ON DIFFERENTIAL TANNAKIAN CATEGORIES AND COLEMAN INTEGRATION

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1. Introduction

As part of my introductory lectures on Coleman integration, to be published in the proceedings of PIA 2010 - The arithmetic of fundamental groups, I discussed the Galois theory of differential equations and the theory of differential Tannakian categories, with a speculative application to Coleman integration in families. The referee of the paper thought it was not appropriate to discuss this material there, so it was left out. However, I still feel that there might be some interest in the material there, in particular in the presentation of Differential Tannakian theory. I am therefore putting the above section, at the moment essentially unaltered, in the present work. Since I did not get to do any serious editing, there is a lot of interesection with the material that did find it into the published work.

The theory of differential Tannakian categories is due to Ovchinikov [6, 7, 8]. The most important application for us, which is also where the theory has its origins, is the theory of parameterized linear differetials developed by Landesman [5]

2. Differential Tannakian categories

We now give a fairly brief introduction to differential Tannakian categories. The Galois theory of differential equations is fairly well known (e.g., [10]). We recall that a differential ring is a ring R (we assume our rings are commutative) equipped with a derivation $\partial : R \to R$. Let K be a differential field. Starting from a linear differential equation

$$\nabla : \partial_x \vec{y} = Ay \; , \tag{1}$$

where A has entries in K, the theory finds an extension differential field K^{∇} over which all solutions of the equation are defined, and considers the group of automorphisms of K^{∇} over K. The resulting Galois groups are algebraic groups. The Tannakian approach to differential Galois theory interprets these Galois groups as the groups associated with the Tannakian subcategory (of all linear differential equations) generated by the given one.

Differential algebraic groups and differential Tannakian categories start showing up when one considers parameterized systems of differential equations. Suppose K is equipped with two commuting derivations ∂_x and ∂_t (e.g., $K\mathbb{C}(x,t)$). If we have a differential equation with respect to ∂_x, we can ignore the t-derivation completely and recover the same theory as before. Instead, we can look for a field extension which still carries two derivations, over which all solutions of the equation exist. This, perhaps surprisingly, gives a fascinatingly different theory. It is best to consider an example (taken from [1, Example 3.1]).

Suppose $K = \mathbb{C}(x,t)$ and our differential equation is

$$\nabla : \partial_x y = \frac{t}{x} y$$

whose solution is $y = cx^t$. Thus, in standard differential Galois theory we would simply add x^t to K. However, since our field K^∇ should be closed with respect to both ∂_x and ∂_t, and since $\partial_t x^t = \log(x)x^t$ we have

$$K^\nabla = K(x^t, \log(x)) \ .$$

We now consider an automorphism σ of K^∇, commuting with the derivations and fixing K. It preserves solutions of the differential equation so

$$\sigma(x^t) = a(t)x^t \ .$$

On the other hand $\sigma(\log(x))$ should be constant with respect to t and differentiate to $1/x$ with respect to x. We therefore have

$$\sigma(\log(x)) = \log(x) + b$$

where $b \in \mathbb{C}$. We now have

$$\begin{aligned}(\partial_t a(t)) \times x^t + a(t)\log(x)x^t &= \partial_t(a(t)x^t) = \partial_t(\sigma(x^t)) \\ &= \sigma(\partial_t x^t) = \sigma(\log(x)x^t) = (\log(x)+b)a(t)x^t\end{aligned}$$

and it follows that $ba(t) = \partial_t a(t)$ hence $b = (\partial_t a(t))/a(t)$. We obtain our differential Galois group

$$\mathrm{Gal}(K^\nabla/K) = \{a(t) \neq 0 \ , \ \partial_t \frac{\partial_t a(t)}{a(t)} = 0\}$$

where the group structure is given by multiplication. This is an example of a *Linear differential algebraic group* (in this case over the field $\mathbb{C}(t)$).

We recall that a differential affine variety over a differential field K (with one derivation ∂_t for simplicity) is the subset of some K^n which is the set of solutions of some differential equation. In other words, if the coordinates are a_1 to a_n, it is defined by the vanishing of a polynomial in the a_i and their derivatives $(\partial_t)^j a_i$.

