Final #1

Mark the correct answer in each part of the following questions.

1. We draw all cards one by one **without replacement** from a full deck of 52 cards. Let $Q_\heartsuit$ denote the number of the step at which the queen of hearts is drawn and $Q_\diamondsuit$ the analogous number for the queen of diamonds.

   (a) $E\left(2^{Q_\heartsuit}\right) =$

   (i) $\frac{2^{52}-1}{52}$.

   (ii) $\frac{2^{52}-1}{51}$.

   (iii) $\frac{2^{52}-1}{26}$.

   (iv) $\frac{2^{52}-1}{13}$.

   (v) None of the above.

   (b) $\rho(Q_\heartsuit, Q_\diamondsuit) =$

   (i) $\frac{-1}{51}$.

   (ii) $\frac{-1}{52}$.

   (iii) $\frac{1}{52}$.

   (iv) $\frac{1}{51}$.

   (v) None of the above.

(c) Let $A_\heartsuit$ denote the event that the queen of hearts is one of the first 10 cards to be drawn and $A_\diamondsuit$ the analogous event for the queen of diamonds. Then $P(A_\diamondsuit|A_\heartsuit) =$

(i) $\frac{10}{53}$.
(ii) $\frac{9}{52}$.
(iii) $\frac{10}{52}$.
(iv) $\frac{9}{51}$.
(v) None of the above.

(d) Suppose now that we draw the cards **with replacement** until each card has been drawn at least once. Denote by $Q'_\heartsuit$ the number of the step at which the queen of hearts is drawn for the first time. Then $V(Q'_\heartsuit)/V(Q_\heartsuit) \in$

(i) $[1/2, 2)$.
(ii) $[2, 4)$.
(iii) $[4, 10)$.
(iv) $[10, 100)$.
(v) None of the above.

2. The number of calves a blue whale has follows a Poisson distribution with parameter 1. Let X be the number of calves of a randomly selected blue whale, and let Y be the number of grandchildren of that same whale. (Assume that the number of calves a whale has and the number of calves each of these calves has are independent.)

(a) $F_{X,Y}(1, 1) =$

(i) $e^{-1} + e^{-2}$.
(ii) $e^{-1} + 2e^{-2}$.
(iii) e^{-2}.
(iv) $2e^{-2}$.
(v) None of the above.

(b) $P(2 \leq X \leq 3|Y = 0) =$

(i) $\dfrac{e^{-1}/2! + e^{-2}/3!}{e^{-1+e^{-1}}}$.

(ii) $\dfrac{e^{-3}/2! + e^{-4}/3!}{e^{-1+e^{-1}}}$.

(iii) $\dfrac{e^{-1}/1! + e^{-2}/2!}{e^{-1+e^{-1}}}$.

(iv) $\dfrac{e^{-3}/1! + e^{-4}/2!}{e^{-1+e^{-1}}}$.

(v) None of the above.

(c) A whale with

- up to two calves – receives a monthly food allowance of 1 ton from the National Security Institute and 3 tax credits from the Income Tax Authority.
- three or four calves – 2 tons and 5 tax credits.
- five calves and above – 3 tons and 7 tax credits.

Denote by Z the food allowance of a random whale (in tons) and by W the number of tax credits of that same whale.

(i) Z and W are independent.

(ii) Z and W are dependent but uncorrelated.

(iii) $\rho(Z, W) = e^{-1}$.

(iv) $\rho(Z, W) = 1$.

(v) None of the above.

(d) Four random whales are selected at random. Denote by $p_{\geq 4}$ the probability that they have altogether at least four calves. We are interested in the best upper bound that follows from Markov's inequality regarding $p_{\geq 4}$.

   (i) No non-trivial bound follows from Markov's inequality in this case.

   (ii) The best upper bound that follows from Markov's inequality is $p_{\geq 4} \leq 1/e$.

   (iii) The best upper bound that follows from Markov's inequality is $p_{\geq 4} \leq 4/e^4$.

   (iv) The best upper bound that follows from Markov's inequality is $p_{\geq 4} \leq 4!/e^4$.

   (v) None of the above.

(e) The exact probability that the four whales from the preceding part have exactly four calves is:

   (i) $1/e^4$.

   (ii) $4/e^4$.

   (iii) $16/(5e^4)$.

   (iv) $32/(3e^4)$.

   (v) None of the above.

