Final #2

Mark the correct answer in each part of the following questions.

1. An urn contains an infinite number of spherical blue diamonds. The radius of the first (in centimeters) is $1/2$ and its value (in millions of shekels) is 1, the radius of the second is $1/4$ and its value is $1/2$, and in general the radius of the i-th is $1/2^i$ and its value is $1/i$. When drawing a random stone from the urn, the probability that it is any specific stone is the same as the radius of the stone.

   (a) A random stone is drawn. Denote by R its radius and by X its value. $V(R) =$

   (i) $2/45$.
   (ii) $2/63$.
   (iii) $2/75$.
   (iv) $2/81$.
   (v) None of the above.

   (b) $E(X) =$

   (i) $\log 2$.
   (ii) $1 - 1/e$.
   (iii) $1/\sqrt{2}$.
   (iv) $1/\sqrt{3}$.
   (v) None of the above.

   (c) Let $Y = 1/X$. Y is distributed

   (i) Binomially.
   (ii) Geometrically.
   (iii) Poisson.

(iv) Negative binomial, but not geometric.

(v) None of the above.

(d) Let $\rho_1 = \rho(Y, R)$. Then $\rho_1 =$

(i) $-\dfrac{\sqrt{2}}{3}$.

(ii) $-\dfrac{\sqrt{5}}{3}$.

(iii) $-\dfrac{\sqrt{7}}{3}$.

(iv) $-\dfrac{\sqrt{8}}{3}$.

(v) None of the above.

(e) Let R', X' be the radius and value of a random stone, measured in millimeters and shekels instead of centimeters and millions of shekels, respectively. Put

$$\rho_2 = \rho(X, R), \qquad C_2 = \mathrm{Cov}(X, R),$$

$$\rho_2' = \rho(X', R'), \qquad C_2' = \mathrm{Cov}(X', R').$$

(i) $\rho_2'/\rho_2 = C_2'/C_2$.

(ii) $\rho_2'/\rho_2 = 10^7 \cdot C_2'/C_2$.

(iii) $\rho_2'/\rho_2 = 10^{-7} \cdot C_2'/C_2$.

(iv) $\rho_2'/\rho_2 = 10^{-14} \cdot C_2'/C_2$.

(v) None of the above.

(f) We draw a random stone from the urn four times **with replacement**. Let N_i be the number of times the i-th stone will be drawn, $i = 1, 2$. Then $F_{N_1,N_2}(2, 2) =$

(i) $\dfrac{163}{2^8}$.

(ii) $\dfrac{165}{2^8}$.

(iii) $\dfrac{167}{2^8}$.

(iv) $\dfrac{169}{2^8}$.

(v) None of the above.

(g) We draw a random stone from the urn 1000 times **with replacement**. The probability for the 10-th stone to be drawn at least once is approximately

(i) $1 - \frac{2}{e}$.

(ii) $\frac{2}{e}$.

(iii) $1 - \frac{1}{e}$.

(iv) $\frac{1}{e}$.

(v) None of the above.

(h) We draw a random stone from the urn infinitely many times **with replacement**. With probability 1,

(i) Some stones will never be drawn.

(ii) Every stone will be drawn at least once, but not infinitely many times.

(iii) Every stone will be drawn at least once. Some stones will be drawn finitely many times and some stones will be drawn infinitely many times.

(iv) All stones will be drawn infinitely many times.

(v) None of the above.

2. Two players – A and B – roll a die each again and again. A rolls until getting a 6. B rolls until getting twice either 1 or 6 (including the possibility of getting once a 1 and once a 6). Let X and Y be the number of rolls of A and of B, respectively. (For example, if A gets $2, 5, 2, 6$ and B gets $3, 4, 3, 6, 2, 1$, then $X = 4$ and $Y = 6$.)

(a) $P(X = Y) =$

(i) $\frac{1}{64}$.

(ii) $\frac{3}{64}$.

(iii) $\frac{5}{64}$.

(iv) $\frac{7}{64}$.

(v) None of the above.

(b) Employing Chebyshev's Inequality, we obtain:

(i) $P(|X-Y| \geq 10) \leq 0.39$.

(ii) $P(|X-Y| \geq 10) \leq 0.4$.

(iii) $P(|X-Y| \geq 10) \leq 0.41$.