The ring of differential functions on the affine space $\mathbb{A}^n$ is the polynomial ring in an infinite number of (formal) variables

$$K\{\mathbb{A}^n\} := K[\{(\partial_t)^j a_i \ , \ j \geq 0 \ , \ i = 1,\ldots,n\}] \ .$$

It has a derivation, extending the one on K, given by

$$\partial_t((\partial_t)^j a_i) = (\partial_t)^{j+1} a_i$$

making it a *differential K-algebra*. Given a differential affine subvariety V of $\mathbb{A}^n$, we can associate with it the radical differential ideal I generated by the defining equations (differential ideal means closed under ∂_t) and its ring of functions $K\{V\} = K\{\mathbb{A}^n\}/I$. Just like in usual algebraic geometry we may now interpret V, or rather its set of K-rational points, as the set of differentiable K-algebra homomorphisms $K\{V\} \to K$ (K-algebra homomorphisms commuting with the derivation). Note that algebraic varieties are a special case of differential varieties but their rings of functions in the two cases are quite different. There are corresponding notions of morphisms between differential varieties $V \to W$, and these give rise to differential algebra morphisms $K\{W\} \to K\{V\}$.

A differential algebraic group over K is defined as a differential subvariety G of some GL_n/K which is closed under multiplication and inversion. By the usual procedure, this gives rise to a structure of a *differential Hopf algebra* on $K\{G\}$, i.e., a Hopf algebra together with a derivation extending the one on K and commuting with all structural morphisms (i.e., both multiplication and co-multiplication between $K\{G\}$ and $K\{G\} \otimes K\{G\}$

commute with the derivation, which is defined by

$$\partial(a\otimes b) = (\partial a)\otimes b + a\otimes(\partial b) \tag{2}$$

on $K\{G\}\otimes K\{G\}$). A homomorphism of differential algebraic groups, i.e., a differentiable algebraic morphism which is also a group homomorphism, induces a homomorphism of differentiable Hopf algebras. An interesting example is provided by the dlog homomorphism

$$\mathrm{d}\log : \mathbb{G}_m \to \mathbb{G}_a \ , \ \mathrm{d}\log(a) = \frac{\partial(a)}{a} \ . \tag{3}$$

Definition 2.1. A representation of a Linear differentiable algebraic group G on a finite dimensional K-vector space V is a differentiable algebraic homomorphism $G \to \mathrm{GL}(V)$.

In this definition we make $\mathrm{GL}(V)$ into a differential algebraic group by identifying it with some GL_n by choosing a basis.

We first remark that even a standard algebraic group has some new representations when viewed as a differential algebraic group. For example, $\mathbb{G}_m$ has a two dimensional representation given by $\left(\begin{smallmatrix} 1 & \mathrm{d}\log \\ 0 & 1 \end{smallmatrix}\right)$.

In terms of the Hopf algebra $K\{G\}$, a representation of G on V is given by a $K\{G\}$-comodule structure on V,

$$\rho : V \to V \otimes_K K\{G\}$$

in the same way as an algebraic G-representation would (no differentials here!). To see this, consider first the comodule corresponding to the standard representation of GL_n on K^n as an algebraic representation, then obtain the comodule structure for K^n with respect to $K\{G\}$ by simply composing with the embedding of the algebraic $K[G]$ in the differential $K\{G\}$. In concrete terms recalling that the comodule structure in the algebraic setting is given by sending v to the function (viewed as an element of $V\otimes K[G]$) $g \mapsto gv$), this gives the comodule structure

$$e_i \mapsto \sum e_j \otimes a_{ji} \tag{4}$$

Now, for a representation of G compose with the Hopf algebra homomorphism $K\{\mathrm{GL}_n\} \to K\{G\}$ to obtain the required $K\{G\}$-comodule structure, from which the representation is easily recoverable.

The fact that no differentials are involved in the Hopf-algebraic description of differential representation is perhaps confusing if G is algebraic, but it is not a contradiction to anything because $K[G]$ is quite different from $K\{G\}$. It nevertheless suggests that a Tannakian description of a differential algebraic group in terms of its category of representations needs to use something outside the structure of Tannakian category on this category of representations.

