On a Question of Leiss regarding the Hanoi Towers Problem

Abstract

The problem of the Towers of Hanoi is generalized in such a way that moves of discs may be made only along edges of a given directed graph G. Leiss [1] has shown that arbitrarily many discs can be moved from the source vertex S to the destination vertex D iff there is a cycle of length 3 or more in the transitive closure of G which can be reached from S and from which D can be reached. We consider graphs which do not satisfy this characterization; therefore there is only a finite number of discs which can be handled. We denote by g_n the maximal such number when G varies over all such graphs with n vertices.

Leiss [1] was interested in the asymptotic behaviour of g_n. The proof of the characterization above may be shown to yield an upper bound, which however exceeds n^n. On the other hand, he showed in [2] that $g_n \geq n^{C \log n}$ for an appropriate constant $0 < C \leq \frac{1}{2}$. He asked whether g_n grows exponentially fast.

We prove that g_n grows *subexponentially* fast. Moreover, there exists a constant C such that $g_n \leq Cn^{\frac{1}{2} \log n}$ for each n. On the other hand, for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that $g_n \geq C_\varepsilon n^{(\frac{1}{2} - \varepsilon) \log n}$ for each n.

References

[1] E.L. Leiss, “Solving the ‘Towers of Hanoi’ on Graphs”, *Journal of Combinatorics, Information and System Sciences* **8** (1983), 81–89.

[2] E.L. Leiss, “Finite Hanoi problems: How many discs can be handled?”, *Congressus Numerantium* **44** (1984), 221–229.