Anticoloring for Toroidal Grids

Abstract

An *anticoloring* of a graph is a coloring of some of the vertices, such that no two adjacent vertices are colored in distinct colors. In the basic *anticoloring problem* we are given an undirected graph G and positive integers $B_1, \ldots, B_k$, and we have to determine whether there exists an anticoloring of G such that B_j vertices are colored in color j, $j = 1, \ldots, k$.

The anticoloring problem with $k = 2$ is the *Black-and-White Coloring (BWC) problem*. We usually refer to the optimization version of the BWC problem, in which we are given a graph G and a positive integer B, and have to color B of the vertices in black, so that there will remain as many vertices as possible which are non-adjacent to any of the B vertices. (These latter vertices are to be colored in white.) We denote by W the maximum possible number of such vertices.

The BWC problem has been introduced and proved to be NP-complete by Hansen, Hertz and Quinodoz.

Problem 0.0.1. *Given positive integers m, n and B, place B black and W white kings on an $m \times n$ toroidal chessboard, so that no black king and white king attack each other, and with W as large as possible.*

In our main theorem, we provide an algorithm for coloring the vertices of an $m \times n$ board, solving our optimality problem. It turns out that the optimal coloring behaves differently depending on the size of B. For small B (up to $n^2/4$ approximately), an optimal coloring may be obtained by coloring an 'almost square' region. For intermediate B (from $n^2/4$ up to $mn - n^2/4$ approximately), an optimal coloring may be obtained by coloring an almost rectangular region, consisting of about B/n adjacent completely full columns. For large B, the optimal coloring is almost the complement of a square. More formally,

Theorem 0.0.1. *Consider Problem 0.0.1, where $m \geq n \geq 1$. An optimal solution may be constructed, depending on the size of B, as follows:*

1. $B \le (\frac{n}{2}-1)^2$: *Let* $a = \left\lceil \sqrt{B} \right\rceil$ *and* $r = B-(a-1)\left\lfloor \frac{B}{a-1} \right\rfloor$. *Color in black the set* $\{1,\ \ldots,\ a-1\} \times \{1,\ \ldots,\ \left\lfloor \frac{B}{a-1} \right\rfloor\} \cup \{a\} \times \{1,\ldots,r\}$.

2. $(\frac{n}{2}-1)^2 < B \le mn-(\frac{n}{2}+1)^2$: *Let* $a = \left\lceil \frac{B}{n} \right\rceil$ *and* $r = B-(a-1)n$. *Color in black the set* $\{1,\ \ldots,\ n\} \times \{1,\ \ldots,\ a-1\} \cup \{1,\ \ldots,\ r\} \times \{a\}$.

3. $mn-(\frac{n}{2}+1)^2 < B$: *Let* $a = \left\lceil \sqrt{mn-B} \right\rceil$ *and* $r = a \cdot \left\lceil \frac{mn-B}{a} \right\rceil - (mn-B)$. *Color in black the set* $\{a+1,\ \ldots,\ n\} \times \{1,\ \ldots,\ m\} \cup \{1,\ \ldots,\ n\} \times \{\left\lceil \frac{mn-B}{a} \right\rceil + 1,\ \ldots,\ m\} \cup \{a\} \times \{1,\ \ldots,\ r\}$.