(iv) $P(|X-Y| \geq 10) \leq 0.42$.

(v) None of the above.

Remark: We mean the best bound that can be obtained. For example, if (i) is correct, then (ii)-(iv) are correct as well, but only (i) should be marked as the correct answer.

(c) Denote by N_3 and N_4 the number of times B rolls a 3 and a 4, respectively. (In the example above, $N_3 = N_4 = 0$.) Then:

(i) N_3 is distributed approximately Poisson.

(ii) $N_3 + 1$ is geometrically distributed.

(iii) $N_3 + 2$ is negative binomially distributed.

(iv) N_3, N_4 are independent.

(v) None of the above.

(d) Let S be the sum of outcomes of A's rolls. Then $E(S|X) =$

(i) $3X + 3$.

(ii) $7X/2 + 5/2$.

(iii) $3X + 6$.

(iv) $7X/2 + 6$.

(v) None of the above.

3. The amount of snow on a random day in Utqiaġvik, Alaska, is

- 0cm – with probability 0.5;
- 2cm – with probability 0.4;

- 3cm – with probability 0.1.

We choose n random days. Let R_n denote the number of days with snow out of these n days and S_n the total amount of snow in these days.

(a) $P(R_n = n | S_n = 3n - 3) =$

  (i) $\dfrac{64\binom{n}{3}}{64\binom{n}{3} + 5n}$.

  (ii) $\dfrac{64\binom{n}{3}}{64\binom{n}{3} + 7n}$.

  (iii) $\dfrac{64\binom{n}{3}}{64\binom{n}{3} + 9n}$.

  (iv) $\dfrac{64\binom{n}{3}}{64\binom{n}{3} + 11n}$.

  (v) None of the above.

(b) $V(S_n | R_n) =$

  (i) $0.12R_n$.

  (ii) $0.16R_n$.

  (iii) $0.2n$.

  (iv) $0.24R_n$.

  (v) None of the above.

(c) We want to bound $P(S_n \geq 2n)$ employing Chernoff's Inequality.

  (i) For every fixed n, as $t \to \infty$, the bound approaches the exact value of $P(S_n \geq 2n)$.

  (ii) The optimal t to use is the same for every value of n, but the bound itself decreases as n increases.

  (iii) Both the optimal t to use and the bound itself are the same for every value of n.

  (iv) For all sufficiently large n, the bound is trivial.

  (v) None of the above.

Solutions

1. (a) We have

$$E(R) = \sum_{i=1}^{\infty} P(R = 1/2^i) \cdot 1/2^i = \sum_{i=1}^{\infty} 1/2^{2i} = 1/3$$

and

$$E(R^2) = \sum_{i=1}^{\infty} P(R = 1/2^i) \cdot (1/2^i)^2 = \sum_{i=1}^{\infty} 1/2^{3i} = 1/7.$$

Therefore

$$V(R) = E(R^2) - E^2(R) = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.$$

Thus, (ii) is true.

(b) We have:

$$E(X) = \sum_{i=1}^{\infty} P(X = 1/i) \cdot 1/i = \sum_{i=1}^{\infty} \frac{1}{i2^i}.$$

We have seen in class that the sum is $\log 2$.

Thus, (i) is true.

(c) Y assumes the values $1, 2, 3, \ldots$ with probabilities of $1/2, 1/4, 1/8, \ldots$, respectively. Hence, $Y \sim G(1/2)$.

Thus, (ii) is true.

(d) We easily find that

$$E(Y) = \frac{1}{1/2} = 2$$

and

$$V(Y) = \frac{1 - 1/2}{(1/2)^2} = 2.$$

Also,

$$E(YR) = \sum_{i=1}^{\infty} P(R = 1/2^i) \cdot i/2^i = \sum_{i=1}^{\infty} i/4^i = \sum_{i=1}^{\infty} \binom{i}{1} /4^i = \frac{(1/4)^1}{(1 - 1/4)^{1+1}} = \frac{4}{9}.$$

Therefore:

$$\operatorname{Cov}(Y,R) = E(YR) - E(Y)E(R) = \frac{4}{9} - 2\cdot\frac{1}{3} = \frac{4}{9} = -\frac{2}{9}.$$

Consequently:

$$\rho(Y,R) = \frac{\operatorname{Cov}(Y,R)}{\sigma_Y\sigma_R} = \frac{-2/9}{\sqrt{2}\sqrt{2/63}} = -\frac{\sqrt{7}}{3}.$$

Thus, (iii) is true.

