Final #1

Mark the correct answer in each part of the following questions.

1. A player rolls a die n times. After each roll, if the die shows "6", the player gets a gold coin, and if it shows any other number – a silver coin. Let X denote the number of times the die showed "6". (For example, if $n = 20$ and the die showed "6" three times and other numbers 17 times, the player gets three gold coins and 17 silver coins.)

   (a) $F_X(n-2) =$

   (i) $1 - \frac{1}{6^n}$.
   (ii) $1 - \frac{n}{6^n}$.
   (iii) $1 - \frac{n+1}{6^n}$.
   (iv) $\binom{n}{n-2} \cdot \left(\frac{1}{6}\right)^{n-2} \cdot \left(\frac{5}{6}\right)^2$.
   (v) None of the above.

   (b) Suppose that $n = 100$. Let M_X denote the moment-generating function of X. Then $M_X(\log 19) =$

   (i) 2^{100}.
   (ii) 3^{100}.
   (iii) 4^{100}.
   (iv) 5^{100}.
   (v) None of the above.

   (c) Denote by Z the number of indices i between 1 and $n-1$, such that the coins the player wins at stages i and $i+1$ are of distinct types. (For example, if the player first won 10 gold coins, then 30 silver coins, and then 60 gold coins, the relevant indices i are 10 and 40, and $Z = 2$.) Then $V(Z) =$

(i) $(n-1)\cdot\frac{65}{324}+(n-1)(n-2)\cdot\frac{10}{81}$.
(ii) $(n-1)\cdot\frac{65}{324}+(n-2)\cdot\frac{10}{81}$.
(iii) $(n-1)\cdot\frac{65}{324}+(n-1)(n-2)\cdot\frac{5}{81}$.
(iv) $(n-1)\cdot\frac{65}{324}+(n-2)\cdot\frac{5}{81}$.
(v) None of the above.

(d) Suppose now that the rules are changed against the player, as follows. He has to toss each of the gold coins he got, and wins the coin only if it lands heads. Denote by Y the number of gold coins the player gets. (Continuing the example from part (a), if two of the gold coins showed a head and the third showed a tail, then $Y=2$.) Then $P(X=0|Y=0)=$

(i) $\left(\frac{5}{11}\right)^n$.
(ii) $\left(\frac{1}{2}\right)^n$.
(iii) $\left(\frac{10}{11}\right)^n$.
(iv) $\left(\frac{11}{12}\right)^n$.
(v) None of the above.

(e)

(i) $Y \sim B(n, 1/24)$.
(ii) $Y \sim B(n, 1/12)$.
(iii) $Y \sim B(n, 5/24)$.
(iv) $Y \sim B(n, 5/12)$.
(v) None of the above.

2. The Tax Agency needs to refund n taxpayers. The first is to get a refund of 1 shekel, the second of 2 shekels, ..., the n-th of n shekels. Assume that n is large. To minimize the total refund actually paid out, the letters are sent using the algorithm of the absent-minded secretary. Denote by A_i the event that the i-th letter is sent to the correct recipient, by X the total amount of money sent to the correct recipients and by Y the number of letters sent to the correct recipients. (For example, if the letters to the first, tenth, and twentieth people are the only letters sent to the correct recipients, then $X=31$ and $Y=3$.)

(a) $P(A_{20}|\overline{A}_{10}) =$

(i) $\frac{1}{n-1}$.

(ii) $\frac{n-1}{n^2}$.

(iii) $\frac{n-2}{(n-1)^2}$.

(iv) $\frac{n-2}{n(n-1)}$.

(v) None of the above.

(b) $P(X = Y) =$

(i) $\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}$.

(ii) $\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) + \frac{1}{n} \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right)$.

(iii) $\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) + \frac{1}{n-1} \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right)$.

(iv) $\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) + \frac{n-2}{(n-1)^2} \cdot \left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right)$.

(v) None of the above.

(c) $E(X|Y) =$

(i) $\frac{n+1}{2} \cdot Y$.

(ii) $\frac{n}{2} \cdot Y$.

(iii) $\frac{n-1}{2} \cdot Y$.

