Lights Out on Trees

Nadav Azaria

June 7, 2004

Abstract

The following is a popular hand-held electronic game by Tiger Electronics, called "Lights Out". It is played on a 5×5 grid of button-bulbs. By pressing a button, its light and those of the (non-diagonally) adjacent buttons are changed (from ON to OFF and vice versa). Given some initial pattern of lights, one has to switch them all OFF by pressing several buttons. Obviously, the game can be played on other boards (indeed, Tiger produced a 6×6 version, and even a $3 \times 3 \times 3$ cube version), and naturally generalizes to any graph G.

Questions related to (the generalization of) "Lights Out" (e.g., which configurations can be turned off for which graphs, how many buttons does one have to press to arrive at a required configuration, what is the smallest number of lights that can be left ON, etc.) have generated a surprising amount of research (see the references below).

We consider the game on trees and address the following question.

Question 0.1. *Which trees have the property that one can pass from any configuration to any other?*

We call such trees *light transitive*. We provide a characterization of light transitive trees. Moreover, we present a linear time algorithm for evaluating the number of reachable configuration in any tree.

The proportion of light transitive graphs on n vertices, out of all graphs on n vertices, converges to a positive number as $n \to \infty$. This

seems not to be the case for labeled trees. It seems that only an exponentially small fraction of the labeled trees on n vertices are light transitive. We present a partial result in this direction.

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