3. In a multi-round game, two dice – one blue and the other red – are rolled again and again until the blue one shows a 6. In addition, a single coin is tossed in each stage. Let X be the number of times the red die shows a 6 in those rolls and Y the number of times the coin shows a head.

(a) $P(X = k) =$

(i)

$$\begin{cases} \frac{5}{11}, & k = 0, \\ \frac{5 \cdot 6}{11^2} \cdot \left(\frac{6}{11}\right)^{k-1}, & k \geq 1. \end{cases}$$

(ii)

$$\begin{cases} \frac{5}{11}, & k = 0, \\ \frac{6^2}{11^2} \cdot \left(\frac{5}{11}\right)^{k-1}, & k \geq 1. \end{cases}$$

(iii)

$$\begin{cases} \frac{6}{11}, & k = 0, \\ \frac{5 \cdot 6}{11^2} \cdot \left(\frac{5}{11}\right)^{k-1}, & k \geq 1. \end{cases}$$

(iv)

$$\begin{cases} \frac{6}{11}, & k = 0, \\ \frac{5^2}{11^2} \cdot \left(\frac{6}{11}\right)^{k-1}, & k \geq 1. \end{cases}$$

(v) None of the above.

(b) Consider the variables $3X$ and Y.

(i) $3X = Y$.

(ii) The variables $3X$ and Y are not equal, but they are identically distributed.

(iii) The variables $3X$ and Y are not identically distributed, but $E(3X) = E(Y)$.

(iv) The variable $3X$ does not have an expectation, whereas Y has.

(v) None of the above.

(c) Suppose now that we repeat the experiment 100 times. Let Z be the number of stages at which the red die has shown no 6. The distribution of Z is:

(i) Hypergeometric.

(ii) Binomial.

(iii) Uniform.

(iv) Negative binomial.

(v) None of the above.

Solutions

1. (a) Clearly, $Q_\heartsuit \sim U[1, 52]$, and therefore:

$$E\left(2^{Q_\heartsuit}\right) = \sum_{k=1}^{52} \frac{1}{52} \cdot 2^k = \frac{1}{52} \cdot 2 \cdot (2^{52} - 1) = \frac{2^{52} - 1}{26}.$$

Thus, (iii) is true.

(b) Put an arbitrary order on the set of all cards. Let $X_i, 1 \leq i \leq 52$, be the number of the step at which the i-th card is drawn. (Thus, $Q_\heartsuit$ and $Q_\diamondsuit$ are two of the X_i-s.) By symmetry, $\mathrm{Cov}(X_i, X_j)$ is the same for every i, j with $i \neq j$. Now notice that the numbers $X_1, X_2, \ldots, X_{52}$ form a permutation of $1, 2, \ldots, 52$, and hence

$$X_1 + X_2 + \cdots + X_{52} = 1 + 2 + \cdots + 52 = 52 \cdot 53/2.$$

It follows that:

$$V(X_1 + X_2 + \cdots + X_{52}) = 0.$$

On the other hand:

$$V(X_1 + X_2 + \cdots + X_{52}) = \sum_{i=1}^{52} V(X_i) + 2 \sum_{i=1}^{51} \sum_{j=i+1}^{52} \mathrm{Cov}(X_i, X_j).$$

By symmetry:

$$V(X_1 + X_2 + \cdots + X_{52}) = 52V(X_1) + 52 \cdot 51 \,\mathrm{Cov}(X_1, X_2).$$

It follows that

$$\mathrm{Cov}(X_1, X_2) = -\frac{1}{51} \cdot V(X_1),$$

which implies

$$\rho(X_1, X_2) = \frac{-V(X_1)/51}{V(X_1)} = -\frac{1}{51}.$$

Thus, (i) is true.

(c) If the queen of hearts is known to be one of the first 10, then each of the other 51 cards has, by symmetry, 9 possibilities of being drawn also among the first 10 out of 51 possibilities altogether. Thus, (iv) is true.

(d) Clearly, $Q'_\heartsuit \sim G(1/52)$, while, as mentioned above, $Q_\heartsuit \sim U[1, 52]$. According to the formulas for the variance of random variables of the families considered in class, we have:

$$\frac{V(Q'_\heartsuit)}{Q_\heartsuit} = \frac{(51/52)/(1/52)^2}{((1+52-1)^2-1)/12} = \frac{51 \cdot 52 \cdot 12}{51 \cdot 53} = \frac{52 \cdot 12}{53}.$$

Thus, (iv) is true.

