Midterm

Mark the correct answer in each part of the following questions.

1. A fair die is rolled n times. Let S_k denote the sum of outcomes in the first k rolls, $1 \le k \le n$.

   (a) $F_{S_3}(16) =$

   (i) $\dfrac{212}{216}$.
   (ii) $\dfrac{213}{216}$.
   (iii) $\dfrac{214}{216}$.
   (iv) $\dfrac{215}{216}$.
   (v) None of the above.

   (b) A player gets one shekel the first time the die shows a 6 (if it ever does), two shekels the second time it shows a 6, ..., n shekels at the nth time. Let X denote the player's total winnings. (For example, if the die shows altogether four times a 6, then $X = 1 + 2 + 3 + 4 = 10$.) Then $E(X) =$

   (i) $\dfrac{1}{72}n^2 + \dfrac{11}{72}n$.
   (ii) $\dfrac{2}{72}n^2 + \dfrac{10}{72}n$.
   (iii) $\dfrac{3}{72}n^2 + \dfrac{9}{72}n$.
   (iv) $\dfrac{4}{72}n^2 + \dfrac{8}{72}n$.
   (v) None of the above.

(c) Suppose that $n = 6$, and let A denote the event whereby the six outcomes are distinct. Then $P(A) =$

(i) $\frac{1}{6^6}$.

(ii) $1 - \frac{1}{6^6}$.

(iii) $\frac{6!}{6^6}$.

(iv) $\frac{1}{6!}$.

(v) None of the above.

(d) Let k and A be as in the preceding part. Consider also the event B whereby the outcome of the first roll is not 1, of the second – not 2, ..., of the sixth – not 6. Then $P(B|A) =$

(i) $\frac{5^6}{6^6}$.

(ii) $\frac{5^5}{6^6}$

(iii) $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$.

(iv) $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$.

(v) None of the above.

2. According to Tolkien ("Lord of the Rings"), on Evenstar, the first, third, fifth, ..., children in a family are considered lucky if they are girls and unlucky if they are boys. With the second, fourth, sixth, ..., children, the situation is the opposite – boys are lucky and girls are unlucky. Families on Evenstar grow until they have three lucky children. Let X denote the number of children of a random family (when it reaches its final size).

(a) $V(X) =$

(i) 3.

(ii) 4.

(iii) 5.

(iv) 6.

(v) None of the above.

(b) Let G denote the event whereby the first child in a random family is a girl. Then $P(G|X = 10) =$

(i) $\frac{1}{2}$.

(ii) $\frac{1}{3}$.

(iii) $\frac{1}{4}$.

(iv) $\frac{1}{5}$.

(v) None of the above.

(c) A random family with 10 children is selected. The probability that, at all steps during the period of growth of the family, there were never more lucky children than unlucky ones, is

(i) $\frac{2}{3}$.

(ii) $\frac{3}{4}$.

(iii) $\frac{4}{5}$.

(iv) $\frac{7}{9}$.

(v) None of the above.

(d) In the Probability for CS Students at Central Evenstar University, there are 50 lucky students and 50 unlucky ones. 50 students were selected randomly and given passing grades, and the other 50 – failing grades. The probability that, among the students who passed the course, exactly 25 are lucky and 25 are unlucky is:

(i) $\dfrac{25!^2 \cdot 50!}{100!}$.

(ii) $\dfrac{50!^4}{25!^4 \cdot 100!}$.

(iii) $\dfrac{50!^3}{25!^2 \cdot 100!}$.

(iv) $\dfrac{25!^4 \cdot 50!^2}{100!^2}$.

(v) None of the above.

3. Let $X \sim P(\lambda)$.

   (a) $P(X = 1 | X > 0) =$

   (i) $\dfrac{\lambda}{e^\lambda - 1}$.

   (ii) $e^{-\lambda}$.

   (iii) $\lambda e^{-\lambda}$.

   (iv) $\dfrac{\lambda}{e^\lambda + 1}$.

   (v) None of the above.