(iv) $\frac{n+1}{2}$.

(v) None of the above.

(d) $E(XY) =$

(i) $\frac{n+1}{2}$.

(ii) $n + 1$.

(iii) $\frac{3(n+1)}{2}$.

(iv) $2(n+1)$.

(v) None of the above.

3. Two players, A and B, toss a coin each again and again until each one's coin shows a head. Let X denote the number of tosses of A and Y the number of tosses of B. Then a coin is tossed $Z = X + Y$ more times. Let W denote the number of heads in these additional tosses. (For example, if A's coin showed a head for the first time at the sixth toss and B's coin at the fourth toss, then $X = 6, Y = 4$, and the coin is tossed $Z = 10$ more times. If out of these 10 tosses there are three heads, then $W = 3$.)

   (a) $\mathrm{Cov}(X^2 + X, Y^2 + Y)$ belongs to the interval:

   (i) $[-100, -1)$.

   (ii) $[-1, -0.01)$.

   (iii) $[-0.01, 0.01)$.

   (iv) $[0.01, 1)$.

   (v) None of the above.

   (b) $\rho(2X + 3Y, 3X + 2Y)$ belongs to the interval:

   (i) $[-1 - 0.01)$.

   (ii) $[-0.01, 0.01)$.

   (iii) $[0.5, 0.9)$.

   (iv) $[0.9, 1]$.

   (v) None of the above.

   (c) Denote $A = \{X = 3\}, B = \{Y = 3\}, C = \{Z = 3\}$. The union bound implies that $P(A \cup B \cup C) \leq$

   (i) $\frac{3}{8}$.

   (ii) $\frac{1}{2}$.

   (iii) $\frac{5}{8}$.

   (iv) $\frac{3}{4}$.

(v) None of the above.

Remark: We mean the best bound that can be obtained. For example, if (i) is correct, then (ii)-(iv) are correct as well, but only (i) should be marked as the correct answer.

(d) $P(W = 0) =$

(i) $\frac{1}{18}$.

(ii) $\frac{1}{16}$.

(iii) $\frac{1}{12}$.

(iv) $\frac{1}{9}$.

(v) None of the above.

(e) $V(W) =$

(i) 2.

(ii) 3.

(iii) 4.

(iv) 6.

(v) None of the above.

(f) $F_{Z,W}(4, 100) =$

(i) $\frac{1}{2}$.

(ii) $\frac{9}{16}$.

(iii) $\frac{5}{8}$.

(iv) $\frac{11}{16}$.

(v) None of the above.

Solutions

1. (a) Clearly, $X \sim B(n, 1/6)$, and therefore:

$$
\begin{aligned}
F_X(n-2) = P(X \leq n-2) &= 1 - P(X = n-1) - P(X = n) \\
&= 1 - \binom{n}{n-1} \left(\frac{1}{6}\right)^{n-1} \left(\frac{5}{6}\right)^{1} - \left(\frac{1}{6}\right)^{n} \\
&= 1 - n \cdot \frac{1}{6^{n-1}} \cdot \frac{5}{6} - \frac{1}{6^n} \\
&= 1 - \frac{5n+1}{6^n}.
\end{aligned}
$$

Thus, (v) is true.

(b) Since $X \sim B(100, 1/6)$, its moment-generating function is:

$$M_X(t) = \left(\frac{5}{6} + \frac{1}{6}e^t\right)^{100}.$$

Substitute $t = \log 19$:

$$M_X(\log 19) = \left(\frac{5}{6} + \frac{1}{6} \cdot 19\right)^{100} = \left(\frac{5+19}{6}\right)^{100} = \left(\frac{24}{6}\right)^{100} = 4^{100}.$$

Thus, (iii) is true.

(c) We have $Z = \sum_{i=1}^{n-1} I_i$, where $I_i = 1$ if the coins at positions i and $i+1$ differ and $I_i = 0$ otherwise.