A Tannakian description of Linear differential algebraic groups was given quite recently by Ovchinikov [6, 7, 8]. There is an alternative approach using model theory due to Kamensky [3, 4]. The papers by Kamensky do an excellent job of describing the categorical formalism. Unfortunately, for proofs they use model theory in a rather “black-box” approach (referring to deep work of Hrushovsky) which sheds little light on the algebraic point of view. We try to give here a minimal account, which we found useful in understanding the situation. We note however, that the above mentioned references due more, in the sense that they show, under some additional assumptions, that the Galois group is a pro-differential algebraic group rather than just a Hopf algebra (this is not equivalent in the differential algebraic setting see [6, p. 8]).

Suppose we are given a Linear differential algebraic group G, which we would like to recover from its category of representations Rep_G. If we take the Hopf-algebraic point of view, the usual Tannakian formalism already reconstructs for us the Hopf algebra $K\{G\}$ (since the category is just that of comodules for that algebra) and so we only need to recover the derivation. This is then not to be found in the category Rep_G itself, as this suffices exactly to recover the Hopf algebra structure by Tannakian duality. It must come from an additional structure on Rep_G. A so called *differential structure*.

For motivating this structure, consider again the differential equation (1) but over the field $\mathbb{C}(x,t)$. Using the fact that the two derivations commute we obtain

$$\partial_x(\partial_t\vec{y}) = \partial_t(A\vec{y}) = (\partial_t A)\vec{y} + A\partial_t\vec{y}\,.$$

This means that we obtain a new differential equation

$$\partial_x \begin{pmatrix} \vec{y} \\ \partial_t\vec{y} \end{pmatrix} = \begin{pmatrix} A & 0 \\ \partial_t A & A \end{pmatrix} \begin{pmatrix} \vec{y} \\ \partial_t\vec{y} \end{pmatrix} \tag{5}$$

which is an extension of the original equation by itself. This can be seen to be a functorial construction, and is the required differential structure.

Definition 2.2. A differential rigid abelian tensor category over the field K is a rigid abelian tensor category $\mathscr{T}$, satisfying the condition that $\mathrm{End}(\mathbb{1}) = K$, together with a functor $D : \mathscr{T} \to \mathscr{T}$ sitting in a short exact sequence

$$0 \to \mathrm{id} \to D \to \mathrm{id} \to 0\,, \tag{6}$$

and which satisfies a certain list of axioms connecting D with the tensor structure (see [6, 7, 8, 3, 4]). We call D the *differential structure* on $\mathscr{T}$.

Remark 2.3. In the description given in [3] one defines a new category consisting of short exact sequences $0 \to M \to N \to M \to 0$ in the category $\mathscr{T}$. This is given a tensor structure as follows: The tensor product of $(0 \to M_1 \to N_1 \to M_1 \to 0)$ with $(0 \to M_2 \to N_2 \to M_2 \to 0)$ is the Baer sum of extensions of the first sequence tensored with M_2 with the second sequence tensored with M_1, both viewed as extensions of $M_1 \otimes M_2$ by itself. The requirements on D are now simply that it defines a tensor functor from $\mathscr{T}$ to this new category.

We note that we do not need to assume that K is a differential field because that will be forced from the axioms. To get a Tannakian theory we need to introduce a differential structure on Vec_K for a differential field K. To see this, recall first that for any ring R, R-module extensions of R by itself are equivalent to derivations of R - The R module associated with a derivation ∂ is R^2 with $r \in R$ acting by multiplying a column vector with $\begin{pmatrix} r & 0 \\ -\partial r & r \end{pmatrix}$.

By tensoring with an arbitrary R-module M we see that a derivation leads to a functor D from the category of R-modules to itself which sits in a short exact sequence as in (6). This holds of course for a differential field K providing the required structure on Vec_K. In concrete terms, for a K-vector space V, $D(V)$ is the abelian group $V \times V$ with the K-vector space structure given by

$$\alpha(v_1, v_2) = (\alpha v_1, \alpha v_2 - \partial(\alpha)v_1)\,. \tag{7}$$

A useful convention is to identify the vector $v \in V$ with $(0,v) \in D(V)$ and to denote the map $v \mapsto (v,0)$ by ∂. This way, the action of the field is given by the following equations

$$\alpha v = \alpha v\,,$$
$$\partial(\alpha v) = (\partial\alpha)v + \alpha\partial v$$

