Midterm

Mark the correct answer in each part of the following questions.

1. A fair die is rolled n times, where n is a large number. After each roll, if the outcome is 6 we add to an (initially empty) urn a white ball, and if the outcome is not 6 we add a black ball. Each ball is marked by the number of the step at which it was put in the urn. Let X denote the number of white balls in the urn at the end of the process and S the sum of numbers written on these balls. (For example, if $n = 7$, and the outcomes are $6, 2, 6, 1, 5, 5, 6$, then $X = 3$ and $S = 1 + 3 + 7 = 11$.)

   (a) Suppose that at the end of the process we draw two balls out of the urn **with replacement**. The probability that both are white is:

   (i) $\frac{1}{36} - \frac{5^2}{36n}$.
   (ii) $\frac{1}{36}$.
   (iii) $\frac{1}{36} + \frac{5}{36n}$.
   (iv) $\frac{1}{36} + \frac{5^2}{36n}$.
   (v) None of the above.

   (b) $P(S = 0 \mid S \leq 1) =$

   (i) $\left(\frac{5}{6}\right)^n$.
   (ii) $\left(\frac{5}{6}\right)^{n-1}$.
   (iii) $\frac{5}{6}$.

(iv) $\binom{n}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{n-1}$.

(v) None of the above.

(c) $E(S \mid X) =$

(i) $\frac{nX}{2}$.

(ii) $\frac{(n+1)X}{2}$.

(iii) $\frac{7nX}{12}$.

(iv) $\frac{7nX}{6}$.

(v) None of the above.

(d) $V(S) =$

(i) $\frac{10n^3 + 15n^2 + 5n}{216}$.

(ii) $\frac{10n^3 + 15n^2 + 10n}{216}$.

(iii) $\frac{10n^3 + 20n^2 + 5n}{216}$.

(iv) $\frac{10n^3 + 20n^2 + 10n}{216}$.

(v) None of the above.

(e) Now suppose that the experiment is extended as follows. The first stage, consisting of n die rolls and placing balls in the urn, is the same as before. Then, in the second stage, we draw indefinitely a ball from the urn, record its color, and return it to the urn. Let p denote the probability that we will never draw a white ball in the whole process. Then $p \in$

(i) $[0, 1/6^n)$.

(ii) $[1/6^n, 1/2^n)$.

(iii) $[1/2^n, (5/6)^n)$.

(iv) $[(5/6)^n, (5/6)^n + 1/6^n)$.

(v) None of the above.

2. Two full decks of cards are given. One card is drawn randomly from each of the decks and put into the other deck. Denote by K and Q the number of kings and of queens in the first deck after these operations. (For example, if the card drawn from the first deck is a king and that drawn from the second is an ace, then $K = 3$ and $Q = 4$.)

   (a) $P(Q = 4) =$

      (i) $\frac{144}{169}$.

      (ii) $\frac{145}{169}$.

      (iii) $\frac{146}{169}$.

      (iv) $\frac{147}{169}$.

      (v) None of the above.

   (b) $\mathrm{Cov}(K, Q) =$

      (i) $\frac{-2}{169}$.

      (ii) $\frac{-1}{169}$.

      (iii) $\frac{1}{169}$.

      (iv) $\frac{2}{169}$.

      (v) None of the above.

   (c) Let Q' denote the final number of queens in the second deck. Then $\rho(Q, Q') \in$

      (i) $[-1, -2/169)$.

      (ii) $[-2/169, -1/169)$.

      (iii) $[-1/169, 0)$.

      (iv) $[0, 2/169)$.

      (v) None of the above.

   (d) Suppose the experiment is repeated $39 \cdot 52$ (each time starting with two full decks). The probability that in at least one of these experiments, each of the drawn cards will be the queen of diamonds of its deck, is approximately

(i) $e^{-1/4}$.
(ii) $e^{-3/4}$.
(iii) $1 - e^{-1/4}$.
(iv) $1 - e^{-3/4}$.
(v) None of the above.

3. (a) Let X, Y be discrete random variables on the same probability space.

(i) If $2X$ and Y are identically distributed then $F_Y(t) = 2F_X(t)$ for every $t \in \mathbf{R}$.
(ii) If $Y = 2X$ then $F_Y(t) = 2F_X(t)$ for every $t \in \mathbf{R}$, but the claim in the preceding part is false in general.
(iii) If $F_Y(2t) = F_X(t)$ for every $t \in \mathbf{R}$, then X and $2Y$ are identically distributed.
(iv) If $F_Y(2t) = F_X(t)$ for every $t \in \mathbf{R}$, then $2X$ and Y are identically distributed.
(v) None of the above.