Since the rolls are independent:

$$E(I_i) = \frac{1}{6} \cdot \frac{5}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{18}.$$

By the representation of Z:

$$V(Z) = \sum_{i=1}^{n-1} V(I_i) + 2 \sum_{1 \leq i < j \leq n-1} \mathrm{Cov}(I_i, I_j).$$

Clearly:

$$V(I_i) = \frac{5}{18}\left(1 - \frac{5}{18}\right) = \frac{5}{18} \cdot \frac{13}{18} = \frac{65}{324}, \qquad 1 \le i \le n-1.$$

Next, we consider the covariances. Consecutive I_i's are dependent because they share a common roll, whereas non-consecutive ones are independent. Now $I_i I_{i+1} = 1$ only if the coins at rolls $i, i+1, i+2$ are either gold-silver-gold or silver-gold-silver. Hence:

$$E(I_i I_{i+1}) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}, \qquad 1 \le i \le n-2,$$

and thus:

$$\text{Cov}(I_i, I_{i+1}) = E(I_i I_{i+1}) - E^2(I_i) = \frac{5}{36} - \left(\frac{5}{18}\right)^2 = \frac{5}{81}.$$

Therefore:

$$V(Z) = (n-1) \cdot \frac{65}{324} + (n-2) \cdot \frac{10}{81}.$$

Thus, (ii) is true.

(d) We have:

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0, Y = 0)}{P(Y = 0)} = \frac{P(X = 0)}{P(Y = 0)}.$$

Clearly:

$$P(X = 0) = \left(\frac{5}{6}\right)^n.$$

For the denominator, by the law of total probability, we have:

$$\begin{aligned} P(Y=0) &= \sum_{k=0}^{n} P(X=k) \cdot P(Y=0 \mid X=k) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \cdot \left(\frac{1}{2}\right)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{12}\right)^k \left(\frac{5}{6}\right)^{n-k} \\ &= \left(\frac{1}{12} + \frac{5}{6}\right)^n \\ &= \left(\frac{11}{12}\right)^n . \end{aligned}$$

Therefore:

$$P(X=0 \mid Y=0) = \frac{\left(\frac{5}{6}\right)^n}{\left(\frac{11}{12}\right)^n} = \left(\frac{10}{11}\right)^n .$$

Thus, (iii) is true.

(e) Relating to the event that a die roll yields "6" and the coin toss following it yields a head as a success, we see that Y is the number of successes in n independent trials with success probability $1/12$ each. Hence $Y \sim B(n, 1/12)$.

Thus, (ii) is true.

2. (a) We have:

$$\begin{aligned} P(A_{20} \mid \overline{A}_{10}) &= \frac{P(A_{20} \cap \overline{A}_{10})}{P(\overline{A}_{10})} \\ &= \frac{P(A_{20})P(\overline{A}_{10} \mid A_{20})}{P(\overline{A}_{10})} \\ &= \frac{\frac{1}{n} \cdot (1 - \frac{1}{n-1})}{1 - \frac{1}{n}} \\ &= \frac{n-2}{(n-1)^2}. \end{aligned}$$

Thus, (iii) is true.

(b) If any taxpayer who needs to get two shekels or more gets it, then the amount paid exceeds the number of refunded taxpayers. Hence the event $X = Y$ occurs if and only if either no letter is sent to the correct address or only the first taxpayer gets the refund. The first of these two events is exactly the event considered in class, whose probability was found to be $P_n := \sum_{k=0}^{n} (-1)^k/k!$. For the second to occur, the first letter has to be sent to the correct address, which occurs with probability $1/n$, and then we have $n-1$ letters left, that are all to be sent to incorrect addresses, which occurs with probability P_{n-1}. Hence the required probability is

$$P_n + \frac{1}{n} \cdot P_{n-1}.$$

Thus, (ii) is true.

(c) Given Y, by symmetry, each letter has a probability of Y/n of having been sent to the correct address. Hence the contribution of taxpayer i to the expected total payments is $i \cdot Y/n$. It follows that:

$$E(X \mid Y) = (1 + 2 + \cdots + n) \cdot \frac{Y}{n} = \frac{n(n+1)}{2} \cdot \frac{Y}{n} = \frac{n+1}{2} \cdot Y.$$

Thus, (i) is true.

