

Midterm

Mark the correct answer in each part of the following questions.

1. We have three urns. The first contains one die, the second two dice, and the third three dice. An urn is selected randomly, and all dice in it are rolled.
 - (a) Suppose it is known that the sum of upfaces in the rolls is 4. The probability that the third urn has been selected is:
 - (i) $\frac{1}{18}$.
 - (ii) $\frac{1}{19}$.
 - (iii) $\frac{1}{20}$.
 - (iv) $\frac{1}{21}$.
 - (v) None of the above.
 - (b) Now suppose we repeat the entire experiment (i.e., selecting an urn and rolling all its dice) six times. The probability that the sum of the outcomes is 5 in exactly two of the six trials is

- (i) $15 \cdot \left(\frac{7}{108}\right)^2 \cdot \left(\frac{101}{108}\right)^4$.
- (ii) $30 \cdot \left(\frac{7}{108}\right)^2 \cdot \left(\frac{101}{108}\right)^4$.
- (iii) $15 \cdot \left(\frac{11}{108}\right)^2 \cdot \left(\frac{97}{108}\right)^4$.
- (iv) $30 \cdot \left(\frac{11}{108}\right)^2 \cdot \left(\frac{97}{108}\right)^4$.
- (v) None of the above.

2. Consider the negligent secretary problem. For $m = 1, 2, \dots$, denote by p_m the probability that, for m letters and corresponding envelopes, none of the letters will be sent to the correct destination.

(a) On each of three distinct days, we send three letters to three addressees, using the algorithm of the absent-minded secretary. Let X be the number of letters (out of all nine letters) sent to the correct destination. Then $F_X(7.5) =$

- (i) $\frac{199}{216}$.
- (ii) $\frac{203}{216}$.
- (iii) $\frac{211}{216}$.
- (iv) $\frac{215}{216}$.
- (v) None of the above.

(b) Now consider the problem with $n \geq 5$ letters and envelopes in a single trial. The probability that exactly three of the letters will be sent to the correct destination is

- (i) $\frac{1}{18}p_{n-3}$.
- (ii) $\frac{n(n-1)(n-2)}{18}p_{n-3}$.
- (iii) $\frac{1}{6}p_{n-3}$.
- (iv) $\frac{n(n-1)(n-2)}{6}p_{n-3}$.
- (v) None of the above.

(c) Mr. John Doe is one of the n addressees. It is known that he received a letter intended to someone else. The probability that none of the letters was sent to the correct address is:

- (i) $\frac{n}{n-1}p_n$.
- (ii) $\frac{n-1}{n}p_n$.
- (iii) p_n .
- (iv) p_{n-1} .
- (v) None of the above.

3. A player draws cards **with replacement** from a full deck until drawing the king of hearts. If he has drawn the card in his n -th trial, he is asked to guess an integer drawn randomly between 1 and n . If he guesses correctly, he gets a prize.

(a) The player's winning probability is

- (i) $\frac{1}{51}$.
- (ii) $\frac{\log 52}{51}$.
- (iii) e^{-51} .
- (iv) $e^{-52/51}$.
- (v) None of the above.

(b) Let X denote the number of kings and queens drawn throughout the game (including the king of hearts drawn at the end). Then $X \sim$

(i) $B(52, 8/52)$.

(ii) $H(8, 8, 44)$.

(iii) $\bar{B}(8, 8/52)$.

(iv) $G(1/8)$.

(v) None of the above.

(c) We repeat the game 104 times. Let Y denote the number of times the king of hearts was drawn right at the first step (so the player got the prize for certain). Then $P(Y = 6) \approx$

(i) $\frac{e^{-2}}{45}$.

(ii) $\frac{2e^{-2}}{45}$.

(iii) $\frac{4e^{-2}}{45}$.

(iv) $\frac{8e^{-2}}{45}$.

(v) None of the above.

4. An urn contains an infinite number of discs, marked by the numbers $1, 2, 3, \dots$. The radius of the first disc is $1/3$, of the second $1/3^2$, of the third $1/3^3$ and so forth. When drawing from the urn, the probability of each disc is proportional to its radius (so that the probability of the n -th ball is $2/3^n$ for $1 \leq n < \infty$). We now draw a disc randomly from the urn infinitely many times **with replacement**.

