Final #1

Mark the correct answer for each part of each question.

1. Ronit and Stav participate in the following game. A die is rolled n times. After each roll, if the outcome is d, Ronit gets d shekels and Stav gets

   - 1 shekel if $d = 1, 2, 3, 4$.
   - 7 shekels if $d = 5$.
   - 10 shekels if $d = 6$.

   Let R and S denote the total winnings of Ronit and Stav, respectively. (For example, if $n = 4$ and the outcomes are $6, 1, 5, 3$, then $R = 6 + 1 + 5 + 3 = 15$ and $S = 10 + 1 + 7 + 1 = 19$.)

   (a) If $n = 4$ then $P(S = 22) =$

   (i) $\frac{5}{81}$.

   (ii) $\frac{7}{81}$.

   (iii) $\frac{8}{81}$.

   (iv) $\frac{10}{81}$.

   (v) None of the above.

   Remark. $S = 22$ if and only if there are either (i) two 6s and two outcomes between 1 and 4, or (ii) three 5s and one outcome between 1 and 4.

(b) If $n = 2$ then $V(S \mid R = 7) =$

(i) 11.

(ii) 12.

(iii) 13.

(iv) 14.

(v) None of the above.

(c) $\mathrm{Cov}(R, S) =$

(i) $\frac{21n}{4}$.

(ii) $\frac{23n}{4}$.

(iii) $\frac{25n}{4}$.

(iv) $\frac{27n}{4}$.

(v) None of the above.

(d) Chebyshev's Inequality gives the following upper bound on $P(|R-S| \geq n)$:

(i) $\frac{13}{3n}$.

(ii) $\frac{14}{3n}$.

(iii) $\frac{16}{3n}$.

(iv) $\frac{17}{3n}$.

(v) None of the above.

Remark: We mean the best bound that can be obtained. For example, if (i) is correct, then (ii)-(iv) are correct as well, but only (i) should be marked as the correct answer.

(e) We want an upper bound on the probability $P(S \geq 7n)$, applying Chernoff's bound. The optimal value of t to get the best possible bound is $t_0 =$

(i) $\frac{1}{6} \cdot \log 8$.

(ii) $\frac{1}{7} \cdot \log 8$.

(iii) $\frac{1}{8} \cdot \log 8$.

(iv) $\frac{1}{9} \cdot \log 8$.

(v) None of the above.

2. In a two-stage experiment, we first draw cards from a full deck one by one **without replacement** until the queen of hearts is drawn. Let X be the number of cards drawn during this stage.

We now return the drawn cards to the deck. In the second stage, we draw X cards from the deck, **now with replacement**. Let $Q_\heartsuit$ denote the number of times the queen of hearts is drawn during the second stage.

(a) $P(Q_\heartsuit = X) =$

(i) $\frac{1 - 1/52^{52}}{52 \cdot 51}$.

(ii) $\frac{1 - 1/52^{51}}{52 \cdot 51}$.

(iii) $\frac{1 - 1/52^{52}}{52^2}$.

(iv) $\frac{1 - 1/52^{51}}{52^2}$.

(v) None of the above.

(b) $P_{X|Q_\heartsuit}(52 \mid 51) \in$

(i) $[0, 1/10^6)$.

(ii) $[1/10^6, 0.01)$.

(iii) $[0.01, 0.04)$.

(iv) $[0.98, 1]$.

(v) None of the above.

(c) The number of discontinuities of the function $F_{Q_\heartsuit}$ is:

(i) 51.

(ii) 52.

(iii) 53.

(iv) Infinite.

(v) None of the above.

(d) $V(Q_\heartsuit) =$

(i) $\dfrac{6 \cdot 51 \cdot 53}{12 \cdot 52^2}$.

(ii) $\dfrac{7 \cdot 51 \cdot 53}{12 \cdot 52^2}$.

(iii) $\dfrac{8 \cdot 51 \cdot 53}{12 \cdot 52^2}$.