(d) We have:

$$\begin{aligned}
E(XY) &= E(E(XY \mid Y)) \\
&= E(YE(X \mid Y)) \\
&= E(Y \cdot \frac{n+1}{2} Y) \\
&= \frac{n+1}{2} E(Y^2) \\
&= \frac{n+1}{2} (E^2(Y) + V(Y)) \\
&= \frac{n+1}{2} \cdot (1+1) \\
&= n+1.
\end{aligned}$$

Thus, (ii) is true.

3. (a) Since X and Y are independent, and $X^2 + X$ is a function of X alone, while Y^2+Y is a function of Y alone, the latter two variables are independent as well. In particular, they are uncorrelated:

$$\mathrm{Cov}(X^2 + X, Y^2 + Y) = 0.$$

Thus, (iii) is true.

(b) Since X and Y are independent and identically distributed,

$$\mathrm{Cov}(2X + 3Y,\ 3X + 2Y) = 2 \cdot 3 \cdot V(X) + 3 \cdot 2 \cdot V(Y) = 12V(X).$$

Also

$$V(2X + 3Y) = 2^2 V(X) + 3^2 V(Y) = 13V(X),$$

and similarly

$$V(3X + 2Y) = 13V(X).$$

Hence:

$$\rho(2X + 3Y,\ 3X + 2Y) = \frac{12V(X)}{13V(X)} = \frac{12}{13}.$$

Thus, (iv) is true.

(c) We are asked to apply the union bound:

$$P(A \cup B \cup C) \leq P(A) + P(B) + P(C).$$

Since $X, Y \sim G(1/2)$, we have $Z \sim \overline{B}(2, 1/2)$. Therefore:

$$P(A) = P(X = 3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8},$$

and $P(B) = 1/8$ for the same reason. Next:

$$P(Z = 3) = \binom{3-1}{2-1}\left(\frac{1}{2}\right)^3 = \frac{1}{4}.$$

By the union bound:

$$P(A \cup B \cup C) \leq \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}.$$

Thus, (ii) is true.

(d) Employing the law of total probability, we obtain:

$$\begin{aligned}
P(W = 0) &= \textstyle\sum_{k=2}^{\infty} P(Z = k) P(W = 0 \mid Z = k) \\
&= \textstyle\sum_{k=2}^{\infty} \binom{k-1}{2-1}\left(\frac{1}{2}\right)^k \cdot \left(\frac{1}{2}\right)^k \\
&= \textstyle\sum_{k=1}^{\infty} \binom{k}{1}\left(\frac{1}{4}\right)^{k+1} \\
&= \tfrac{1}{4} \cdot \tfrac{(1/4)^1}{(1-1/4)^2} \\
&= \tfrac{1}{9}.
\end{aligned}$$

Thus, (iv) is true.

(e) We have $W \mid Z \sim B(Z, 1/2)$, so that

$$E(W \mid Z) = Z \cdot \frac{1}{2},$$

and

$$V(W \mid Z) = Z \cdot \frac{1}{2} \cdot \frac{1}{2}.$$

Consequently:

$$
\begin{aligned}
V(W) &= E(V(W \mid Z)) + V(E(W \mid Z)) \\
&= E(Z/4) + V(Z/2) \\
&= \tfrac{1}{4} \cdot \tfrac{2}{1/2} + \tfrac{1}{4} \cdot \tfrac{2 \cdot 1/2}{(1/2)^2} \\
&= 2.
\end{aligned}
$$

Thus, (i) is true.

(f) We have:

$$
\begin{aligned}
F_{Z,W}(4, 100) &= P(Z \leq 4, W \leq 100) \\
&= P(Z \leq 4) \\
&= P(Z = 2) + P(Z = 3) + P(Z = 4) \\
&= \tbinom{2-1}{2-1} \left(\tfrac{1}{2}\right)^2 + \tbinom{3-1}{2-1} \left(\tfrac{1}{2}\right)^3 + \tbinom{4-1}{2-1} \left(\tfrac{1}{2}\right)^4 \\
&= \tfrac{11}{16}.
\end{aligned}
$$

Thus, (iv) is true.