   (b) $E\left(\binom{X}{3} \cdot X!\right) =$

   (i) $e^{-\lambda}\dfrac{\lambda^3}{(1-\lambda)^3}$ for $\lambda < 1$.

   (ii) $e^{-\lambda}\dfrac{\lambda^3}{(1-\lambda)^4}$ for $\lambda < 1$.

   (iii) $e^{-\lambda}\dfrac{\lambda^4}{(1-\lambda)^3}$ for $\lambda < 1$.

   (iv) Does not exist for any $\lambda > 0$.

   (v) None of the above.

4. Consider Banach's matchbox problem, discussed in class.

   (a) Suppose that, at each step, the probability of selecting a match from one of the pockets is proportional to the number of matches in that pocket. Let X denote the number of matches the smoker has left when the first pocket becomes empty. (Note that, unlike in class, we do not wait until he tries to take a match from one of the pockets and finds it empty. We refer to the time when one pocket becomes empty.) Then $E(X) =$

(i) 2.

(ii) $\dfrac{2N+1}{N+2}$.

(iii) $\dfrac{2N-1}{N}$.

(iv) $\dfrac{2N}{N+1}$.

(v) None of the above.

(b) Return to the version discussed in class, where each pocket has a probability of 1/2 at each step. Suppose that $N = 50$ and the smoker uses all $2N = 100$ every single day. He continues this procedure for $3 \cdot \binom{100}{50}$ days. Let Y denote the number of days in which one of the pockets emptied when the other was still completely full. Then $P(Y = 6) \approx$

(i) e^{-1}.

(ii) $\dfrac{2^2 e^{-2}}{2!}$.

(iii) $\dfrac{3^3 e^{-3}}{3!}$.

(iv) $\dfrac{6^6 e^{-6}}{6!}$.

(v) None of the above.

Solutions

1. (a) We have

$$\begin{aligned} F_{S_3}(16) &= P(S_3 \leq 16) \\ &= 1 - P(S_3 = 17) - P(S_3 = 18) \\ &= 1 - \frac{3}{6^3} - \frac{1}{6^3} \\ &= \frac{212}{216}. \end{aligned}$$

Thus, (i) is correct.

(b) Let Y denote the number of 6-s. Then:

$$X = 1 + 2 + \cdots + Y = \frac{Y(Y+1)}{2}.$$

Now $Y \sim B(n, 1/6)$, so that by the formulas for the expectation and variance of binomial variables we get:

$$\begin{aligned} E(X) &= E\left(\tfrac{Y(Y+1)}{2}\right) \\ &= \frac{E(Y^2)}{2} + \frac{E(Y)}{2} \\ &= \frac{n \cdot 1/6 \cdot (1 - 1/6) + (n \cdot 1/6)^2}{2} + \frac{n \cdot 1/6}{2} \\ &= \frac{n^2 + 11n}{72}. \end{aligned}$$

Thus, (i) is correct.

(c) There are 6^6 equi-probable outcomes, of which $6!$ consist of distinct faces showing in the 6 rolls. Hence, the required probability is $6!/6^6$.

Thus, (iii) is correct.

(d) Given that all 6 outcomes are distinct, the possibilities correspond to all 6! bijections between the set of rolls and the set $\{1,2,3,4,5,6\}$. The requirement, whereby the outcome of no roll is equal to the number of the roll in the sequence, is analogous to the requirement in the negligent secretary problem that no letter is sent to the correct destination. It follows that the required probability is

$$1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}.$$

Thus, (iii) is correct.

2. (a) Consider the birth of a lucky child as a success and that of an unlucky child as a failure. Then X is the number of trials until having 3 successes. Hence $X \sim \bar{B}(3,1/2)$, so that:

$$V(X)=\frac{3\cdot(1-1/2)}{(1/2)^2}=6.$$

Thus, (iv) is correct.