2. (a) We have:

$$\begin{aligned} F_{X,Y}(1,1) &= P(X \le 1, Y \le 1) \\ &= P(X=0) + P(X=1, Y=0) + P(X=Y=1) \\ &= e^{-1} + e^{-1} \cdot e^{-1} + e^{-1} \cdot e^{-1} \\ &= e^{-1} + 2e^{-2}. \end{aligned}$$

Thus, (ii) is true.

(b) First note that:

$$\begin{aligned} P(Y=0) &= \sum_{k=0}^{\infty} P(X=k)P(Y=0|X=k) \\ &= \sum_{k=0}^{\infty} e^{-1}\frac{1^k}{k!} \cdot \left(e^{-1}\right)^k \\ &= e^{-1}\sum_{k=0}^{\infty} \frac{(e^{-1})^k}{k!} \\ &= e^{-1}e^{e^{-1}} \\ &= e^{-1+e^{-1}}. \end{aligned}$$

Therefore:

$$\begin{aligned} P(2 \le X \le 3 | Y = 0) &= \frac{P(2 \le X \le 3, Y = 0)}{P(Y = 0 | X = k)} \\ &= \frac{e^{-1} \cdot 1^2/2! \cdot (e^{-1})^2 + e^{-1} \cdot 1^3/3! \cdot (e^{-1})^3}{e^{-1+e^{-1}}} \\ &= \frac{e^{-2}/2! + e^{-3}/3!}{e^{e^{-1}}}. \end{aligned}$$

Thus, (ii) is true.

(c) Z and W are linearly related:

$$W = 2Z + 1.$$

Therefore, $\rho(Z, W) = 1$.

Thus, (iv) is true.

(d) According to the formula for the expectation of a Poissonian random variable, the expected number of calves of a random whale is 1. Denoting by X_1, X_2, X_3, X_4 the number of calves of the selected whales, we get by Markov's inequality:

$$\begin{aligned} p_{\ge 4} &= P(X_1 + X_2 + X_3 + X_4 \ge 4) \\ &\le \frac{E(X_1 + X_2 + X_3 + X_4)}{4} \\ &= \frac{4}{4} \\ &= 1. \end{aligned}$$

Thus, (i) is true.

(e) We have $X_1 + X_2 + X_3 + X_4 \sim P(4)$, so that

$$P(X_1 + X_2 + X_3 + X_4 = 4) = e^{-4} \cdot \frac{4^4}{4!} = \frac{32}{3} \cdot e^{-4}.$$

Thus, (iv) is true.

3. (a) Let Z be the number of times the blue die is rolled. By the law of total probability:

$$\begin{aligned} P(X=k) &= \sum_{n=1}^{\infty} P(Z=n)P(X=k|Z=n) \\ &= \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} \cdot \binom{n}{k} \left(\frac{5}{6}\right)^{n-k} \left(\frac{1}{6}\right)^{k} \\ &= \frac{1}{5^{k+1}} \sum_{n=1}^{\infty} \binom{n}{k} \left(\frac{5^2}{6^2}\right)^{n}. \end{aligned}$$

For $k = 0$ we obtain a geometric series:

$$P(X=0) = \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{5^2}{6^2}\right)^{n} = \frac{1}{5}\cdot\frac{5^2/6^2}{1-5^2/6^2} = \frac{5}{11}.$$

For $k \geq 1$ we use the lemma proved in class to obtain:

$$P(X=0) = \frac{1}{5^{k+1}}\cdot\frac{(5^2/6^2)^k}{(1-5^2/6^2)^{k+1}} = \frac{5^{k-1}6^2}{11^{k+1}}.$$

Thus, (ii) is true.

(b) The two variables do not have the same distribution as they do not even have the same set of possible values. Whereas Y may assume any non-negative integer value, $3X$ assumes only values divisible by 3. We now calculate the expectations of the two variables. Let $Y_n = 1$ if the experiment lasts at least n rounds, and the coin shows a head at the n-th round, and $Y_n = 0$ otherwise, $1 \leq n < \infty$. Then

$$Y = \sum_{n=1}^{\infty} Y_n,$$

and therefore

$$E(Y) = \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1}\cdot\frac{1}{2} = 3.$$

A similar calculation shows that $E(3X) = 3$ as well.

Thus, (iii) is true.

(c) At each of the 100 repetitions, there is some positive probability p that the red die will show no 6. Since the experiments are independent, $Z \sim B(100, p)$.

Thus, (ii) is true.