The functoriality is given sending $T : V \to W$ to $(T,T) : D(V) \to D(W)$. Note however that in terms of standard bases this description is misleading: Suppose that B is a matrix with entries in K such that multiplying on the left by B gives a linear map $B : K^n \to K^m$. Then, in terms of the standard bases provided, e.g. for K^n by $(e_1,0),\dots,(e_n,0),(0,e_1),\dots,(0,e_n)$ the matrix of $D(B)$ is going to be

$$\begin{pmatrix} B & 0 \\ \partial(B) & B \end{pmatrix} \tag{8}$$

The above description immediately suggests the extension of D to differential algebraic representations and to the category of Hopf-comodules. Indeed, applying D to the standard representation of GL_n on K^n we get, by (8), the representation of GL_n on $K^n \oplus K^n$ given by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ \partial(A) & A \end{smallmatrix}\right)$ whose associated comodule is given by

$$\begin{aligned} e_i &\mapsto \sum e_j \otimes a_{ji} \\ \partial e_i &\mapsto \sum (\partial e_j \otimes a_{ji} + e_j \otimes \partial a_{ji}) \end{aligned}$$

from which we get the extension of D to comodules:

$$D(\rho)(v) = \rho(v)\ ,\ D(\rho)\partial v = \partial(\rho(v))$$

where ∂ acts on a tensor product in the obvious way (2).

Definition 2.4. A differential tensor functor $\mathscr{T}_1 \xrightarrow{F} \mathscr{T}_2$ between two differential rigid abelian tensor categories, with differential structures D_1 and D_2 respectively, is a tensor functor together with a natural isomorphism $D_2 \circ F \cong F \circ D_1$ compatible in the obvious way with the short exact sequence (6). A morphism of differential tensor functors $\alpha : F \to F'$ is a natural transformation of tensor functors which commutes with D in the sense that the diagram

$$\begin{array}{ccc} F \circ D_1 & \xrightarrow{\alpha} & F' \circ D_1 \\ \downarrow & & \downarrow \\ D_2 \circ F & \xrightarrow{D_2(\alpha)} & D_2 \circ F' \end{array}$$

commutes.

Example 2.5. Quite clearly the forgetful functor $\mathrm{Rep}_G \to \mathrm{Vec}_K$ is a differential tensor functor. Another example is solutions of differential equations. Consider the functor Sol that takes a differential equation ∇ as in (1) over the field $\mathbb{C}(x,t)$ to its space of solutions in $\mathbb{C}(x,t)^\nabla$ considered as a vector space over $\mathbb{C}(t)$. Then, according to (5), we can map $D(\mathrm{Sol}(\nabla))$ to $\mathrm{Sol}(D(\nabla))$ using the formula

$$(\vec{y}_1,\vec{y}_2) \mapsto (\vec{y}_1,\vec{y}_2 + \partial_t \vec{y}_1)$$

(note that to make this a $\mathbb{C}(t)$ linear map we exactly need to give $D(\mathrm{Sol}(\nabla))$ the vector space structure (7)).

Definition 2.6. A differential fiber functor on a differential rigid abelian tensor category is a differential tensor functor ω to Vec_K. If the category has a fiber functor it is called neutral Tannakian.

Theorem 2.7 ([8, Theorem 1])**.** *Let $\mathscr{T}$ be a neutral differential Tannakian category with the differential fiber functor ω. Then $\mathscr{T}$ is equivalent to the category* Rep_G *of finite dimensional representations of an affine differential group scheme G. Furthermore, for a* *differential K-algebra F we have*

$$G(F) = \mathrm{Aut}(\omega \otimes F) \tag{9}$$

where Aut *here means automorphisms of the differential tensor functor in the sense of Definition 2.4.*

We sketch a proof of this result. Standard Tannakian theory tells us that $\mathscr{T}$ is equivalent to the category of representations of a certain affine group scheme, or equivalently to the category of comodules over some Hopf algebra H. The result will follow if we construct a derivation on H in such a way that the functor D on $\mathscr{T}$ corresponds to the functor D on the category of H-comodules.

