid X > a/2) = \frac{1}{2} \neq \frac{3}{4} = P(X > a/4).$$

Hence, (D) is false too.

Thus, (i) is true.

5. (a) Note that the density functions $f_1, f_2, f_3$ of $X_1, X_2, X_3$, respectively, are given by:

$$f_1(x) = \begin{cases} \frac{1}{2x^{3/2}}, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$f_2(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$f_3(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore

$$E(X_1) = \int_1^\infty t \cdot \frac{1}{2t^{3/2}} dt = \int_1^\infty \frac{1}{2\sqrt{t}} dt = \infty.$$

Similarly, $E(X_2) = \infty$, while $E(X_3) = \int_1^\infty \frac{2}{t^2} dt = 2$. Moreover,

$$P(X_3 \geq a) = 1 - F_3(a) = \begin{cases} \frac{1}{a^2}, & a \geq 1, \\ 1, & \text{otherwise}. \end{cases}$$

Taking $C = 1$ we obtain $P(X_3 \geq a) \leq C/a$ for every $a > 0$. Therefore $X_3$ satisfies Markov's Inequality. Similarly,

$$P(X_2 \geq a) = 1 - F_2(a) = \begin{cases} \frac{1}{a}, & a \geq 1, \\ 1, & \text{otherwise}. \end{cases}$$

Again, taking $C = 1$ we obtain $P(X_2 \geq a) \leq C/a$ for every $a > 0$. Therefore, $X_2$ also satisfies Markov's Inequality. However,

$$P(X_1 \geq a) = 1 - F_1(a) = \begin{cases} \frac{1}{\sqrt{a}}, & a \geq 1, \\ 1, & \text{otherwise}. \end{cases}$$

Hence $P(X_1 \geq a) \leq C/a$ if and only if $C \geq \sqrt{a}$. Therefore, $X_1$ does not satisfy Markov's Inequality, since there is no constant $C > 0$ such that $P(X_1 \geq a) \leq C/a$ for every $a > 0$.
Thus, (iii) is true.

(b) Obviously, all the density functions are even, and there is no problem with the existence of expectation for each random variable. Therefore, $E(X_1) = E(X_2) = E(X_3) = \mu = 0$. Moreover, for $X_1$ we have

$$\begin{aligned} V(X_1) &= E(X_1^2) = \int_{-\infty}^{\infty} x^2 f_{X_1}(x)dx \\ &= 2\int_0^{\infty} x^3 e^{-x^2}dx = 2\int_0^{\infty} te^{-t}dt = 1 < \infty, \end{aligned}$$

and in particular $X_1$ satisfies Chebyshev's Inequality.
Now:

$$P(|X_2| \geq \varepsilon) = 2\int_{\varepsilon}^{\infty} \frac{1}{x^3}dx = \frac{1}{\varepsilon^2}.$$

Therefore, $X_2$ also satisfies Chebyshev's Inequality with $C = 1$. However,

$$P(|X_3| \geq \varepsilon) = 2\theta\int_{\varepsilon}^{\infty} \frac{1}{x^{5/2}}dx = \frac{4\theta}{3}\frac{1}{\varepsilon^{1.5}}.$$

Hence $P(|X_3| \geq \varepsilon) \leq C/\varepsilon^2$ if and only if $C \geq \frac{4\theta}{3} \cdot \sqrt{\varepsilon}$. Therefore, $X_3$ does not satisfy Chebyshev's Inequality, since there is no constant $C > 0$ such that the inequality takes place for every $\varepsilon > 0$. Thus, (ii) is true.

6. (a)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_{-1}^{1}\int_{-1}^{1}(3+2x-y)dxdy \\ &= C \cdot 12. \end{aligned}$$

Hence $C = \frac{1}{12}$.
Thus, (iv) is true.

(b)

$$P(X>0|Y<0) = \frac{P(X>0,Y<0)}{P(Y<0)}$$

$$= \frac{C\int_{-1}^{0}\int_{0}^{1}(3+2x-y)dxdy}{C\int_{-1}^{0}\int_{-1}^{1}(3+2x-y)dxdy}$$

$$= \frac{9}{14}.$$

Thus, (iv) is true.

(c)

$$P(X\cdot Y>0) = C\int_{0}^{1}\int_{0}^{1}(3+2x-y)dxdy$$

$$+ C\int_{-1}^{0}\int_{-1}^{0}(3+2x-y)dxdy$$

$$= 6C.$$

Thus, (iii) is true.

(d) The marginal density function $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$= \begin{cases} C\int_{-1}^{1}(3+2x-y)dy, & -1\le x\le 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} \frac{3+2x}{6}, & -1\le x\le 1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-1}^{1} x\cdot\frac{3+2x}{6}dx = \frac{2}{9},$$

and

$$E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{3+2x}{6} dx = \frac{1}{3}.$$

Therefore

$$V(X) = E(X^2) - E^2(X) = \frac{1}{3} - \left(\frac{2}{9}\right)^2 = \frac{23}{81}.$$

Similarly, one can verify that

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \begin{cases} \frac{3-y}{6}, & -1 \le y \le 1, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

and $E(Y) = -\frac{1}{9}$, $E(Y^2) = \frac{1}{3}$ and $V(Y) = \frac{26}{81}$.

Moreover, from the previous part one can easily see that the density function of $X \cdot Y$ is even on the interval $[-1, 1]$, which implies that $E(X \cdot Y) = 0$. Therefore:

$$\rho(X,Y) = \frac{E(X \cdot Y) - E(X) \cdot E(Y)}{\sqrt{V(X) \cdot V(Y)}} = \frac{0 - \frac{2}{9} \cdot \left(-\frac{1}{9}\right)}{\sqrt{\frac{23}{81} \cdot \frac{26}{81}}} = \sqrt{\frac{2}{299}}.$$

Thus, (iii) is true.

(e)

$$\begin{aligned} \psi(1) &= E(e^X) \\ &= \int_{-\infty}^{\infty} e^x \cdot f_X(x)dx \\ &= \frac{1}{6}\left(3\int_{-1}^{1} e^x dx + 2\int_{-1}^{1} e^x x dx\right) \\ &= c\left(6e + \frac{2}{e}\right). \end{aligned}$$

Thus, (iv) is true.