(a) Let A_n denote the event that the disc drawn at stage n is the n -th disc. Denote $A = \bigcup_{n=1}^{\infty} A_{2n}$. (Note that we took only the even indices.) From the union bound it follows that $P(A) \leq$

- (i) $\frac{1}{4}$.
- (ii) $\frac{1}{3}$.
- (iii) $\frac{1}{2}$.
- (iv) $\frac{5}{6}$.
- (v) None of the above.

Remark: We mean the best bound that can be obtained. For example, if (i) is correct, then (ii)-(iv) are correct as well, but only (i) should be marked as the correct answer.

- (b) Let B denote the event that no disc from the 101st on will ever be drawn. Then $P(B) \in$
 - (i) $[0, 1/2 \cdot 3^{100})$.
 - (ii) $[1/2 \cdot 3^{100}, 1/3^{100})$.
 - (iii) $[1/3^{100}, 2/3^{100})$.
 - (iv) $[2/3^{100}, 1/3^{99})$.
 - (v) None of the above.

Solutions

1. (a) Denote by U the index of the selected urn and by S the sum of outcomes. We have

$$P(U = 1) = P(U = 2) = P(U = 3) = \frac{1}{3},$$

and it is easy to verify that

$$\begin{aligned}P(S = 4 | U = 1) &= \frac{1}{6}, \\P(S = 4 | U = 2) &= \frac{3}{36} = \frac{1}{12}, \\P(S = 4 | U = 3) &= \frac{3}{216} = \frac{1}{72}.\end{aligned}$$

By the law of total probability:

$$\begin{aligned}P(S = 4) &= \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{12} + \frac{1}{3} \cdot \frac{1}{72} \\&= \frac{1}{18} + \frac{1}{36} + \frac{1}{216} \\&= \frac{12 + 6 + 1}{216} \\&= \frac{19}{216}.\end{aligned}$$

Therefore,

$$\begin{aligned}P(U = 3 | S = 4) &= \frac{\frac{1}{3} \cdot \frac{1}{72}}{19/216} \\&= \frac{1/216}{19/216} \\&= \frac{1}{19}.\end{aligned}$$

Thus, (ii) is correct.

- (b) Similarly to the computation in the preceding part:

$$\begin{aligned}P(S = 5) &= \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{4}{36} + \frac{1}{3} \cdot \frac{6}{216} \\&= \frac{1}{3} \cdot \frac{6 + 4 + 1}{36} \\&= \frac{11}{108}.\end{aligned}$$

Considering the event of obtaining a sum of 5 as a success, we need to find the probability of having two successes out of six independent trials with a success probability of $11/108$. The required probability is therefore:

$$\begin{aligned} p &= \binom{6}{2} \left(\frac{11}{108}\right)^2 \left(\frac{97}{108}\right)^4 \\ &= 15 \cdot \left(\frac{11}{108}\right)^2 \left(\frac{97}{108}\right)^4. \end{aligned}$$

Thus, (iii) is correct.

2. (a) On a single day with three letters, the probability for all letters to be sent to the correct destination is $1/3! = 1/6$. Notice that it is impossible for exactly two letters to be sent to the correct destination. Thus, $X \leq 7.5$ unless, in all three days, all letters are sent to the correct destination. Hence:

$$\begin{aligned} F_X(7.5) &= P(X \leq 7.5) \\ &= 1 - \left(\frac{1}{6}\right)^3 \\ &= 1 - \frac{1}{216} \\ &= \frac{215}{216}. \end{aligned}$$

Thus, (iv) is correct.

- (b) The probability that three specific letters are sent to the correct destination and all others are not is

$$\frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdot p_{n-3}.$$

Summing over all $\binom{n}{3}$ choices of such triples (which are pairwise disjoint), we obtain

$$\binom{n}{3} \cdot \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdot p_{n-3} = \frac{1}{6} p_{n-3}.$$

Thus, (iii) is correct.

- (c) Let A_{all} and A_J denote the events that all letters were sent to incorrect addresses and that John Doe's letter was sent to an incorrect address, respectively. We have:

$$\begin{aligned} P(A_{\text{all}} | A_J) &= \frac{P(A_{\text{all}})}{P(A_J)} \\ &= \frac{p_n}{1 - 1/n} \\ &= \frac{n}{n-1} p_n. \end{aligned}$$

Thus, (i) is correct.