(e) We have $X' = 10^6 X$ and $R' = 10R$. Therefore,

$$C_2' = \operatorname{Cov}(X',R') = 10^6\cdot 10\cdot \operatorname{Cov}(X,R) = 10^7 C_2$$

and

$$\rho_2' = \rho(X',R') = \rho(X,R) = \rho_2.$$

It follows that

$$\rho_2'/\rho_2 = 10^{-7}C_2'/C_2.$$

Thus, (iii) is true.

(f) We need to find the probability of the intersection of the events $A_1 = \{N_1 \le 2\}$ and $A_2 = \{N_2 \le 2\}$. Employing the formula for the probability of a union of events, we obtain:

$$P(A_1\cap A_2) = 1 - P(\bar{A}_1\cup\bar{A}_2) = 1 - P(\bar{A}_1) - P(\bar{A}_2) + P(\bar{A}_1\cap\bar{A}_2.$$

Now $N_1 \sim B(4,1/2)$ and $N_2 \sim B(4,1/4)$, so that

$$P(A_1\cap A_2) = 1 - \binom{4}{1}\frac{1}{2^4} - \frac{1}{2^4} - \binom{4}{1}\frac{1}{4^3}\frac{3}{4} - \frac{1}{4^4} =$$

Thus, (i) is true.

(g) At each round, the probability of the 10-th stone to be drawn is $1/2^{10}$. Hence, the number of times that stone will be drawn in 1000 rounds is distributed $B(1000,1/2^{10})$, which is approximately $P(1000\cdot 1/2^{10})$. Hence, the required probability is approximately the probability of a $P(1000\cdot 1/2^{10})$-distributed random variable being non-zero, that is

$$1 - e^{-1000/2^{10}} \approx 1 - e^{-1}.$$

Thus, (iii) is true.

(h) Let F denote the event whereby some stone i is drawn only finitely many times. We have $F = \cup_{i=1}^{\infty} F_i$, where $F_i, 1 \le i < \infty$ is the event whereby stone i is drawn only finitely many times. Now each F_i may be written in the form $\cup_{m=1}^{\infty} F_{im}$, where F_{im} denotes the event that stone i is not seen after the m-th drawing. Now

$$P(F_{im}) = \prod_{k=m+1}^{\infty} (1 - 1/2^i) = 0.$$

It follows now from the union bound that $P(F) = 0$.

Thus, (iv) is true.

2. (a) The variables X and Y are distributed $G(1/6)$ and $\bar{B}(2, 1/3)$, respectively. Hence,

$$\begin{aligned} P(X = Y) &= P\left(\cup_{n=2}^{\infty} \{X = Y = n\}\right) \\ &= \sum_{n=2}^{\infty} \left(\frac{5}{6}\right)^{n-1} \frac{1}{6} \cdot \binom{n-1}{2-1} \left(\frac{2}{3}\right)^{n-2} \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{36} \sum_{n=2}^{\infty} \binom{n-1}{1} \left(\frac{5}{9}\right)^{n-1} \\ &= \frac{1}{36} \sum_{m=1}^{\infty} \binom{m}{1} \left(\frac{5}{9}\right)^{m} \\ &= \frac{1}{36} \cdot \frac{5/9}{(1-5/9)^2} \\ &= \frac{5}{64}. \end{aligned}$$

Thus, (iii) is true.

(b) By the formulas for the expected value and variance of geometric and negative binomial variables, we obtain

$$E(X) = \frac{1}{1/6} = 6, \qquad V(X) = \frac{1 - 1/6}{1/6^2} = 30$$

and

$$E(Y) = \frac{2}{1/3} = 6, \qquad V(Y) = \frac{2(1-1/3)}{1/3^2} = 12.$$

In particular, using also the independence of X and Y:

$$E(X-Y) = 6-6 = 0, \qquad V(X-Y) = 30+12 = 42.$$

Chebyshev's Inequality yields:

$$\begin{aligned} P(|X-Y| \geq 10) &= P(|(X-Y)-E(X-Y)| \geq 10) \\ &\leq \frac{42}{10^2} \\ &= 0.42. \end{aligned}$$

(c) When counting the 3-s until B gets twice either 1 or 6, the other three possible outcomes during the process – 2, 4, and 5 – do not count. Thus, we consider 1 and 6 as a success each, 3 as a failure, and ignore all other rolls on the way. The probability of success in each trial is 2/3, and N_3 counts the failures until obtaining two successes. Hence, $N_3 + 2 \sim \bar{B}(2, 2/3)$.