(b) Let X denote the number of heads in two tosses of a coin and Y_1, Y_2 the outcomes of two rolls of a die. Put $Z_1 = XY_1$ and $Z_2 = XY_2$. Then $F_{Z_1,Z_2}(1, 2) =$

(i) $\frac{10}{36}$.
(ii) $\frac{11}{36}$.
(iii) $\frac{12}{36}$.
(iv) $\frac{13}{36}$.
(v) None of the above.

(c) We toss a coin until it shows a head 3 times and roll a die until it shows 6 for the first time. Let X and Y denote the number of tosses of the coin and rolls of the die, respectively. Then $P(X = Y) =$

(i) $\frac{24}{7^3}$.

(ii) $\frac{25}{7^3}$.

(iii) $\frac{26}{7^3}$.

(iv) $\frac{27}{7^3}$.

(v) None of the above.

Solutions

1. (a) We have $X \sim B(n, 1/6)$. If $X = k$, then the probability of drawing twice a white ball is $(k/n)^2$. By the law of total probability, the probability p of the event in question is:

$$\begin{aligned} p &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \cdot \left(\frac{k}{n}\right)^2 \\ &= \frac{1}{n^2} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \cdot k^2. \end{aligned}$$

The sum on the right-hand side is exactly $E(X^2)$. Consequently:

$$\begin{aligned} p &= \frac{1}{n^2} \left(n \cdot \frac{1}{6} \cdot \frac{5}{6} + \left(n \cdot \frac{1}{6} \right)^2 \right) \\ &= \frac{1}{36} + \frac{5}{36n} \end{aligned}$$

Thus, (iii) is correct.

(b) The event $\{S = 0\}$ occurs if and only if no 6 appears in the n rolls, so

$$P(S = 0) = \left(\frac{5}{6}\right)^n.$$

The event $\{S = 1\}$ occurs if and only if exactly one 6 appears and it is on the first roll; this has probability

$$P(S = 1) = \frac{1}{6} \left(\frac{5}{6}\right)^{n-1}.$$

Therefore

$$
\begin{aligned}
P(S=0 \mid S \le 1) &= \frac{P(S=0)}{P(S \le 1)} \\
&= \frac{P(S=0)}{P(S=0)+P(S=1)} \\
&= \frac{(5/6)^n}{(5/6)^n + (1/6)(5/6)^{n-1}} \\
&= \frac{5}{6}.
\end{aligned}
$$

Thus, (iii) is correct.

(c) Let $N_i = i$ if the i-th ball is white and $N_i = 0$ otherwise, $1 \le i \le n$. Then $S = \sum_{i=1}^n N_i$. By symmetry, the probability of each ball being white, given X, is X/n. It follows that

$$E(N_i \mid X) = \frac{X}{n} \cdot i,$$

and therefore

$$
\begin{aligned}
E(S \mid X) &= \textstyle\sum_{i=1}^n E(N_i \mid X) \\
&= \textstyle\sum_{i=1}^n \frac{X}{n} \cdot i \\
&= \frac{n(n+1)}{2} \cdot \frac{X}{n} \\
&= \frac{(n+1)X}{2}.
\end{aligned}
$$

Thus, (ii) is correct.

(d) With N_i as in the previous part, we note that $N_i = iI_i$, where I_i

is the indicator that the i-th ball is white. Since $I_i \sim B(1, 1/6)$,

$$\begin{aligned} V(S) &= V\left(\sum_{i=1}^n iI_i\right) \\ &= \sum_{i=1}^n i^2 V(I_i) \\ &= \sum_{i=1}^n i^2 \cdot \frac{1}{6} \cdot \frac{5}{6} \\ &= \frac{5}{36} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{10n^3 + 15n^2 + 5n}{216}. \end{aligned}$$

Thus, (i) is correct.

(e) Given X, the probability of each ball during the second stage to be black is $1 - X/n$. If $X = 0$ then, with probability 1, all balls will be black. If $X > 0$, then the probability for all balls to be black is

$$(1 - X/n) \cdot (1 - X/n) \cdot (1 - X/n) \cdots = 0.$$

Hence the event that all balls are black occurs if and only $X = 0$, the probability for which is $(5/6)^n$.

Thus, (iv) is correct.

2. (a) The event $\{Q = 4\}$ occurs if and only if both drawn cards are queens or neither of them is. Hence

$$P(Q = 4) = \left(\frac{12}{13}\right)^2 + \left(\frac{1}{13}\right)^2 = \frac{144}{169} + \frac{1}{169} = \frac{145}{169}.$$

Thus, (ii) is correct.