We now recall [2] that the Hopf algebra H may be described concretely as an "algebra of matrix coefficients". An element in such an algebra is provided by a pair $(T, \mathscr{E})$ where $T \in \mathscr{T}$, $\mathscr{E} \in \omega(T) \otimes \omega(T)^*$, where $\omega(T)^*$ is the K-dual of $\omega(T)$. When $\mathscr{T}$ is the category of representations of an affine group scheme G and $\omega(T)$ is just the underlying vector space to $T \in \mathscr{T}$, then a pair $(T, v \otimes w^*)$ is to be thought of as corresponding to the function on G given by $g \mapsto w^*(gv)$. One identifies two pairs $(T_1, \mathscr{E}_1)$ and $(T_2, \mathscr{E}_2)$ if there exists a map $f : T_1 \to T_2$ in such a way that both $\mathscr{E}_1$ and $\mathscr{E}_2$ are obtained from an element of $\omega(T_1) \otimes \omega(T_2)^*$ via the obvious maps. Clearly, two identified elements give rise to identical functions. One can easily derive from this formulas for the multiplication and comultiplication.

This description immediately suggests the way to define the derivation on H. Indeed, by (8) the derivatives of the matrix coefficients associated with the representation T are visible in $D(T)$. A bit of thought gives the following formula for the derivative of matrix coefficients

$$\partial(T, v \otimes w^*) = (D(T), (\partial v) \otimes (\partial w^*))$$

(if one is puzzled by the fact that ∂ is applied twice, note that the K-algebra structure of H can be obtained by multiplying with elements of K either v or w^*). It now becomes a routine check to see that ∂ is indeed a derivation of the Hopf algebra H and it is quite obvious by the way we defined it that the D on H-comodules and on vector spaces correspond.

Finally, the description of the differential points of G is fairly easy. A point in the usual sense with values in F corresponds on the one hand to an algebra morphism $H \to F$ and on the other hand to an automorphism of $\omega \otimes F$. Now, both the condition that the homomorphism preserve the derivation, and the condition that the automorphism is an automorphism of differential functors boil down to saying that "if α_T is represented by the matrix A, then $\alpha_{D(T)}$ is represented by the matrix $\left(\begin{smallmatrix} A & 0 \\ \partial A & 0 \end{smallmatrix}\right)$."

3. Applications to Coleman integration in families

A major difficulty with the above construction is that it is dependent on the choice of a derivation. If we want to get a theory which takes all derivations into account (like connections do) we are led, after some thought into making the following construction.

Recall that we are assuming a situation $\pi : X \to S$ and that S is one dimensional. Suppose M is a vector bundle on X equipped with a relative connection

$$\nabla : M \to M \otimes \Omega^1_{X/S}$$

which is integrable. Suppose we can lift ∇ to an absolute connection $\tilde{\nabla} : M \to M \otimes \Omega^1_X$. Because ∇ is integrable, the curvature of $\tilde{\nabla}$, which in general lies in $\Omega^2_X \otimes \mathrm{End}(M)$, actually lies in its first filtered part $F^1 \otimes \mathrm{End}(M)$ and so we may define

$$C = \varpi \tilde{\nabla}^2 \in \Omega^1_S \otimes \Omega^1_{X/S} \otimes \mathrm{End}(M) .$$

We define a new module with connection $D = D_{\tilde{\nabla}} = D(M,\nabla)_{\tilde{\nabla}}$ where $D = M \oplus \Omega^1_S \otimes M$ and the connection is defined by

$$\nabla_D(m_1, \alpha \otimes m_2) = (\nabla m_1, \alpha \otimes \nabla m_2 - C \times M) .$$

This connection is integrable. It is independent of $\tilde{\nabla}$ up to a canonical isomorphism: Suppose $\tilde{\nabla}' = \tilde{\nabla} + A$ is another lift. Here $A \in \Gamma(X, \pi^{-1}\Omega^1_S \otimes \mathrm{End}(M))$ because it projects to 0 in relative forms. Then it is easy to compute that the corresponding curvature is $C' = C + \nabla(A)$ (where ∇ takes Ω^1_S as constants and acts in the induced way on $\mathrm{End}(M)$). Then we get a canonical horizontal isomorphism between $D_{\tilde{\nabla}}$ and $D_{\tilde{\nabla}'}$ given by

$$(m_1, \alpha \otimes m_2) \mapsto (m_1, \alpha \otimes m_2 + Am_1) .$$

Consequently we can glue these objects, coming from different local liftings of ∇, to obtain a global object $D(M,\nabla)$. Clearly, there is a short exact sequence of vector bundles with relative connections,

$$0 \to \Omega^1_S \otimes M \to D(M,\nabla) \to M \to 0$$

because all the horizontal isomorphisms constructed commute with these short exact sequences. Clearly, the construction D is functorial.