(iv) $\dfrac{9 \cdot 51 \cdot 53}{12 \cdot 52^2}$.

(v) None of the above.

(e) $E(XQ_\heartsuit) =$

(i) $\dfrac{1}{52} \cdot \left(\dfrac{52^2 - 1}{12} + \dfrac{51^2 - 1}{2^2} \right)$.

(ii) $\dfrac{1}{52} \cdot \left(\dfrac{52^2 - 1}{12} + \dfrac{51^2}{2^2} \right)$.

(iii) $\dfrac{1}{52} \cdot \left(\dfrac{52^2 - 1}{12} + \dfrac{53^2 - 1}{2^2} \right)$.

(iv) $\dfrac{1}{52} \cdot \left(\dfrac{52^2 - 1}{12} + \dfrac{53^2}{2^2} \right)$.

(v) None of the above.

3. Banach's smoker has decided to change the number of matches he carries and the way he arranges them. He now starts with n red, n green, and n blue matches, and puts them all in one pocket. Let G and B denote the number of green and blue matches, respectively, remaining in his pocket after the last red match has been drawn.

   (a) The two quantities $P(G > 0, B > 0)$ and $P(G > 0)P(B > 0)$ are related as follows.

      (i) $P(G > 0, B > 0) = \frac{3}{4} \cdot P(G > 0)P(B > 0)$.

      (ii) $P(G > 0, B > 0) = P(G > 0)P(B > 0)$.

      (iii) $P(G > 0, B > 0) = \frac{4}{3} \cdot P(G > 0)P(B > 0)$.

      (iv) $P(G > 0, B > 0) = \frac{3}{2} \cdot P(G > 0)P(B > 0)$.

      (v) None of the above.

   (b) The limiting distribution of $B + 1$ as $n \to \infty$ is:

      (i) $G(1/3)$.

      (ii) $G(1/2)$.

      (iii) $G(2/3)$.

      (iv) $G(3/4)$.

      (v) None of the above.

   (c) Let S denote the sum of the step numbers at which red matches are drawn. (For example, if $n = 4$, and the red matches are drawn at steps $2, 4, 11$, and 12, then $S = 2 + 4 + 11 + 12 = 29$.) Then $E(S) =$

      (i) $\frac{3n^2 - 3n}{2}$.

      (ii) $\frac{3n^2 - n}{2}$.

      (iii) $\frac{3n^2 + n}{2}$.

      (iv) $\frac{3n^2 + 3n}{2}$.

      (v) None of the above.

Solutions

1. (a) We have

$$P(S=22) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{4}{6}\right)^2 + \binom{4}{3}\left(\frac{1}{6}\right)^3\left(\frac{4}{6}\right) = \frac{7}{81}.$$

Thus, (ii) is true.

(b) For $n = 2$, the ordered roll pairs with sum 7 are

$$(1,6),(2,5),(3,4),(4,3),(5,2),(6,1).$$

The corresponding S values are $11, 8, 2, 2, 8, 11$. Hence, S takes the values $2, 8, 11$ with probabilities $1/3$ each. Thus,

$$E(S \mid R=7) = \frac{11+8+2}{3} = 7$$

and

$$V(S \mid R=7) = \frac{(11-7)^2 + (8-7)^2 + (2-7)^2}{3} = 14.$$

Thus, (iv) is true.

(c) Write $R = \sum_{i=1}^n R_i$ and $S = \sum_{i=1}^n S_i$, where R_i and S_i are the winnings of Ronit and Stav, respectively, in the ith roll, $1 \le i \le n$. Since R_i and S_j are independent for $i \neq j$, we have:

$$\begin{aligned}
\text{Cov}(R,S) &= \text{Cov}\left(\sum_{i=1}^n R_i, \sum_{i=1}^n S_i\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(R_i, S_j) \\
&= n \cdot \text{Cov}(R_1, S_1).
\end{aligned}$$