(b) Let K_1, K_2 be the indicators that the cards drawn from the first and second decks, respectively, are kings, and let Q_1, Q_2 be the analogues for queens. Then

$$K = 4 - K_1 + K_2, \qquad Q = 4 - Q_1 + Q_2,$$

and therefore

$$\begin{aligned}
\text{Cov}(K,Q) &= \text{Cov}(4-K_1+K_2, 4-Q_1+Q_2) \\
&= \text{Cov}(-K_1+K_2, -Q_1+Q_2) \\
&= \text{Cov}(-K_1,-Q_1)+\text{Cov}(-K_1,Q_2)+\text{Cov}(K_2,-Q_1) \\
&\quad +\text{Cov}(K_2,Q_2).
\end{aligned}$$

The two pairs of variables K_1, Q_2 and K_2, Q_1 are clearly independent, and thus have zero covariances. By symmetry, $\text{Cov}(K_1, Q_1) = \text{Cov}(K_2, Q_2)$. Now:

$$\begin{aligned}
\text{Cov}(K_1,Q_1) &= E(K_1Q_1) - E(K_1)E(Q_1) \\
&= P(K_1 = Q_1 = 1) - P(K_1 = 1)P(Q_1 = 1) \\
&= 0 - \frac{1}{13}\cdot\frac{1}{13} \\
&= -\frac{1}{169}.
\end{aligned}$$

Altogether:

$$\text{Cov}(K,Q) = -\frac{2}{169}.$$

Thus, (i) is correct.

(c) Whatever happens throughout the trial, the total number of queens in both decks together is 8, so that $Q' = 8-Q$. Since Q' is a linear function of Q, and the first coefficient is negative, we get

$$\rho(Q,Q') = -1.$$

Thus, (i) is correct.

(d) In a single experiment, the probability that the card drawn from a given deck is the queen of diamonds is $1/52$, so the probability that both drawn cards are the queen of diamonds of their decks is $1/52^2$. Let X denote the number of times this occurs when we repeat the experiment $39 \cdot 52$ times. In terms of X, the event in question is $\{X \geq 1\}$. Clearly, $X \sim B(39 \cdot 52, 1/52^2)$. Employing the Poissonian approximation for the binomial, we get that X is distributed approximately

$$P(39 \cdot 52 \cdot 1/52^2) = P(3/4).$$

Therefore:

$$P(X \geq 1) = 1 - P(X = 0) \approx 1 - e^{-3/4}.$$

Thus, (iv) is correct.

3. (a) The function $2F_X(t)$ cannot possibly be a distribution function as its limit at ∞ is 2 rather than 1.

   Suppose $F_Y(2t) = F_X(t)$ for all $t \in \mathbf{R}$. Then for any $u \in \mathbf{R}$,

$$F_Y(u) = F_Y(2 \cdot u/2) = F_X(u/2) = P(X \leq u/2) = P(2X \leq u) = F_{2X}(u).$$

   Hence $2X$ and Y are identically distributed.

   Thus, (iv) is correct.

   (b) Clearly, $X \sim B(2, 1/2)$ and $Y_1, Y_2 \sim U[1, 6]$. Now, if $X = 0$, then $Z_1 = Z_2 = 0$, so the event $\{Z_1 \leq 1, Z_2 \leq 2\}$ certainly occurs. If $X = 1$, then the event occurs if and only if $Y_1 = 1$ and $Y_2 \leq 2$. Finally, if $X = 2$, the event cannot occur. It follows that:

$$\begin{aligned} F_{Z_1,Z_2}(1,2) &= P(Z_1 \leq 1, Z_2 \leq 2) \\ &= P(X=0) + P(X=1)P(Y_1=1)P(Y_2 \leq 2) \\ &= \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{2}{6} \\ &= \frac{1}{4} + \frac{1}{36} \\ &= \frac{5}{18}. \end{aligned}$$

   Thus, (i) is correct.

(c) We have $X \sim \overline{B}(3, 1/2)$ and $Y \sim G(1/6)$. Hence:

$$\begin{aligned}
P(X=Y) &= \sum_{n=3}^{\infty} P(X=Y=n) \\
&= \sum_{n=3}^{\infty} P(X=n)P(Y=n) \\
&= \sum_{n=3}^{\infty} \binom{n-1}{2} \left(\frac{1}{2}\right)^{n} \cdot \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \\
&= \frac{1}{12} \sum_{n=3}^{\infty} \binom{n-1}{2} \left(\frac{5}{12}\right)^{n-1} \\
&= \frac{1}{12} \sum_{n=2}^{\infty} \binom{n}{2} \left(\frac{5}{12}\right)^{n} \\
&= \frac{1}{12} \cdot \frac{\left(\frac{5}{12}\right)^2}{\left(1-\frac{5}{12}\right)^3} \\
&= \frac{5^2}{7^3}.
\end{aligned}$$

Thus, (ii) is correct.