# Lights Out on Graphs

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## Abstract

Consider the following problem. Let $G = (V, E)$ be a given graph. Suppose that at each vertex there is a light bulb and a switch. Toggling the switch at a vertex, we flip the light at this vertex and all its neighbors – those that were off are turned on and vice versa. A *configuration* of lights is a point of $\{0, 1\}^V$, where a 0 coordinate indicates that the light at the corresponding vertex is off, while a 1 means that it is on.

**Question 0.1.** *Given two configurations, decide whether it is possible to pass from one to the other by some sequence of switch toggles.*

**Question 0.2.** *Which graphs have the property that one can pass from any configuration to any other?*

The game was introduced by Sutner , where the game was called the $\sigma^+$-*game*. (The "+" superscript denotes that each switch controls not only the neighboring vertices, but also the vertex it resides at.) It may also be found as a hand-held electronic game by Tiger electronics, called "Lights Out". This game is played on a 5 × 5 grid of buttons and light-bulbs, where at start-up a random configuration is generate, and the goal is to turn all lights out. Now, in principle, the first question can be easily answered. Let $M(G)$ be the adjacency matrix of $G$, where we take the diagonal elements to be 1. Consider $\mathcal{C} = \{0, 1\}^V$ as a vector space over the field $\{0, 1\}$. Clearly, if the system is at a configuration $C$ and we toggle the switch at vertex $v$, we arrive at the configuration corresponding to the sum in $\mathcal{C}$ of $C$ and the row of $M(G)$ corresponding to $v$. Thus, we can pass from a configuration $C_1$ to a configuration $C_2$ if and only if the vector $C_2 - C_1$ is the sum of several rows of $M(G)$ or, equivalently, if the equation $M(G)x = C_2 - C_1$ has a solution $x \in \{0, 1\}^V$. Thus we shall assume the initial configuration is always the all-off configuration and ask which configurations can be reached. In view of the preceding observations, we can arrive at each configuration if and only if $M(G)$ is non-singular. A graph is *light-transitive* if each configuration can be reached. Note that, in a light-transitive graph, each configuration may be obtained in an essentially unique way.

While the above discussion provides a simple algorithm for answering Question 0.1 and, given a specific graph, for answering Question 0.2 as well, it does not give an insight to understanding which graph-theoretical properties of $G$ are relevant. More generally, the *light deficiency* $\delta(G)$ of $G$, is the dimension of the kernel of $M(G)$. Thus, there exist $2^{|V|-\delta(G)}$ configurations that can be reached.

**Question 0.3.** *How does* $\delta(G)$ *depend on the graph-theoretical properties of* $G$*?*

The following result, due to Sutner, is quite surprising.

**Theorem 0.4.** *Let* $G$ *be any graph. We can always pass from a configuration* $C$ *to the complementary configuration* $\bar{C}$*.*

In other words, we can always arrive at the all-on configuration. Moreover, this configuration may well be the only (non-trivial) configuration, as can be seen by considering the complete graph. Aiming for a better understanding of Theorem 0.4, several more proofs were presented.