Recall that $E(R_1) = 7/2$. Now

$$E(S_1) = \frac{1+1+1+1+7+10}{6} = \frac{7}{2}$$

and

$$E(R_1S_1) = \frac{1\cdot 1 + 2\cdot 1 + 3\cdot 1 + 4\cdot 1 + 5\cdot 7 + 6\cdot 10}{6} = \frac{35}{2}.$$

Therefore

$$\text{Cov}(R_1, S_1) = \frac{35}{2} - \frac{7}{2}\cdot\frac{7}{2} = \frac{21}{4}.$$

Hence $\text{Cov}(R, S) = 21n/4$.

Thus, (i) is true.

(d) Let $D = R - S = \sum_{i=1}^{n}(R_i - S_i)$. Then $E(D) = 0$ and

$$\begin{aligned} E\left((R_1 - S_1)^2\right) &= \frac{0^2 + 1^2 + 2^2 + 3^2 + (-2)^2 + (-4)^2}{6} \\ &= \frac{17}{3}. \end{aligned}$$

Hence $V(D) = 17n/3$, and by Chebyshev,

$$P(|R - S| \geq n) = P(|D| \geq n) \;\leq\; \frac{V(D)}{n^2} \;=\; \frac{17}{3n}.$$

Thus, (iv) is true.

(e) For each roll,

$$M_{S_i}(t) = E(e^{tS_1}) = \frac{4e^t + e^{7t} + e^{10t}}{6}, \qquad t > 0.$$

Chernoff's bound gives

$$\begin{aligned} P(S \geq 7n) &\leq \frac{M_S(t)}{e^{7nt}} \\ &= \left(\frac{4e^t + e^{7t} + e^{10t}}{6e^{7t}}\right)^n \\ &= \left(\frac{4e^{-6t} + 1 + e^{3t}}{6}\right)^n, \qquad t > 0. \end{aligned}$$

To minimize the right-hand side, put

$$f(t) = 4e^{-6t} + 1 + e^{3t}, \qquad t \geq 0.$$

Then

$$f'(t) = -24e^{-6t} + 3e^{3t} = 0 \;\Rightarrow\; e^{9t} = 8 \;\Rightarrow\; t_0 = \frac{1}{9}\log 8.$$

(To see that t_0 is indeed a minimum point, notice that $f'(0) = -21 < 0$, so that f initially decrease, while $f(t) \underset{t\to\infty}{\longrightarrow} \infty$; hence f must reach its minimum at the unique point where $f'(t) = 0$.) Thus, (iv) is true.

2. (a) Clearly, $X \sim U[1, 52]$ and $Q_\heartsuit \mid X \sim B(Q_\heartsuit, 1/52)$. Therefore

$$\begin{aligned} P(Q_\heartsuit = X) &= \sum_{k=1}^{52} P(X = k)\, P(Q_\heartsuit = k \mid X = k) \\ &= \sum_{k=1}^{52} \frac{1}{52}\left(\frac{1}{52}\right)^k \\ &= \frac{1}{52^2} \cdot \frac{1 - 1/52^{52}}{1 - 1/52} \\ &= \frac{1 - 1/52^{52}}{52 \cdot 51}. \end{aligned}$$

Thus, (i) is true.

(b) Given that $Q_\heartsuit = 51$, only $X = 51$ or $X = 52$ is possible. We have:

$$\begin{aligned}
P_{X|Q_\heartsuit}(52 \mid 51) &= \frac{P(X = 52, Q_\heartsuit = 51)}{P(Q_\heartsuit = 51)} \\
&= \frac{\frac{1}{52} \cdot P(Q_\heartsuit = 51 \mid X = 52)}{\frac{1}{52} \cdot P(Q_\heartsuit = 51 \mid X = 51) + \frac{1}{52} \cdot P(Q_\heartsuit = 51 \mid X = 52)} \\
&= \frac{\binom{52}{51}\left(\frac{1}{52}\right)^{51}\left(\frac{51}{52}\right)}{\left(\frac{1}{52}\right)^{51} + \binom{52}{51}\left(\frac{1}{52}\right)^{51}\left(\frac{51}{52}\right)} \\
&= \frac{51}{1 + 51} \\
&= \frac{51}{52}.
\end{aligned}$$

Thus, (iv) is true.