If N_3 were approximately $P(\lambda)$-distributed, we would expect its expectation and variance to be approximately the same as those of N_3, namely

$$\lambda \approx E(N_3) = \frac{2}{2/3} - 2 = 1$$

and

$$\lambda \approx V(N_3) = \frac{2(1-2/3)}{(2/3)^2} = \frac{3}{2}.$$

If $N_3 + 1$ were $G(p)$-distributed, we would similarly have

$$\frac{1}{p} = E(N_3+1) = 2$$

and

$$\frac{1-p}{p^2} = V(N_3+1) = \frac{3}{2},$$

which two requirements are incompatible.

N_3, N_4 are dependent as we can easily check, for example, that

$$P(N_3 = 0) = P(N_4 = 0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9},$$

while

$$P(N_3 = N_4 = 0) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \neq P(N_3 = 0)P(N_4 = 0).$$

Thus, (iii) is true.

(d) The first $X - 1$ rolls of A produce numbers uniformly distributed between 1 and 5. Hence, the expected outcome of each of these rolls is

$$\frac{1+5}{2} = 3.$$

The last roll gives a 6. Altogether:—

$$E(S|X) = (X-1) \cdot 3 + 6 = 3X + 3.$$

Thus, (i) is true.

3. (a) The event $S_n = 3n - 3$ may occur in two ways:

- There are $n - 3$ days with 3 cm of snow in each and three days with 2 cm each.
- There are $n - 1$ days with 3 cm of snow in each and one day with none.

Therefore:

$$\begin{aligned} P(R_n = n | S_n = 3n-3) &= \frac{P(R_n = n, S_n = 3n-3)}{P(S_n = 3n-3)} \\ &= \frac{\binom{n}{3} 0.1^{n-3} 0.4^3}{\binom{n}{3} 0.1^{n-3} 0.4^3 + \binom{n}{1} 0.1^{n-1} 0.5} \\ &= \frac{64\binom{n}{3}}{64\binom{n}{3} + 5n}. \end{aligned}$$

Thus, (i) is true.

(b) Given that there is snow on some day, the conditional probabilities for 2 cm and 3 cm are 0.8 and 0.2, respectively. Hence,

$$S_n - 2R_n | R_n \sim B(n, 0.2).$$

It follows that:

$$V(S_n | R_n) = R_n \cdot 0.2 \cdot 0.8 = 0.16 R_n.$$

Thus, (ii) is true.

(c) S_n may be written in the form

$$S_n = X_1 + X_2 + \cdots + X_n,$$

where X_i is the amount of snow on day $i, 1 \le i \le n$. Now

$$M_{X_i}(t) = E\left(e^{tX_i}\right) = 0.5e^{t\cdot 0} + 0.4e^{t\cdot 2} + 0.1e^{t\cdot 3},$$

and since the X_i-s are independent,

$$M_{S_n}(t) = \left(0.5 + 0.4e^{2t} + 0.1e^{3t}\right)^n.$$

Chernoff's Inequality yields:

$$\begin{aligned} P(S_n \ge 2n) &\le \frac{M_{S_n}(t)}{e^{2nt}} \\ &= \left(0.5e^{-2t} + 0.4 + 0.1e^{t}\right)^n, \qquad t > 0. \end{aligned}$$

To optimize the inequality, denote:

$$f(t) = 0.5e^{-2t} + 0.4 + 0.1e^{t}, \qquad t \ge 0.$$

We have

$$f'(t) = -e^{-2t} + 0.1e^{t}.$$

Equating $f'(t)$ to 0, and noting that $f'(0) < 0$ and $f(t)_{t\to\infty} \longrightarrow \infty$, we see that the optimal t is given by

$$t_0 = \frac{\log 10}{3}.$$

The resulting bound is:

$$P(S_n \ge 2n) \le \left(0.5 \cdot 10^{-2/3} + 0.4 + 0.1 \cdot 10^{1/3}\right)^n.$$

Thus, (ii) is true.