For vector bundles over S we can make an analogous functorial construction. For such a vector bundle M define $D(M) = M \oplus \Omega^1_S \otimes M$ with the $\mathscr{O}_S$-module structure

$$s \times (m_1, \alpha \otimes m_2) = (sm_1, s\alpha \otimes m_2 + ds \otimes m_1) .$$

Suppose now that X and S are residue discs. Then, mimicking the constructions in Example 2.5 we have a well behaved solutions functor

$$\mathrm{Sol} : \{ \text{ Relative connections } (M, \nabla : M \to \Omega^1_{X/S}) \} \to \{ \text{ Vector bundles on } S \}$$

given by taking horizontal sections, and a map

$$D \circ \mathrm{Sol} \to \mathrm{Sol} \circ D \ , \ (m_1, \alpha \otimes m_2) \mapsto (m_1, \alpha \otimes m_2 + \tilde{\nabla} m_1) \tag{10}$$

with $\tilde{\nabla}$ the local lifting of ∇ used for the construction of $D(M)$, where m_1 and m_2 are horizontal sections for ∇, implying that $\tilde{\nabla} m_1 \in \Omega^1_S \otimes M$.

There is some way to go before we can incorporate these constructions into a functioning Tannakian differential Tannakian formalism. The main problem is of having a good Tannakian theory over rings (but see [11, 9] for some progress on these matters). Assuming such a formalism it seems reasonable to prove the following.

Conjecture 3.1. *For any two residue discs in X there is a unique differentiable path invariant under Frobenius between the two solution functors.*

Note that a differentiable path, in the sense of differential Tannakian categories should really be thought of as a horizontal path, only that there is no connection on paths.

We expect the proof of this conjecture to roughly follow the method described in out paper at the PIA proceedings. The Lie algebra will inherit a connection, and its graded pieces are going to be dominated again by tensor powers of $H^1_{\mathrm{dR}}(X/S)^*$ with the connection induced by the Gauss-Manin connection. Differential paths are now going to be related with horizontal sections on the Lie algebra, over some residue disc in S. But these are now well behaved vector spaces over the ground field so we can apply Frobenius as before to complete the argument.

Let us close by observing the relation of this conjecture with the condition (**??**). In the notation introduced before stating this condition, suppose we have a closed $\omega \in \Omega^1_{X/S}$ and we lift it to a $\tilde{\omega} \in \Omega_X$. correspondingly we have the connection ∇ and its lift $\tilde{\nabla}$ given by

$$\nabla = \mathrm{d}_r - \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix} , \; \tilde{\nabla} = \mathrm{d} - \begin{pmatrix} 0 & 0 \\ \tilde{\omega} & 0 \end{pmatrix} .$$

The curvature of $\tilde{\nabla}$ is going to be $\left(\begin{smallmatrix} 0 & 0 \\ -d\tilde{\omega} & 0 \end{smallmatrix}\right)$. We can now compute that the connection ∇_D is going to be given by the following formula,

$$\nabla_D \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \alpha \otimes \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \right) = \left(\nabla \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \alpha \otimes \nabla \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varpi \, \mathrm{d}\, \tilde{\omega} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) .$$

Now we check what it means for $\left(\begin{smallmatrix} 1 \\ F_\omega \end{smallmatrix}\right)$ to be a horizontal section at the residue discs of x and z, say, which is compatible with respect to translation by a differential path. Appropriately translating the condition in Definition 2.4 we find that it simply means that another horizontal section that translates under the same path is the image of $\left(\begin{smallmatrix} 1 \\ F_\omega \end{smallmatrix}\right)$ under (10) which is

$$\left(\begin{pmatrix} 1 \\ F_\omega \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{d}\, F_\omega - \tilde{\omega} \end{pmatrix} \right) .$$

In other words, $\mathrm{d}\, F_\omega - \tilde{\omega}$ is a Coleman integral, which is just (**??**).

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