(c) The variable $Q_\heartsuit$ assumes all the values $0, 1, \ldots, 52$ with positive probabilities, and only these values. Hence it has jumps at all these 53 points and is constant along all intervals in-between and outside.

Thus, (iii) is true.

(d) We have:

$$\begin{aligned}
V(Q_\heartsuit) &= E\left(V(Q_\heartsuit \mid X)\right) + V\left(E(Q_\heartsuit \mid X)\right) \\
&= E\left(X \cdot \frac{1}{52} \cdot \frac{51}{52}\right) \ + \ V\left(\frac{X}{52}\right) \\
&= \frac{51}{52^2}E(X) \ + \ \frac{1}{52^2}V(X) \\
&= \frac{51}{52^2} \cdot \frac{1+52}{2} \ + \ \frac{52^2-1}{12 \cdot 52^2} \\
&= \frac{1}{52^2}\left(\frac{51 \cdot 53}{2} + \frac{51 \cdot 53}{12}\right) \\
&= \frac{7 \cdot 51 \cdot 53}{12 \cdot 52^2}.
\end{aligned}$$

Thus, (ii) is true.

(e) We have:

$$\begin{aligned}
E(XQ) &= E\left(X\,E(Q \mid X)\right) \\
&= E\left(\tfrac{X^2}{52}\right) \\
&= \tfrac{1}{52}\left(V(X) + E^2(X)\right) \\
&= \tfrac{1}{52}\left(\tfrac{52^2-1}{12} + \tfrac{53^2}{2^2}\right).
\end{aligned}$$

Thus, (iv) is true.

3. (a) By symmetry, the last match to be drawn is equally likely to be red, green, or blue. Moreover, between the matches of any two of the colors, it is equally likely for each to end first. The event $\{G > 0\}$ means that the red matches end before the green ones, so that $P(G > 0) = 1/2$. Similarly, $P(B > 0) = 1/2$. The event $\{G > 0, B > 0\}$ means that the red matches are the first to end, and hence $P(G > 0, B > 0) = 1/3$. It follows that:

$$P(G > 0, B > 0) = \frac{1}{3} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{4}{3}P(G > 0)\,P(B > 0).$$

Thus, (iii) is true.

(b) When considering B, we are interested only in the order in which the red and blue matches are drawn, and may ignore the green matches. For any fixed $k \geq 0$, the event $\{B = k\}$ means that the last k matches (out of the $2n$ red and blue ones) are blue and the match before them is red. Hence

$$P(B = k) = \frac{n}{2n} \cdot \frac{n-1}{2n-1} \cdots \frac{n-k+1}{2n-k+1} \cdot \frac{n}{2n-k}.$$

Each of the factors on the right-hand side converges to $1/2$ as $n \to \infty$, so that the product converges to $1/2^{k+1}$. Thus,

$$P(B + 1 = \ell) = P(B = \ell - 1) \xrightarrow[n\to\infty]{} \frac{1}{2^\ell}, \qquad \ell > 0.$$

Thus, (ii) is true.

(c) Let R_i be the indicator that the i-th match is red, $1 \leq i \leq 3n$. Clearly, $E(R_i) = 1/3$ and

$$S = \sum_{i=1}^{3n} i\, R_i.$$

Therefore,

$$E(S) \;=\; \sum_{i=1}^{3n} i\, E(R_i) \;=\; \frac{1}{3} \sum_{i=1}^{3n} i \;=\; \frac{1}{3} \cdot \frac{3n(3n+1)}{2} \;=\; \frac{3n^2 + n}{2}.$$

Thus, (iii) is true.