(b) We have:

$$P(G|X=10)=\frac{P(G\cap\{X=10\})}{P(X=10)}=\frac{P(G)P(X=10|G)}{P(X=10)}.$$

The denominator is easily calculated by the formula for the probability function of a negative binomial random variable. Now, under the condition that G has occurred, the event $X=10$ means that the second success, when counting from the second birth on, will occur at the ninth trial. namely, the second factor in the numerator is the probability that a $\bar{B}(2,1/2)$-distributed random variable will assume the value 9. Consequently:

$$P(G|X=10)=\frac{1/2\cdot\binom{9-1}{2-1}(1/2)^2(1/2)^7}{\binom{10-1}{3-1}(1/2)^3(1/2)^7}=\frac{8}{36}=\frac{2}{9}.$$

A simpler solution is as follows. The fact that $X=10$ means that there are 10 children, out of which two of the first nine, as well as the tenth, are lucky. By symmetry, all pairs of two children out of the first nine are equally likely to be the lucky ones. In particular, each child has a probability of 2/9 being lucky. Thus, the first child has a probability of 2/9 being a girl.

Thus, (v) is correct.

(c) The question is what is the probability that, under the assumption that out of the first nine children two are lucky and seven are not, the unlucky children lead (weakly) at all steps. The question is equivalent to the ballot problem with $m = 7, n = 2$, and therefore the required probability is

$$\frac{m - n + 1}{m + 1} = \frac{6}{8} = \frac{3}{4}.$$

Thus, (ii) is correct.

(d) Denote by Y the number of lucky students who pass the course. Clearly, $Y \sim H(50, 50, 50)$, and therefore

$$P(Y = 25) = \frac{\binom{50}{25}\binom{50}{25}}{\binom{100}{50}} = \frac{50!^4}{25!^4 \cdot 100!}.$$

Thus, (i) is correct.

3. (a) We have:

$$P(X = 1 | X > 0) = \frac{P(X = 1)}{P(X > 0)} = \frac{P(X = 1)}{1 - P(X = 0)} = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}} = \frac{\lambda}{e^{\lambda} - 1}.$$

Thus, (i) is correct.

(b) For $\lambda < 1$:

$$\begin{aligned} E\left(\binom{X}{3} \cdot X!\right) &= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \cdot \binom{k}{3} \cdot k! \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \binom{k}{3} \lambda^k \\ &= e^{-\lambda} \frac{\lambda^3}{(1 - \lambda)^4}. \end{aligned}$$

Thus, (ii) is correct.

4. (a) Refer to the matches in one of the pockets as $\#1, \#2, \dots, \#N$ and to those in the other as $\#(N+1), \#(N+2), \dots, \#2N$. Let $X_i = 1$ if match $\#i$ has not been used by the time all matches in the other pocket have been used and $X_i = 0$ otherwise, $1 \le i \le 2N$. Clearly, all X_i-s are identically distributed and $X = \sum_{i=1}^{2N} X_i$. The event, say, $\{X_1 = 1\}$, occurs if and only if all N matches $\#(N+1), \#(N+2), \dots, \#2N$, are used before match $\#1$. By our assumptions, at each step, all unused matches have the same probability of being selected. Hence, all possible orders of the matches are equally likely. This implies that

$$P(X_1 = 1) = \frac{1}{N+1}.$$

It follows that:

$$\begin{aligned} E(X) &= \textstyle\sum_{i=1}^{2N} E(X_i) \\ &= 2N \cdot \frac{1}{N+1} \\ &= \frac{2N}{N+1}. \end{aligned}$$

Thus, (iv) is correct.

(b) Considering a day in which one of the pockets empties while the other is still full, we may view Y as the number of successes in $3 \cdot \binom{100}{50}$ independent trials. In each trial, the smoker has 2^{50} possibilities as to the pocket to be searched to get each of the first 50 matches. Only for two of these possibilities does one of the pockets become empty while the other is still full. Hence, $Y \sim B(3 \cdot \binom{100}{50}, 2/2^{50})$. Since the product $3 \cdot \binom{100}{50} \cdot 2/2^{50}$ is nowhere near one of the numbers 1, 2, 3, and 6, the distribution of Y is not close to a Poisson distribution with any of these parameters.

Thus, (v) is correct.