3. (a) Let N be the number of trials until the king of hearts is first drawn. Then:

$$P(N = n) = \left(\frac{51}{52}\right)^{n-1} \frac{1}{52}, \quad n = 1, 2, \dots$$

Given $N = n$, the player guesses correctly with probability $\frac{1}{n}$. Hence the probability of winning is

$$\begin{aligned} \sum_{n=1}^{\infty} P(N = n) \cdot \frac{1}{n} &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{51}{52}\right)^{n-1} \frac{1}{52} \\ &= \frac{1}{52} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{51}{52}\right)^n. \end{aligned}$$

Now recall that for $|x| < 1$:

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x).$$

Thus:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{51}{52}\right)^n &= -\log\left(1 - \frac{51}{52}\right) \\ &= \log 52. \end{aligned}$$

Hence the required probability is $\frac{1}{52} \log 52$.

Thus, (ii) is correct.

- (b) The process may be considered as consisting of sub-processes, each of which consists of any number of drawing of cards that are neither kings nor queens, and ending with drawing a king or queen. Consider each sub-process as a success if the final card is the king of hearts, and as a failure otherwise. Then X is the number of such sub-processes until the first success. By symmetry, the probability of success is $1/8$, and hence $X \sim G(1/8)$.

We also provide an alternative computational solution. Let Z denote the number of trials until the king of hearts is drawn. For $i \geq 1$:

$$\begin{aligned}
 P(X = i) &= \sum_{n=i}^{\infty} P(X = i \mid Z = n) P(Z = n) \\
 &= \sum_{n=i}^{\infty} \binom{n-1}{i-1} \left(\frac{7}{51}\right)^{i-1} \left(\frac{44}{51}\right)^{n-i} \left(\frac{51}{52}\right)^{n-1} \cdot \frac{1}{52} \\
 &= \frac{1}{52} \cdot \left(\frac{7}{44}\right)^{i-1} \sum_{n=i}^{\infty} \binom{n-1}{i-1} \left(\frac{44}{52}\right)^{n-1}.
 \end{aligned}$$

Putting $n' = n - 1, i' = i - 1$, we have:

$$\begin{aligned}
 \sum_{n=i}^{\infty} \binom{n-1}{i-1} \left(\frac{44}{52}\right)^{n-1} &= \sum_{n'=i'}^{\infty} \binom{n'}{i'} \left(\frac{44}{52}\right)^{n'} \\
 &= \frac{(44/52)^{i'}}{(1 - 44/52)^{i'+1}} \\
 &= \frac{52}{8} \cdot \left(\frac{44}{8}\right)^{i'}.
 \end{aligned}$$

Hence:

$$\begin{aligned}
 P(X = i) &= \frac{1}{52} \cdot \left(\frac{7}{44}\right)^{i-1} \cdot \frac{52}{8} \cdot \left(\frac{44}{8}\right)^{i'} \\
 &= \frac{1}{8} \cdot \left(\frac{7}{8}\right)^{i-1}.
 \end{aligned}$$

Thus, (iv) is correct.

- (c) Now we consider the event of first drawing the king of hearts as a success. Thus, Y is the number of successes in a series of 104

independent trials with success probability $1/52$ each. Hence $Y \sim B(104, \frac{1}{52})$. By the Poissonian approximation of the binomial, Y is distributed approximately $P(\lambda)$ with

$$\lambda = 104 \cdot \frac{1}{52} = 2.$$

It follows that:

$$\begin{aligned} P(Y = 6) &\approx e^{-2} \cdot \frac{2^6}{6!} \\ &= e^{-2} \cdot \frac{64}{720} \\ &= \frac{4e^{-2}}{45}. \end{aligned}$$

Thus, (iii) is correct.

4. (a) By the union bound:

$$\begin{aligned} P(A) &\leq \sum_{n=1}^{\infty} P(A_{2n}) \\ &= \sum_{n=1}^{\infty} \frac{2}{9^n} \\ &= \frac{2}{9} \cdot \frac{1}{1 - 1/9} \\ &= \frac{1}{4}. \end{aligned}$$

Thus, (i) is correct.

(b) The probability that a draw yields a disc from the 101st on is:

$$\begin{aligned} \sum_{n=101}^{\infty} \frac{2}{3^n} &= \frac{2}{3^{101}} \cdot \frac{1}{1 - 1/3} \\ &= \frac{1}{3^{100}}. \end{aligned}$$

Hence the probability of drawing a disc among the first 100 ones is $1 - 1/3^{100}$. The event in question means that all draws yield discs among the first 100.

Hence:

$$P(B) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{3^{100n}}\right) = 0.$$

Thus, (i) is correct.